Automatica, Vol. 26, No. 2, pp. 303–309, 1990
Printed in Great Britain.

0005-1098/90 $3.00 + 0.00
Pergamon Press plc

Design of Robust Linear State Feedback Laws: Ellipsoidal Set-theoretic Approach*

SHENG-DE WANG†, ‡ and TE-SON KUO†

A method based on ellipsoidal set-theoretic approach is developed for designing an optimal boundedness control law for a class of linear systems with uncertain parameters.

Key Words—Robust control; robustness; linear system; state feedback; bounding ellipsoids.

Abstract—This paper is concerned with the problem of designing robust state feedback laws for linear systems with uncertain parameters. The parameter uncertainties under consideration are assumed to be measurable, unknown but bounded and present in both state and input matrices. Using the ellipsoidal set-theoretic approach, we formulate and solve the optimal boundedness control problem. A linear state feedback law is developed such that the system states are minimally bounded in appropriate sense.

1. INTRODUCTION

THE PROBLEM OF designing a state feedback control that guarantees the desired performance of a class of uncertain linear systems has been investigated by several authors (Leitmann, 1979; Eslami and Russell, 1980; Corless and Leitmann, 1981; Noldus, 1982; Barmish, 1983, 1985). Among these, the so-called uniform ultimate boundedness control approach (Corless and Leitmann, 1981) has been developed to guarantee, for all admissible uncertainties, that the system states enter and remain within a small region around the origin after a finite interval of time. For such approaches, the resultant feedback control law is often a nonlinear function of the states (Leitmann, 1979; Corless and Leitman, 1981; Barmish, 1985).

There is also a variety of results which can be used to design linear controllers for linear systems with parameter uncertainty. Thorp and Barmish (1981) used the Lyapunov direct method to derive a stabilizing linear state feedback law. On the other hand, Chang and Peng (1972) used the concept of fuzzy dynamical programming and proposed the guaranteed cost control approach for desigining a linear state feedback law which stabilizes the system for all prescribed uncertainties. Vinkler and Wood (1979) generalized the method and proposed a counterpart which makes use of a multistep design procedure. In these methods, the main attention is focused on the system matrix uncertainty and hence the input matrix uncertainty is not explicitly treated.

In the work of Eslami (1982), a robust method was proposed to minimize the sensitivity of linear systems with large parameter variations. The difference between the actual and the nominal values of the system response was considered as the sensitivity measure of the plant. An optimal controller gain was developed such that a combined-quadratic cost consisting of the standard regulator cost and the sensitivity measure due to system parameter variations is minimized.

In this paper we treat the problem similarly to that of Eslami (1982), but from a different point of view. We start from the representation of unknown but bounded uncertainties and use the concept of ellipsoidal set-theoretic approach (Schweppe, 1973) to formulate the optimal boundedness control problem. A linear state-feedback law is developed such that the system states are minimally bounded in some appropriate sense.

A set-theoretic approach to solving constrained control problems has been presented by Usoro *et al.* (1982, 1984) for systems with control and state constraints which are subject to input disturbances. The objective in Usoro *et al.* (1982, 1984) was to construct a control system that uses only available control effort to keep the system states within prescribed bounds. The present framework differs from the previous research in that, in this paper, the parameter

* Received 3 May 1988; revised 8 February 1989; received in final form 14 March 1989. The original version of this paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor R. V. Patel under the direction of Editor H. Kwakernaak.
† Department of Electrical Engineering, National Taiwan University, Taipei 10764, Taiwan, R.O.C.
‡ To whom correspondence should be addressed.

perturbations are considered and the objective is to minimize the "size" of the bounding ellipsoid which contains system states.

In this paper the notation A' denotes the transpose of the matrix A, and I is the identity matrix. For the sake of brevity, the arguments of time functions are sometimes suppressed.

2. SYSTEM UNCERTAINTIES AND BOUNDING ELLIPSOIDS

In this section we start from the representation of unknown but bounded uncertainties and use the concepts of bounding ellipsoids to model the system states.

