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On Minimum-Fuel Control of Affine Nonlinear Systems

JING-SIN LIU, KING YUAN, AND WEI-SONG LIN

Abstract—**The minimum-fuel control problem is investigated for a class of multiinput affine nonlinear systems whose associated Lie algebra is nilpotent. Interesting consequences of the maximum principle are deduced for such systems.**

I. Introduction

Optimal control theory [7] provides a systematic design method for modern control systems and thus plays an important role in linear control theory (more specifically, the linear quadratic regulator and linear quadratic Gaussian control theories). Roughly speaking, the success of optimal control theory in the context of linear systems is due to the ease of computation of the optimal control law. On the other hand, until now there has been a lack of systematic and reliable procedures for solving nonlinear optimal control problems. This is unfortunate, but some attempts have been made to resolve these difficulties; in [2] a Lie algebraic approach has been used to derive a set of quasi-linear partial differential equations which the optimal feedback law must satisfy.

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J.-S. Liu and W.-S. Lin are with the Department of Electrical Engineering, National Taiwan University, Taipei, Taiwan, Republic of China.
K. Yuan is with the Department of Mechanical Engineering, National Taiwan University, Taipei, Taiwan, Republic of China.
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Application of this new computing method for the optimal control of regulation of satellite angular momentum has been reported in [8] recently. The Lie brackets of vector fields have become a main mathematical tool in nonlinear control theory [9] and optimal control theory [10]. In this note we consider the following optimal control problem:

the performance index to be minimized is:

$$J(x_0, u)=\frac{1}{2}\int_0^T u^T u\,dt+K(x(T)) \tag{1}$$

subject to the smooth affine system dynamics

$$\begin{aligned}\dot{x}&=f(x)+\sum_{i=1}^{m} g_i(x)u_i, \qquad x(0)=x_0\\ &=f(x)+g(x)u \end{aligned}\tag{2}$$

where $T > 0$ is the fixed end time, the system's vector fields f, g_i, and K are all smooth, x is a real n-vector, u_i is a scalar control, $i = 1, \cdots, m$. Since in most cases the cost integrand $u^T u$ of (1) can be identified as the energy expended, we call the optimal control problem (1), (2) a minimum-fuel control problem for (2). Associated with the nonlinear system (2) we define the Lie algebra L

$L :=$ the Lie algebra generated by the system vector fields

$$\{f, g_1 \cdots, g_m\} \tag{3}$$

i.e., L is the set consisting of f, $g_1, \cdots, g_m$ and all possible Lie brackets generated by f, $g_1, \cdots, g_m$ and their linear combinations. The following notations are also used in this note:

$$\begin{aligned} ad_L^0 &:= L\\ ad_L L&=[L, L]\\ &=\{[X, Y] : X\in L, Y \in L\}\\ ad_L^{k+1}L&=ad_L ad_L^k L \end{aligned}$$

where $[\cdot, \cdot]$ is the Lie bracket.

The minimum-fuel control problem of a scalar input bilinear system was studied in [1], assuming that the Lie algebra L is nilpotent. The purpose of this note is to generalize results of [1] to a more general class of multiinput systems described by (2) with nilpotent Lie algebra L. The organization of this note is as follows. In the next section, some preliminary notions and definitions will be given; then the motivations of our work on the problem (1), (2) will be explained. Then we will concentrate on the solution to the optimal control problem if (3) is nilpotent. It will be seen that some results of [1] are due not only to the nilpotent property of L but also to the fact that the systems considered are single-input in which case the solution is greatly simplified. In the last section we make several conclusions.

II. Preliminaries and Motivation

In this note, we are especially interested in system (2) with special structure: the Lie algebra L is nilpotent. Recall [5] that a Lie algebra L is nilpotent if there exists a positive integer k such that

$$ad_L^k L=0. \tag{4}$$

Note that other equivalent definitions are available [5] and the one adopted here is the same as that used in [1]. A system of the form (2) can be very complex to allow for the application of existing control theory, e.g., optimal control theory. A possible first step to overcome the difficulties arising from the system's complex structure is to approximate system (2) locally by a system with simpler structure. A system with simpler structure must be more mathematically tractable to facilitate the subsequent control design problem. It has been shown [3], [4] that under some nonrestrictive conditions, a system of the affine form (2), with or without the drift term $f(x)$, can be locally approximated by a nilpotent one with the same form of state equations as (2). Therefore, it is useful to study the control problem of an affine smooth nonlinear control system with nilpotent Lie algebra L before we thoroughly investigate the general control problem. For brevity, the nonlinear system described by the state equations (2) is called a nilpotent nonlinear control system if its associated system's Lie algebra L is nilpotent.

