Generalized Commuting Matrices and Their Eigenvectors for DFTs, Offset DFTs, and Other Periodic Operations

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***Abstract*—It is well known that some matrices (such as Dickinson-Steiglitz's matrix) can commute with the discrete Fourier transform (DFT) and that one can use them to derive the complete and orthogonal DFT eigenvector set. Recently, Candan found the general form of the DFT commuting matrix. In this paper, we further extend the previous work and find the general form of the commuting matrix for any periodic, quasi-periodic, and offset quasi-periodic operations. Using the general commuting matrix, we can derive the complete and orthogonal eigenvector sets for offset DFTs, DCTs of types 1, 4, 5, and 8, DSTs of types 1, 4, 5, and 8, discrete Hartley transforms of types 1 and 4, the Walsh transform, and the projection operation (the operation that maps a whole vector space into a subspace) successfully. Moreover, several novel ways of finding DFT eigenfunctions are also proposed. Furthermore, we also extend our theories to the continuous case, i.e., if a continuous transform is periodic, quasi-periodic, or offset quasi-periodic (such as the FT and some cyclic operations in optics), we can find the general form of the commuting operation and then find the complete and orthogonal eigenfunctions set for the continuous transform.**

***Index Terms*—Commuting matrix, discrete Fourier transform (DFT), discrete fractional Fourier transform, discrete sinusoid transform, eigenfunction, eigenvector, offset DFT, Walsh transform.**

I. Introduction

The discrete Fourier transform (DFT) is defined as

$$\mathbf{F}[m,n] = (1/\sqrt{N}) \cdot e^{-j2\pi mn/N}, \quad 0 \le m,n \le N-1. \quad (1)$$

Since the DFT has repeated eigenvalues, it has infinite number of eigenvectors. The commuting matrix is very helpful for finding the DFT eigenvectors with some specific form. For example, in [1], Dickinson and Steiglitz found that the DFT and the following matrix $\mathbf{S}$ commute

$$\mathbf{SF} = \mathbf{FS} \quad (2)$$

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where

$$\begin{aligned}\mathbf{S}[n,n] &= 2\cos(2\pi n/N)\\ \mathbf{S}[n,n+1] &= \mathbf{S}[n+1,n] = 1 \text{ for } 0 \le n \le N-2\\ \mathbf{S}[N-1,0] &= \mathbf{S}[0,N-1] = 1\\ \mathbf{S}[m,n] &= 0 \text{ otherwise.}\end{aligned} \quad (3)$$

Since (i) $\mathbf{S}$ commutes with $\mathbf{F}$ and (ii) all the eigenvalues of $\mathbf{S}$ are distinct, the eigenvectors of $\mathbf{S}$ are also the eigenvectors of $\mathbf{F}$. Moreover, since (iii) the $\mathbf{S}$ matrix approximates the equation of

$$\frac{d^2}{dt^2} h_p(t) - t^2 h_p(t) = \tau h_p(t), \quad \text{where } \tau = -2p-1 \quad (4)$$

and the solution of (4) is the p^{th} order Hermite-Gaussian function (HGF), the DFT eigenvectors derived from $\mathbf{S}$ are near to the samplings of HGFs, which are the eigenfunctions of the continuous Fourier transform (FT). Based on the results in [1], the discrete fractional Fourier transform (DFRFT) was derived using the eigenvector decomposition method [5], [6].

In addition to Dickinson and Steiglitz's work, recently, in [4], we found that the following matrix $\mathbf{T}$ can also commute with the DFT matrix $\mathbf{F}$:

$$\begin{aligned}\mathbf{T}[n,n] &= \cos^2(n\pi/N), \text{ for } 0 \le n \le N-1\\ \mathbf{T}[n,n+1] &= \mathbf{T}[n+1,n] = \frac{\cos[n\pi/N]\cos[(n+1)\pi/N]}{2\cos(\pi/N)}\\ &\quad \text{for } 0 \le n \le N-2 \quad (5)\\ \mathbf{T}[N-1,0] &= \mathbf{T}[0,N-1] = 0.5, \ \mathbf{T}[m,n] = 0 \text{ otherwise}\\ \mathbf{TF} &= \mathbf{FT}. \quad (6)\end{aligned}$$

The $\mathbf{T}$ matrix is the improved form of Grünbaum's matrix [2], which commutes with the central form DFT. We find that the $\mathbf{T}$ matrix can even better approximate the differential equation in (4) than the $\mathbf{S}$ matrix and the DFT eigenvectors derived from $\mathbf{T}$ can approximate the HGFs with even less error.