2.1. *System equations and assumptions on uncertainties*

The uncertain linear systems under consideration are described by state equations of the form

$$\dot{x}(t) = [A + \Delta A(w(t))]x(t) + [B + \Delta B(v(t))]u(t) \tag{1}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control vector and $w(t) \in R^q$, $v(t) \in R^s$ are vectors of uncertain parameters. In this paper, the following assumptions concerning the uncertainties are made on system (1).

(A1) $\Delta A(\cdot): R^q \to R^{n\times n}$ and $\Delta B(\cdot): R^s \to R^{n\times m}$ are continuous matrix functions which are assumed to be linear in their arguments.

(A2) The uncertain parameters

$$w(\cdot): R \to W \subset R^q$$
$$v(\cdot): R \to V \subset R^s$$

are Lebesgue measurable, where W and V are prescribed compact convex subsets of appropriate spaces.

While the assumption that W and V are compact convex sets may seem severe, a class of bounded sets is. For example,

$$W = \{w(t) \mid w_{\min} \le \|w(t)\| \le w_{\max}\}$$

and

$$W = \{w(t) \mid (w_i)_{\min} \le w_i(t) \le (w_i)_{\max},\ w_i(t) \text{ is } i\text{th component of } w(t)\}$$

are compact convex sets.

For the system described by (1), we consider a linear state feedback law of the form

$$u(t) = K(t)x(t). \tag{2}$$

The closed-loop system then becomes

$$\dot{x}(t) = (A + BK)x(t) + \Delta A(w)x(t) + \Delta B(v)Kx(t). \tag{3}$$

We observe that the dynamic equation in (3) can be viewed as a state equation driven by a state-dependent disturbance due to parameter variations, which can be written in the form

$$\dot{x}(t) = (A + BK)x(t) + \tau(x(t), t) \tag{4}$$

where $\tau(x(t), t)$ is the state-dependent disturbance defined as

$$\tau(x(t), t) = \Delta A(w)x + \Delta B(v)K(t)x. \tag{5}$$

We assume that the initial state $x(0)$ is bounded as

$$x(0) \in \Omega_x(0) \equiv \{x \mid (x - x_0)'\Psi^{-1}(x - x_0) \le 1\} \tag{6a}$$

or, in terms of its support function (Schweppe, 1973),

$$x(0) \in \Omega_x(0) \equiv \{x \mid x'\eta \le x_0 + \sqrt{\eta'\Psi\eta};\ \text{for all } \eta,\ \eta'\eta = 1\} \tag{6b}$$

where x_0 is the center of $\Omega_x(0)$ and Ψ is a positive semidefinite matrix defining ellipsoidal set $\Omega_x(0)$ of possible initial states. Note that in the ellipsoid defined by (6a) Ψ is required to be positive definite, whereas in (6b) a positive semidefinite Ψ is sufficient to specify an elllpsoid (actually, a degenerate ellipsoid). In the following, we will use the form of (6a) to specify an ellipsoidal set, but with the understanding that positive semidefinite Ψ is sufficient to define an ellipsoid in terms of its support function. Furthermore, we assume that the disturbance $\tau(x(t), t)$ at time t is contained within an ellipsoidal set $\Omega_\tau(t)$, i.e.

$$\tau(x(t), t) \in \Omega_\tau(t) \equiv \{\tau \mid \tau' Q^{-1}(t)\tau \le 1\} \tag{7}$$

where the center of $\Omega_\tau(t)$ is assumed to be at the origin and $Q(t)$ is the positive semidefinite matrix defining the ellipsoid $\Omega_\tau(t)$. Since $\tau(x(t), t)$ is state-dependent, it is obvious that matrix $Q(t)$ must be related to the state bounding ellipsoid. We will determine matrix Q in the following subsection.