III. Optimal Control Problem

In this section, we solve the optimal control problem (1), (2) under the assumption that L is nilpotent, i.e., we consider the minimum-fuel control problem for nilpotent control system (2). For such an optimal control problem, we formulate the associated Hamiltonian

$$\begin{aligned} H&=p^T(f+gu)-\frac{1}{2}u^T u\\ &=H(x, p, u) \end{aligned}\tag{5}$$

where p is the $n \times 1$ real costate vector. For the Hamiltonian given by (5) we have the Hamiltonian system given below

$$\begin{aligned} \dot{x}&=f(x)+g(x)u, \qquad x(0)=x_0\\ \dot{p}&=-\frac{\partial H}{\partial x}(x, p, u)\\ &=-\frac{\partial}{\partial x}(f+gu)^T p, \qquad p(T)=-\frac{\partial K}{\partial x}(x(T)) \end{aligned}\tag{6}$$

and an $m \times 1$ output is also added to (6)

$$\begin{aligned} y&=\frac{\partial H}{\partial u}\\ &=\begin{bmatrix} p^T g_1-u_1\\ \vdots\\ p^T g_m-u_m\end{bmatrix}. \end{aligned}\tag{7}$$

Note that the system (6), (7) forms a Hamiltonian system in the canonical coordinate system (x, p). The optimal control problem (1), (2) with Hamiltonian (5) is regular (or nondegenerate) since

$$\frac{\partial^2 H}{\partial u^2}=-I$$

is nonsingular for any (x, p, u). The optimal control u^* satisfies the necessary condition

$$y=0$$

or equivalently,

$$\left.\frac{\partial H}{\partial u}\right|_{u^*} := 0. \tag{8}$$

From this equation (8) we get explicitly

$$u_i^*=p^T g_i(x), \qquad i=1, 2, \cdots, m. \tag{9}$$

Let

$$H^*(x, p) := H(x, p, u^*)$$

denote the optimal Hamiltonian, then the resulting Hamiltonian system (6) after replacing $H(x, p, u)$ by $H^*(x, p)$ will be called an optimal Hamiltonian.

From (7), the ith $(i = 1, 2, \cdots, m)$ output is

$$y_i=p^T g_i-u_i$$

then the first time derivative of y_i is

$$\dot{y}_i = \dot{p}^T g_i + p^T \dot{g}_i - \dot{u}_i$$
$$= -p^T \frac{\partial}{\partial x}(f+gu)g_i + p^T \frac{\partial g_i}{\partial x}(f+gu) - \dot{u}_i$$
$$= p^T[f+gu,\ g_i] - \dot{u}_i$$
$$= p^T ad_F g_i - \dot{u}_i$$

or

$$\dot{u}_i = p^T ad_F g_i - \dot{y}_i \tag{10}$$

where

$$F := f + gu. \tag{11}$$

The following simple Lemma is of use in the subsequent derivations.

Lemma: Let Y be a vector and p the optimal costate vector. Then

$$\frac{d}{dt}(p^T Y) = p^T ad_F Y$$

where F is as defined in (11) and the time derivative is calculated along the system's trajectory.

Proof: The time derivative of the function $p^T Y$ along the system's trajectory is

$$\frac{d}{dt}(p^T Y) = \dot{p}^T Y + p^T \dot{Y}$$
$$= -p^T \frac{\partial F}{\partial x} Y + p^T \frac{\partial Y}{\partial x} F$$
$$= p^T ad_F Y.$$

From this Lemma, we easily obtain the second time derivative of u_i by differentiating (10):

$$\ddot{u}_i = p^T ad_F^2 g_i - \ddot{y}_i. \tag{12}$$

In general, we have for each $i = 1, 2, \cdots, m$

$$u_i^{(k)} = p^T ad_F^k g_i - y_i^{(k)}, \qquad k = 0, 1, 2, \cdots. \tag{13}$$

Thus, the necessary condition $y_i^{(k)} = 0$, $k = 0, 1, 2, \cdots$, for optimality of u_i^* is equivalent to

$$u_i^{*(k)} = p^T ad_F^k g_i|_{u*}, \qquad k = 0, 1, 2, \cdots.$$

The above derivations give the following.