Moreover, in [11], Candan proposed that the commuting matrix of the DFT can be generated from an arbitrary $N \times N$ matrix

$$\begin{aligned}\mathbf{K} &= \mathbf{M} + \mathbf{FMF^{-1}} + \mathbf{F^2MF^{-2}} + \mathbf{F^3MF^{-3}}\\ &\quad \text{where } \mathbf{M} \text{ is an arbitrary } N \times N \text{ matrix} \quad (7)\\ \mathbf{KF} &= \mathbf{FK}. \quad (8)\end{aligned}$$

In fact, any commuting matrix of the DFT can be expressed in the form as in (7), which was proved in [11]. In other words, (7) gives a complete characterization for the DFT commuting matrix. For example, when $M[n,n] = \cos(2\pi n/N)$ and $M[m,n] = 0$ if $m \neq n$, $\mathbf{K}$ in (7) becomes the $\mathbf{S}$ matrix. When $M[n,n+1] = \cos(2\pi(n+1/2)/N)$, $M[n+1,n] = \cos(2\pi(n-1/2)/N)$, and $M[m,n] = 0$

otherwise, $\mathbf{K}$ in (7) becomes the $\mathbf{T}$ matrix. Furthermore, Candan derived that, if

$$\mathbf{M} = \sum_{m=1}^{k} (-1)^{m-1} \frac{2\left[(m-1)!\right]^2}{(2m)!} \Delta_m \text{ where}$$
$$\Delta_m = \underbrace{\Delta * \Delta * \Delta * \cdots\cdots * \Delta}_{m \text{ times}} \tag{9}$$
$$\begin{aligned} \Delta[n,n] &= -2 \\ \Delta[n,n-1] &= \Delta[n-1,n] = \Delta[1,N] = \Delta[N,1] = 1 \\ \Delta[m,n] &= 0 \text{ otherwise}, \quad *\text{means convolution} \\ &k \text{ is the largest number that} \\ &\text{satisfies } 2k+1 \leq N \end{aligned} \tag{10}$$

then the DFT eigenvectors obtained from the commuting matrix generated from $\mathbf{M}$ can very well approximate the samples of the HGF, especially for the lower order HGF.

Recently, as described in [19], Santhanam found another matrix that can commute with the central form DFT. (In Section V-B, we modify this matrix so that the modified version can commute with the standard definition of the DFT.)

In this paper, we will generalize the previous works to a much greater extent.

(A) In addition to the DFT, we find that, for **any periodic operation**, we can use a formula similar to (7) to find the commuting matrix **from an arbitrary $N \times N$ matrix**. We can then use the commuting matrix to find the complete-orthogonal eigenvector set of the periodic operation. See Section II-A.

(B) In addition to the periodic operation, we also find the general form of the commuting matrix for the **quasi-periodic** and the **offset quasi-periodic** operations. See Section II-B.

(C) Several important properties of the commuting matrix are discussed in Section II-C. Moreover, if a suitable $\mathbf{M}$ (the matrix used for generating the commuting matrix) is chosen properly, the eigenvectors we derive can be a **real** and **orthogonal** eigenvector set. See Section II-D and Table I.

(D) Since (i) the offset DFT, (ii) the discrete cosine transform, (iii) the Hartley transform, (iv) the projection operation, (v) the Walsh transform (Hadamard transform), and (vi) the matrix with a smaller number of eigenspaces, we can use the proposed method to find the commuting matrices of these operations and find the complete and orthogonal eigenvector sets. See Sections III and IV.

(E) We also further explore the commuting matrix of the original DFT. We generalize (3) and (5) and find the multiple diagonal forms of $\mathbf{S}$ and $\mathbf{T}$. We also modify the results in [19] and propose the n^2 matrix, which can commute with the noncentral standard form DFT. When the order is low, the DFT eigenvectors derived from the n^2 matrix can very well approximate the HGF. See Section V.

(F) Moreover, we extend our result into the **continuous case**, i.e., for **any** periodic, quasi-periodic, or offset quasi-periodic continuous operation, we can find the commuting operation from an **arbitrary** continuous operator. Then, using the commuting operator, we can derive the eigenfunction sets of these continuous operations. See Section VI.