2.2. *Bounding ellipsoids*

Following the development of set-theoretic formulation (Schweppe, 1973; Usoro *et al.*, 1982), it can be shown that the state $x(t)$ of the system described by (4) at time t is contained within an ellipsoidal set $\Omega_x(t)$ and is given by

$$x(t) \in \Omega_x(t) \equiv \{x \mid (x - x_c(t))'\Gamma^{-1}(t) \times (x - x_c(t)) \le 1\} \tag{8}$$

where $\Gamma(t)$ satisfies the matrix differential equation

$$\dot{\Gamma} = (A + BK)\Gamma + \Gamma(A + BK)' + \zeta\Gamma + (Q/\zeta) \tag{9a}$$

with

$$\Gamma(0) = \Psi \quad (9b)$$

where $\zeta(t) > 0$ is a free parameter which enters in the construction of the bounding ellipsoid, and $x_c(t)$, the center of $\Omega_x(t)$, is described by the nominal system equation

$$\dot{x}_c(t) = (A + BK)x_c(t)$$

with

$$x_c(0) = x_0.$$

To complete the description of the state bounding ellipsoid, we need to know the matrix $Q(t)$, which can be determined as follows. It can be seen from (5) that the disturbance $\tau(x(t), t)$ due to parameter variation is the vector sum of the two terms $\tau_1 = \Delta A(w)x$ and $\tau_2 = \Delta B(v)Kx$. Since τ_1 and τ_2 are linear transformations of x, it can be shown (Schweppe, 1973) that these two terms are bounded, respectively, by

$$\tau_1 \in \Omega_{\tau_1} \equiv \{\tau_1 \mid \tau_1'[\Delta A(w)\Gamma\Delta A'(w)]^{-1}\tau_1 \leq 1\} \quad (10)$$

and

$$\tau_2 \in \Omega_{\tau_2} \equiv \{\tau_2 \mid \tau_2'[\Delta B(v)K\Gamma K'\Delta B'(v)]^{-1}\tau_2 \leq 1\} \quad (11)$$

where the centers of $\Omega_{\tau 1}$ and $\Omega_{\tau 2}$ are assumed to be at the origin since $\Delta A(\cdot)$ and $\Delta B(\cdot)$ are uncertain terms and it seems to be reasonable to assume it varies around the origin. Unfortunately, the vector sum of two ellipsoids is generally not an ellipsoid. It is desirable to find a bounding ellipsoid which contains the vector sum of the two ellipsoids described by (10) and (11). Following the development of Schweppe (1973), the bounding ellipsoid which contains the vector sum of (10) and (11) is described by (6), where $Q(t)$ is given by

$$Q(t) = \gamma_1\Delta A(w)\Gamma\Delta A'(w) + \gamma_2\Delta B(v)K\Gamma K'\Delta B'(v) \quad (12)$$

with γ_1 and γ_2 being positive scalars satisfying

$$\frac{1}{\gamma_1} + \frac{1}{\gamma_2} = 1. \quad (13)$$

It is seen that simply taking $\gamma_1 = \gamma_2 = 2$ is a satisfactory choice. When combined with (12), the matrix differential equation (9) describing Γ becomes

$$\dot{\Gamma} = \left(A + BK + \frac{\zeta}{2}I\right)\Gamma + \Gamma\left(A + BK + \frac{\zeta}{2}I\right)' + \frac{\Delta A(w)\Gamma\Delta A'(w)}{(\zeta/\gamma_1)} + \frac{\Delta B(v)K\Gamma K'\Delta B'(v)}{(\zeta/\gamma_2)} \quad (14a)$$

with the initial condition

$$\Gamma(0) = \Psi. \quad (14b)$$

Let the positive scalars α and β be defined by

$$\alpha \equiv \frac{\zeta}{\gamma_1}, \qquad \beta \equiv \frac{\zeta}{\gamma_2}.$$

From (13), it is obvious that $\zeta = \alpha + \beta$. With this relation and the above definitions, (14) can be rewritten as

$$\dot{\Gamma} = \left(A + BK + \frac{\alpha+\beta}{2}I\right)\Gamma + \Gamma\left(A + BK + \frac{\alpha+\beta}{2}I\right)' + \frac{1}{\alpha}\Delta A(w)\Gamma\Delta A'(w) + \frac{1}{\beta}\Delta B(v)K\Gamma K'\Delta B'(v) \quad (15a)$$

with

$$\Gamma(0) = \Psi. \quad (15b)$$

where α and β are positive scalars which provide the designer some degrees of freedom in the construction of the bounding ellipsoid.