Proposition: The necessary conditions for optimality of u_i^* are that along the flow of the optimal Hamiltonian H^*

$$u_i^{*(k)} = p^T ad_F^k g_i|_{u*}, \qquad k = 0, 1, 2, \cdots, i = 1, 2, \cdots, m. \tag{14}$$

Remark: The hierarchy of conditions (13) are also given in [2] for a more general class of criterions and systems, written for a slightly different problem (Mayer problem) in a less explicit form. The conditions

$$u_i^* = p^T g_i(x) \tag{15}$$

$$\dot{u}_i^* = p^T[f + gu^*,\ g_i] \tag{16}$$

were also derived in [6] in which (16) was obtained from direct differentiation (with respect to time) of (15). The derivation of [6, eq. (15)] was from the trivial symmetry (or trivial (energy) conservation law)

$$\{H^*, H^*\} = 0$$

where $\{\cdot, \cdot\}$ is the Poisson bracket of smooth functions, since H^* is a first integral of optimal Hamiltonian system. In this regard, we can view the conditions given in the Proposition as representations of energy conservation law.

From the Lemma, the following Corollary is clear.

Corollary: If L satisfies the nilpotence condition

$$ad_L^k = 0$$

for some positive integer k, then for any vector field $X \in ad_L^{k-1}L$, $p^T X(x)$ is a constant.

There are three special cases of interest to be considered.

Case 1 (Commutative Case or $k = 1$): In this case, for each $i, j = 1, 2, \cdots, m$

$$[f,\ g_i] = 0,$$

$$[g_j,\ g_i] = 0.$$

Since from (14)

$$\dot{u}_i^* = p^T ad_F g_i|_{u*}$$

by the Corollary, we have

$$\dot{u}_i^* = 0,$$

i.e., the minimum-fuel control for a nilpotent control system with $ad_L = 0$ is a constant vector: $u^* = C$. The computation of this constant u^* can proceed as follows. In this case, the minimum cost is

$$J^*(x_0) := J(x_0, u^*)$$
$$= \frac{1}{2}\int_0^T u^{*T}u^*\, dt + K(x(T))$$
$$= \frac{T}{2}\sum_{i=1}^m C_i^2 + K(x(T)) \tag{17}$$

and the system dynamics become

$$\dot{x} = f(x) + g(x)C,\ x(0) = x_0. \tag{18}$$

From (18) we can (numerically or analytically) solve for $x(t)$ and thus $x(T)$, then (17) is a function of C_i only. For optimality of C_i, we must require that

$$\frac{dJ^*}{dC_i} = TC_i + \frac{\partial K}{\partial C_i}(x(T))$$
$$= 0, \qquad i = 1, 2, \cdots, m.$$

These constitute a set of m algebraic equations in m unknowns: $C_1, \cdots, C_m$; C can then be solved by standard methods.

Remark: For the present Case 1, the result is the same as that of [1] for a single input bilinear system.

Case 2 ($k = 2$ or $ad_L^2 = 0$): In particular, $ad_F^2 g_i = 0$, $i = 1, 2, \cdots, m$. In view of (14): we have, by the Corollary,

$$\ddot{u}_i^* = p^T ad_F^2 g_i$$
$$= 0.$$

Therefore the open-loop optimal control is

$$u_i^*(t) = C_i + d_i t, \qquad i = 1, 2, \cdots, m$$

for some constants C_i and d_i. Note that the result obtained here is also analogous to that of [1] for single input bilinear systems. Consequently, the two cases just considered are natural extensions of [1] to the more general problem (1), (2) considered.