TABLE I

WAYS TO CHOOSE $\mathbf{M}$ FOR DERIVING THE REAL, COMPLETE, AND ORTHOGONAL EIGENVECTOR SET. WHEN $\mathbf{B}$ IS PERIODIC OR QUASI-PERIODIC, $\mathbf{B_1} = \mathbf{B}$. WHEN $\mathbf{B}$ IS OFFSET QUASI-PERIODIC, $\mathbf{B_1} = \mathbf{B} + \sigma\mathbf{I}$

Conditions	Related theorem	Better ways to choose $\mathbf{M}$	Forms of $\mathbf{A}$	Forms of derived eigenvectors
(1) $\mathbf{B_1} = \pm\mathbf{B_1}^{\mathbf{H}}$	Theorem 9	$\mathbf{M} = \mathbf{M}^{\mathbf{H}}$	$\mathbf{A} = \mathbf{A}^{\mathbf{H}}$	complete and orthogonal
(2) $\mathbf{B_1}^{\mathbf{H}}\mathbf{B_1} = \rho\cdot\mathbf{I}$	Theorem 10	$\mathbf{M} = \mathbf{M}^{\mathbf{H}}$	$\mathbf{A} = \mathbf{A}^{\mathbf{H}}$	complete and orthogonal
(3) $\mathbf{B_1} = \pm\mathbf{B_1}^{\mathbf{T}}$	Theorem 11	$\mathbf{M} = \mathbf{M}^{\mathbf{T}}$	$\mathbf{A} = \mathbf{A}^{\mathbf{T}}$	complete and $\mathbf{E}^{\mathbf{T}}\mathbf{E} = \mathbf{D_1}$
(4) $\mathbf{B_1} = \pm\mathbf{B_1}^{\mathbf{T}}$ and $\mathbf{B_1}$ is real	Th. 14, (31) & (36), (43) & (45)	$\mathbf{M} = \mathbf{M}^{\mathbf{T}}$ and $\mathbf{M}$ is real	$\mathbf{A} = \mathbf{A}^{\mathbf{T}}$ and $\mathbf{A}$ is real	real, complete, and orthogonal
(5) $\mathbf{B_1} = \pm\mathbf{B_1}^{\mathbf{H}}$ and $\mathbf{B_1}^{\mathbf{T}}\mathbf{B_1} = \rho\cdot\mathbf{I}$	Th. 14, (31) & (36), (44) & (46)	$\mathbf{M} = \mathbf{M}^{\mathbf{T}}$ and $\mathbf{M}$ is real	$\mathbf{A} = \mathbf{A}^{\mathbf{T}}$ and $\mathbf{A}$ is real	real, complete, and orthogonal
(6) $\mathbf{B_1}^{\mathbf{T}}\mathbf{B_1} = \mathbf{I}$ and $\mathbf{B_1}$ is real	Th. 14, (34) & (38), (43) & (45)	$\mathbf{M} = \mathbf{M}^{\mathbf{T}}$ and $\mathbf{M}$ is real	$\mathbf{A} = \mathbf{A}^{\mathbf{T}}$ and $\mathbf{A}$ is real	real, complete, and orthogonal
(7) $\mathbf{B_1} = \pm\mathbf{B_1}^{\mathbf{T}}$ and $\mathbf{B_1}^{\mathbf{H}}\mathbf{B_1} = \rho\cdot\mathbf{I}$	Th. 14, (34) & (38), (44) & (46)	$\mathbf{M} = \mathbf{M}^{\mathbf{T}}$ and $\mathbf{M}$ is real	$\mathbf{A} = \mathbf{A}^{\mathbf{T}}$ and $\mathbf{A}$ is real	real, complete, and orthogonal

II. COMMUTING MATRICES FOR PERIODIC, QUASI-PERIODIC, AND OFFSET QUASI-PERIODIC OPERATIONS

A. Case of Periodic Operations

If an operation $\mathbf{B}$ satisfies

$$\mathbf{B}^p = \mathbf{I} \tag{11}$$

then we call $\mathbf{B}$ a **periodic operation** with period p (There are alternative names for these types of operations, such as the p^{th} root of the unity matrix and the cyclic discrete transform [15].) For example, since $\mathbf{F^4} = \mathbf{I}$, the DFT is a periodic operation with period 4.