The ellipsoidal set described by (8) and (15) is a bounding ellipsoid of the system states incurred by the state feedback law $u = Kx$ and the system uncertainties $\Delta A(\cdot)$ and $\Delta B(\cdot)$. With the state feedback control $u = Kx$, if the system state is bounded by $\Omega_x(t)$ in (8), then the control u is bounded by an ellipsoid $\Omega_u(t)$ which is simply a linear transformation of the ellipsoid $\Omega_x(t)$. It can be shown that

$$\Omega_u = \{u \mid (u - Kx_c)'(K\Gamma K')^{-1}(u - Kx_c) \leq 1\}.$$

Remark 1. It is noted that the matrix $\Gamma(t)$ in (8) and (14) determine the size and shape of the bounding ellipsoid of the system states. The orientation of $\Omega_x(t)$ is determined by the eigenvectors of $\Gamma(t)$, and the lengths of semi-major axes of $\Omega_x(t)$ are determined by eigenvalues of $\Gamma(t)$. It is noted that the trace of $\Gamma(t)$, $\mathrm{tr}\,(\Gamma(t))$, may serve as a measure of the "mean size" of the bounding ellipsoid.

3. OPTIMAL BOUNDEDNESS CONTROL

In this section we formulate the optimal boundedness control problem and present a solution approach for the problem. The infinite time case for time-invariant uncertain systems is also considered.

3.1. *Problem formulation*

Using the results of Section 2, we can state the optimal boundedness control problem as a constrained optimization problem as follows: For a given system described by (1) and control weighting factor $\rho > 0$, find the linear feedback gain K and the free parameters α and β to minimize the cost functional;

$$\max_{\substack{w \in W \\ v \in V}} J(K; w, v, t_0) \tag{16}$$

with

$$J(K; w, v, t_0) \equiv \int_{t_0}^{T} [\operatorname{tr}(\Gamma) + \rho \operatorname{tr}(K\Gamma K')]\, dt \tag{17}$$

subject to constraint (15).

Remark 2. The inclusion of the second terms in cost functional (17) is to take into account the minimization of the control effort. The weighting factor ρ should not be determined in advance.

The problem formulated above is minimax in nature. Before solving this problem, it is worthwhile investigating some useful properties associated with the problem. We show in the following theorem that the cost functional $J(K; w, v, t_0)$ is convex with respect to w and v under some appropriate conditions.

Theorem 1. If the assumptions (A1)–(A2) hold, then $J(K; w, v, t_0)$ is convex in w and v.

Proof (*see Appendix A*). Since a convex scalar function of a vector defined on a compact convex set attains its maximum on the extreme points of the set, we immediately have

$$\min_{K} \max_{\substack{w \in W \\ v \in V}} J(K; w, v, t_0) = \min_{K} \max_{\substack{w \in \partial W \\ v \in \partial W}} J(K; w, v, t_0)$$

where ∂W and ∂V denote the boundaries of W and V, respectively. If, furthermore, the sets W and V are convex polyhedrons, the search for the maximizing uncertainty w and v can be reduced to the sets of the vertices of W and V.

Based on the above observation, the maximization part of the problem will be solved by making use of an exhaustive search over a set of finite vertices assuming that the uncertain sets W and V are convex polyhedrons.

3.2. *Solution of the minimization problem*

The constrained minimization problem formulated above can be solved by making use of the Lagrange solution approach. The Hamiltonian for this constrained minimization is defined by

$$H(\Gamma, K, \rho, t) = \operatorname{tr}(\Gamma) + \rho \operatorname{tr}(K\Gamma K') + \operatorname{tr}(\Lambda'\dot{\Gamma}) \tag{18}$$

where Λ is an $n \times n$ matrix Lagrange multiplier. On using the gradient matrix calculation formulae (Athans, 1968), the Lagrange approach requires the following optimality conditions:

(i)
$$0 = \frac{\partial H}{\partial K} = 2\rho K\Gamma + 2B'\Lambda\Gamma + \left(\frac{2}{\beta}\right)\Delta B'(v)\Lambda\Delta B(v)K\Gamma; \tag{19}$$