Case 3 ($k = 3$ or $ad_L^3 = 0$): In particular, $ad_F^3 g_i = 0$. It can be seen from (14) that

$$u_i^{*(3)} = 0, \qquad i = 1, 2, \cdots, m$$

and by the Corollary and (12):

$$\ddot{u}_i^* = a^i + \sum_{j=1}^m b_j^i u_j^* + \sum_{k=1}^m C_k^i u_k^* + \sum_{j=1}^m \sum_{k=1}^m d_{jk}^i u_j^* u_k^* \qquad i = 1, 2, \cdots, m \tag{19}$$

where the constants are defined by

$$a^i = p^T[f, [f, g_i]],$$

$$b^i_j = p^T[f, [g_j, g_i]],$$

$$C^i_k = p^T[g_k, [f, g_i]],$$

$$d^i_{jk} = p^T[g_j, [g_k, g_i]].$$

This set of m nonlinear coupled second-order differential equations given by (19) may be solved for u_i^*. However, usually analytic closed-form solution to $u_i^*(t)$ is not possible and we must accept a numerical solution using numerical integration techniques. Note that the second and final terms on the right-hand side of (19) are due to the multiinput nature of the system in addition to the nilpotence structure imposed on the system (cf. the first equation in [1, p. 897]).

For single input systems (i.e., $m = 1$) $b^i_j = d^i_{jk} = 0$, (19) is then reduced to a linear constant second-order ordinary differential equation

$$\ddot{u}^* = C_1 + C_2 u^* \tag{20}$$

where

$$C_1 = p^T[f, [f, g]],$$

$$C_2 = p^T[g, [f, g]].$$

The general solution of (20) is

$$u^*(t) = \begin{cases} k_1(e^{k_2 t} - e^{-k_2 t}) - \dfrac{C_1}{C_2}, & \text{if } C_2 \neq 0 \\ \dfrac{1}{2} C_1 t^2 + C_3 t + C_4, & \text{if } C_2 = 0 \end{cases}$$

where C_i's and k_i's are constants; k_2 satisfies the characteristic equation of (20)

$$k_2^2 = C_2.$$

Remark: For the single-input nilpotent control system with $ad^3_L = 0$, the minimum-fuel control is given by (20) which is a generalization of [1]. For the multiinput system, the situation is far more complicated as can be seen from (19). The solution of $u^*(t)$ from (19) is not a simple task and numerical integration is helpful in the present case. We can then safely say that the result of [1] is not only due to the nilpotent structure of the system's Lie algebra L but also due to the fact that the systems considered are single input. Although it is well known [5] that every finite-dimensional nilpotent Lie algebra has a matrix representation, it is not convenient to analyze the problem in the matrix setting. This is different from the bilinear case in which the matrix representation is provided in the problem.

IV. Conclusion

We have considered the optimal control problem (1), (2) when the system (2) under investigation is such that its Lie algebra L defined in (3) is nilpotent, i.e., $ad^k_L L = 0$ for some positive integer k. The key equations for optimal control u^* are (14) which constitute a hierarchy of necessary conditions for u^*. These equations play a crucial role in obtaining the open-loop optimal control $u^*(t)$ at least for $k = 1, 2, 3$ which were studied in this note. The result of [1] for single input bilinear system was then naturally generalized to system (2); and it was also stressed that their results are due to two properties of systems considered by them: single input and nilpotent L. Since those systems (2) with nilpotent L are of special interest as mentioned in the second section, other control aspects of such systems need extensive and intensive research in the future.

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Optimal Control Via Fourier Series of Operational Matrix of Integration

Y. ENDOW

Abstract—**The state equations of an optimal regulator problem are given in terms of the truncated Fourier series and the associated operational matrix of integration. An effective computational algorithm is developed to calculate the expansion coefficients of the derivatives of state variables for saving computer storage and time and minimizing the computational error. An illustrative example is also given, and satisfactory computational results are obtained.**

I. Introduction

In recent years orthogonal functions have been used by a number of researchers to solve control problems. The objective is to obtain efficient algorithms, and hence to use the computational capacity of computers. The main characteristic of this technique is that it reduces the differential equation involved in the problem to an algebraic equation in terms of the orthogonal functions and the operational matrix of integration associated with these functions. Typical examples of the orthogonal functions are the Walsh [1], block-pulse [2], Laguerre [3], Legendre [4], Chebyshev [5], [6], Fourier [7], [8], and polynomial [9] functions.

In this note the Fourier series operational matrix of integration is used to determine an optimal control for a linear regulator problem. This approach has advantages due mainly to the use of sinusoidal functions since they are widely used in engineering fields and their properties are well known. In addition, the algorithm is comparatively simple and does not require excessive memories so it is suitable for microprocessors.

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The author is with the Department of Industrial Engineering, Chuo University, Kasuga, Bunkyo-ku, Tokyo, Japan.
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