(ii)
$$-\dot{\Lambda} = \frac{\partial H}{\partial \Gamma} = I + \rho K'K + \Lambda\left(A + BK\frac{\alpha+\beta}{2}I\right) + \left(A + BK + \frac{\alpha+\beta}{2}I\right)'\Lambda + \left(\frac{1}{\alpha}\right)\Delta A'(w)\Lambda\Delta A(w) + \left(\frac{1}{\beta}\right)K'\Delta B'(v)\Lambda\Delta B(v)K \tag{20a}$$

with

$$\Lambda(T) = 0; \tag{20b}$$

(iii)
$$\dot{\Gamma} = \frac{\partial H}{\partial \Lambda} = \left(A + BK + \frac{\alpha+\beta}{2}I\right)\Gamma + \Gamma\left(A + BK + \frac{\alpha+\beta}{2}I\right)' + \left(\frac{1}{\alpha}\right)\Delta A(w)\Gamma\Delta A'(w) + \left(\frac{1}{\beta}\right)\Delta B(v)K\Gamma K'\Delta B'(v) \tag{21a}$$

with

$$\Gamma(t_0) = \Psi; \tag{21b}$$

(iv)
$$\frac{\partial H}{\partial \alpha} = \operatorname{tr}(\Gamma\Lambda) - \frac{1}{\alpha^2}\operatorname{tr}(\Delta A(w)\Gamma\Delta A'(w)\Lambda) = 0; \tag{22}$$

(v)
$$\frac{\partial H}{\partial \beta} = \operatorname{tr}(\Gamma\Lambda) - \frac{1}{\beta^2}\operatorname{tr}(\Delta B(v)K\Gamma K'\Delta B'(v)\Lambda) = 0. \tag{23}$$

From (19), we have

$$\left[\left(\rho I + \left(\frac{1}{\beta}\right)\Delta B'(v)\Lambda\Delta B(v)\right)K - B'\Lambda\right]\Gamma = 0.$$

Since this equation must hold for all $\Gamma(t)$, it reduces to

$$K = -R^{-1}(\Lambda)B'\Lambda \tag{24}$$

where $R(\Lambda)$ is defined as

$$R(\Lambda) \equiv \rho I + \left(\frac{1}{\beta}\right)\Delta B'(v)\Lambda\Delta B(v). \tag{25}$$

Substituting (24) into (20) and (21), and defining

$$Q(\Lambda) \equiv I + \left(\frac{1}{\alpha}\right)\Delta A'(w)\Lambda\Delta A(w) \tag{26}$$

we have the following equations:

$$-\dot{\Lambda} = \Lambda\left(A + \frac{\alpha+\beta}{2}I\right) + \left(A + \frac{\alpha+\beta}{2}\right)'\Lambda - \Lambda BR^{-1}(\Lambda)B\Lambda + Q(\Lambda) \tag{27a}$$

with terminal condition

$$\Lambda(T) = 0 \tag{27b}$$

and

$$\dot{\Gamma} = \left(A - BR^{-1}(\Lambda)B'\Lambda + \frac{\alpha+\beta}{2}I\right)\Gamma + \Gamma\left(A - BR^{-1}(\Lambda)B'\Lambda + \frac{\alpha+\beta}{2}I\right)' + \left(\frac{1}{\alpha}\right)\Delta A(w)\Gamma\Delta A'(w) + \left(\frac{1}{\beta}\right)\Delta B(v)R^{-1}(\Lambda)B'\Lambda\Gamma\Lambda BR^{-1}(\Lambda)\Delta B'(v) \tag{28a}$$

with the initial condition

$$\Gamma(t_0) = \Psi. \tag{28b}$$

Furthermore, from (22) and (23) we have

$$\alpha = \left[\frac{\operatorname{tr}(\Gamma\Delta A'(w)\Lambda\Delta A(w))}{\operatorname{tr}(\Gamma\Lambda)}\right]^{1/2} \tag{29}$$

$$\beta = \left[\frac{\operatorname{tr}(\Gamma K'\Delta B'(v)\Lambda\Delta B(v)K)}{\operatorname{tr}(\Gamma\Lambda)}\right]^{1/2}. \tag{30}$$

Using the Hamilton–Jacobi approach (Bell *et al.*, 1982; Sage, 1968), we can show (see Appendix B) that the optimizing cost functional $J(K; w, v, t_0)$ defined by (17) is given by

$$J(K; w, v, t_0) = \operatorname{tr}(\Lambda'(t_0)\Psi) \tag{31}$$

where $\Psi = \Gamma(t_0)$ is the positive semidefinite matrix defining the bounding ellipsoid of the initial state as described by (6), and $\Lambda(t_0)$ is the solution of (27) evaluated at time t_0.

Once the minimization problem has been solved the optimizing cost functional can be evaluated using (31). The minimization problem posed above has therefore been reduced to solving a set of simultaneous nonlinear equations given by (27)–(30). Observe that (27) is a Ricatti-type equation with $R(\Lambda)$ and $Q(\Lambda)$ being linear functions of Λ and that (28) is a linear matrix differential equation. If the solution of this set of equations is obtained, then the linear feedback gain matrix is given by (24).

In general, it is difficult to solve a set of nonlinear equations. But, fortunately, owing to the special structure of the equations in (27)–(30), the difficulty may be alleviated considerably. To show the special structure in (27)–(30), it may be pointed out that the coupling between (27) and (28) arises from conditions (29) and (30). If the parameters α and β are given, then we can solve (27) for Λ, and subsequently solve (28) for Γ. Based on these observations, an iterative procedure may be used to solve the set of nonlinear equations.

3.3. *The infinite time case*

For a linear time-invariant system with uncertain but constant parameters w and v, the steady-state solution of (27) will approach a constant one, which satisfies the following algebraic Ricatti-type equation:

$$0 = \Lambda\left(A + \frac{\alpha+\beta}{2}I\right) + \left(A + \frac{\alpha+\beta}{2}I\right)'\Lambda - \Lambda BR^{-1}(\Lambda)B\Lambda + Q(\Lambda). \tag{32}$$

This corresponds to the solution of an infinite time optimal boundedness problem, i.e. $t_0 \to -\infty$ in (17). It should be noted that, even in the infinite time case, the optimal values of the parameters α and β given by (29) and (30), respectively, are still time-varying, since they depend on $\Gamma(t)$. In many cases, it is desirable to use fixed values for α and β. To this end, it is suggested to use the following sub-optimal but constant α and β:

$$\alpha = \left[\frac{\operatorname{tr}(\Delta A'(w)\Lambda\Delta A(w))}{\operatorname{tr}(\Lambda)}\right]^{1/2} \tag{33}$$

$$\beta = \left[\frac{\operatorname{tr}(K'\Delta B'(v)\Lambda\Delta B(v)K)}{\operatorname{tr}(\Lambda)}\right]^{1/2}. \tag{34}$$

Thus, the problem is reduced to solving the Ricatti-type equation (32), and the corresponding cost functional is given by

$$J(K; w, v, t_0) = \operatorname{tr}(\Lambda'\Psi), \quad t_0 \to -\infty$$

where the optimal feedback gain is given by $K = -R^{-1}(\Lambda)B'\Lambda$.

4. A NUMERICAL EXAMPLE

To illustrate the design procedure, we consider the example given in Wang *et al.* (1987), which is described by the following linear uncertain system:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1+w & 1+w \end{bmatrix} x + \begin{bmatrix} 0 \\ 1+v \end{bmatrix} u.$$

The uncertain parameters are bounded as $w \in W$ and $v \in V$, where

$$W = \{w \mid |w| \leq 1\}$$

$$V = \{v \mid |v| \leq 0.2\}.$$

Obviously, the sets W and V are convex polyhedrons. The optimal boundedness problem is solved for this example. By Theorem 1, it is obvious that the maximizing uncertainties occur at $|w| = 1$ and $|v| = 0.2$. The optimal feedback gain K, the free parameters α and β, and the corresponding cost J are numerically determined to be $K = [-5.43\ \ -6.61]$, $\alpha = 0.81$, $\beta = 1.177$ and $J = 291.8$, where the matrix Ψ has been assumed to be the identity matrix in the evaluation of the cost J. It can be seen from the results obtained in this example that it is somewhat less conservative than that obtained in Wang *et al.* (1987). This is expected since the criterion used in this paper is a scalar measure [see (17)], whereas in Wang *et al.* (1987) the upper bound of the cost functional is minimized in matrix ordering sense.

5. CONCLUDING REMARKS

In this paper, we have formulated and solved the optimal boundedness problem. The formulation is based on the ellipsoidal set-theoretic approach. It is of interest to note that the results obtained in this paper are very similar to those in our previous research (Wang *et al.*, 1987), which is based on the linear quadratic (LQ) approach. Note that the free parameters α and β have the same meaning in these two approaches: both provide degrees of freedom for the construction of the upper bounds. Although this is surprising, it may be expected from the conceptual similarity between the LQ upper bound and the state bounding ellipsoids.

The results presented in this paper may be used in the following two ways:

(1) The bound-minimizing control law can be used as a stabilizing control provided that the stability with respect to large parameter variations is the main concern.

(2) It can be viewed as a sensitivity reduction control law if the main concern is performance degradation due to small parameter variations.

REFERENCES

Athans, M. (1968). The matrix minimum principle. *Information Control*, **11**, 592–606.

Barmish, B. R. (1983). Stabilization of uncertain systems via linear control. *IEEE Trans. Aut. Control*, **AC-28**, 848–850.

Barmish B. R. (1985). Necessary and sufficient conditions for quadratic stabilizability of an uncertain linear system. *J. Optimiz. Theory Appl.*, **46**, 399–408.

Bell, D. J., P. A. Cook and N. Munro (1982). *Design of Modern Control Systems*. Peter Peregrinus, Stevenage.

Bellman, R. E. (1970). *Introduction to Matrix Analysis*. McGraw-Hill, New York.

Chang, S. S. L. and T. Peng (1972). Adaptive guaranteed cost control of systems with uncertain parameters. *IEEE Trans. Aut. Control*, **AC-17**, 474–483.

Corless, M. J. and G. Leitmann (1981). Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamic systems. *IEEE Trans. Aut. Control*, **AC-26**, 1139–1144.

D'Appolito, J. A. and C. E. Hutchinson (1972). A minimax approach to the design of low sensitivity state estimators. *Automatica*, **8**, 599–608.

Eslami, M. (1982). On sensitivity minimization algorithms with quadratic performances. *Proc. Control Decision Conf.*, pp. 637–641.

Eslami, M. and D. L. Russell (1980). On stability with large parameter variations: stemming from the direct method of Lyapunov. *IEEE Trans. Aut. Control*, **AC-25**, 1231–1234.

Leitmann, G. (1979). Guaranteed asymptotic stability for some linear systems with bounded uncertainties. *J. Dynam. Syst., Meas., Control*, **101**, 212–216.

Noldus, E. (1982). Design of robust state feedback laws. *Int. J. Control*, **35**, 935–944.

Sage, A. P. (1968). *Optimum systems Control*. Prentice-Hall, Englewood Cliffs, NJ.

Schweppe, F. C. (1973). *Uncertain Dynamic Systems*. Prentice-Hall, Englewood Cliffs, NJ.

Thorp, J. S. and B. R. Barmish (1981). On guaranteed stability of uncertain linear systems via linear control. *J. Optimiz. Theory Appl.*, **35**, 559–579.

Usoro, P. B. (1984). Set-theoretic analysis of linear control systems with unknown-but-bounded inputs. *Proc. Am. Control Conf.*, pp. 1344–1351.

Usoro, P. B., F. C. Schweppe, L. A. Gould and D. N. Wormley (1982). A Lagrange approach to set-theoretic control synthesis. *IEEE Trans. Aut. Control*, **AC-27**, 393–399.

Vinkler, A. and L. J. Wood (1979). Multistep guaranteed cost control of linear systems with uncertain parameters. *J. Guidance Control*, **2**, 449–456.

Wang, S. D., T. S. Kuo, Y. H. Lin, C. F. Hsu and Y. T. Juang (1987). Robust control design for linear systems with uncertain parameters. *Int. J. Control*, **46**, 1557–1567.

APPENDIX A: PROOF OF THEOREM 1

In order to prove Theorem 1, we should invoke the following lemma.

*Lemma A*1 (*D'Appolito and Hutchinson*, 1972). If $f(x, y, t)$ is a continuous scalar function of x, y and t and if $f(x, y, t)$ is convex in x for every y and t, with second order partials with respect to x continuous in y and t, then

$$g(x, y) = \int_{t_0}^{T} f(x, y, t)\, dt$$

is a convex function of x for every y.

Theorem 1. If assumptions (A1)–(A2) in Section 2 hold, then $J(K; w, v, t_0)$ is convex in w and v.

Proof. By Lemma A1 and the fact that the integrand in the cost functional (17) is linear in $\mathrm{tr}(\Gamma)$, it remains to show that $\mathrm{tr}(\Gamma)$ is convex in w and v. It is well known that the solution to the matrix differential equation (15) can be written in the form

$$\Gamma(t) = \int_0^t \Phi(t)\left[\frac{1}{\alpha}\Delta A(w)\Gamma\Delta A'(w) + \frac{1}{\beta}\Delta B(v)k\Gamma K'\Delta B'(v)\right]\Phi'(t)\,\mathrm{d}t \tag{A1}$$

where $\Phi(t)$ is the transition matrix associated with $A + BK + (\alpha+\beta)I/2$. Taking the trace operation on both sides of (A1) and using the gradient matrix calculation formulae (Athans, 1968) and the trace property $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, we have

$$\frac{\partial\, \mathrm{tr}(\Gamma)}{\partial \Delta A} = 2(\Phi'\Phi \otimes \Gamma) \tag{A2}$$

and

$$\frac{\partial\, \mathrm{tr}(\Gamma)}{\partial \Delta B} = 2(\Phi'\Phi \otimes K\Gamma K') \tag{A3}$$

where $\otimes$ denotes the Kronecker product (Bellman, 1970). Since $\Phi'\Phi$, Γ and $K\Gamma K'$ are semidefinite, the gradients in (A2) and (A3) are positive semidefinite (Bellman, 1970), and $\mathrm{tr}(\Gamma)$ is convex in $\Delta A(\cdot)$ and $\Delta B(\cdot)$. By the assumptions (A1)–(A2) given in Section 2, $\mathrm{tr}(\Gamma)$ is also convex in w and v. This completes the proof.

APPENDIX B: OPTIMIZING THE COST FUNCTIONAL

In this appendix, we will show that the optimizing cost functional is given by (31). To this end, we assume that the cost functional defined in (17) has the following optimal value:

$$J(K; w, v, t_0) = \int_{t_0}^{T} [\mathrm{tr}(\Gamma) + \rho\, \mathrm{tr}(K\Gamma K')]\,\mathrm{d}t = F(\Gamma, t) \tag{B1}$$

i.e. $F(\Gamma, t)$ is the value of the cost function evaluated along an optimal $\Gamma(t)$ at a general initial time t. With the Hamiltonian defined in (18) and the cost functional given by (B1), the Hamilton–Jacobi equation (Sage, 1968) can be written as

$$\frac{\partial F}{\partial t} + H(\Gamma, K, \rho, t) = 0 \tag{B2}$$

or

$$\frac{\partial F}{\partial t} + \mathrm{tr}(\Gamma) + \rho\, \mathrm{tr}(K\Gamma K') + \mathrm{tr}(\Lambda'\dot{\Gamma}) = 0 \tag{B3}$$

where K is the optimal feedback gain given by (24). Substituting (15a) for $\dot{\Gamma}$ into (B3) and noting that

$$\frac{\partial F}{\partial t} = \mathrm{tr}[\dot{\Lambda}'(t)\Gamma]$$

and that (B3) must hold for all Γ, we obtain (27). This completes the proof.