Theorem 1: Suppose that $\mathbf{B}$ is an $N \times N$ **periodic operation** with period p. Then, for arbitrary $N \times N$ matrix $\mathbf{M}$, the following matrix $\mathbf{A}$ generated from $\mathbf{M}$ will commute with $\mathbf{B}$

$$\begin{aligned} \mathbf{A} = &\mathbf{M} + \mathbf{BMB^{-1}} + \mathbf{B^2MB^{-2}} + \cdots\cdots\cdots \\ &+ \mathbf{B^{p-1}MB^{-p+1}} \end{aligned} \tag{12}$$
$$\mathbf{AB} = \mathbf{BA}. \tag{13}$$

Proof:

$$\begin{aligned} \mathbf{AB} = &(\mathbf{M} + \mathbf{BMB^{-1}} + \mathbf{B^2MB^{-2}} + \cdots\cdots \\ &+ \mathbf{B^{p-1}MB^{-p+1}})\mathbf{B} \\ = &\mathbf{MB} + \mathbf{BM} + \mathbf{B^2MB^{-1}} + \cdots\cdots\cdots \\ &+ \mathbf{B^{p-1}MB^{-p+2}} \end{aligned} \tag{14}$$
$$\begin{aligned} \mathbf{BA} = &\mathbf{B}(\mathbf{M} + \mathbf{BMB^{-1}} + \mathbf{B^2MB^{-2}} + \cdots\cdots\cdots \\ &+ \mathbf{B^{p-2}MB^{-p+2}} + \mathbf{B^{p-1}MB^{-p+1}}) \end{aligned} \tag{15}$$
$$\begin{aligned} = &\mathbf{BM} + \mathbf{B^2MB^{-1}} + \mathbf{B^3MB^{-2}} + \cdots\cdots\cdots \\ &+ \mathbf{B^{p-1}MB^{-p+2}} + \mathbf{B^pMB^{-p+1}} \\ = &\mathbf{BM} + \mathbf{B^2MB^{-1}} + \mathbf{B^3MB^{-2}} + \cdots\cdots \\ &+ \mathbf{B^{p-1}MB^{-p+2}} + \mathbf{MB} \\ &(\text{since } \mathbf{B^p} = \mathbf{I} \text{ and } \mathbf{B^{-p+1}} = \mathbf{B}). \\ = &\mathbf{AB}. \end{aligned} \tag{16}$$

#

From the above theorem, we can derive the commuting matrices for the transforms that satisfy $\mathbf{B}^p = \mathbf{I}$, such as the offset DFT when $a = b$ (see Section III), the discrete cosine transform, the Hartley transform, and the discrete Walsh (Hadamard) transform (see Section IV). Note that (7) is a special case of (12). Theorem 1 is a further generalization of Candan's work [11].

B. Cases of Quasi-Periodic and Offset Quasi-Periodic Operations

Theorem 1 can be further generalized. If a matrix $\mathbf{B}$ satisfies

$$\mathbf{B}^p = C \cdot \mathbf{I}, \quad \text{where } C \text{ is some constant} \tag{17}$$

then we call it a **quasi-periodic** operation. For example, the offset DFT when $a \neq b$ (See Section III) is a quasi-periodic operation.

Theorem 2: If an $N \times N$ matrix $\mathbf{B}$ is **quasi-periodic**, then, since in (16), $\mathbf{B^p M B^{-p+1}} = C\mathbf{M}C^{-1}\mathbf{B} = \mathbf{MB}$, and the following commutative relation is still satisfied:

$$\begin{aligned}\mathbf{AB} = \mathbf{BA} \quad \text{where } \mathbf{A} &= \mathbf{M} + \mathbf{BMB^{-1}} + \mathbf{B^2MB^{-2}} \\ &\quad + \cdots\cdots\cdots + \mathbf{B^{p-1}MB^{-p+1}} \\ &\mathbf{M} \text{ is an arbitrary } N \times N \text{ matrix.}\end{aligned} \tag{18}$$

Proof:

$$\begin{aligned}\mathbf{AB} &= \mathbf{MB} + \mathbf{BM} + \mathbf{B^2MB^{-1}} + \cdots\cdots\cdots \\ &\quad + \mathbf{B^{p-1}MB^{-p+2}} \\ &= \mathbf{BB^{-1}MB} + \mathbf{BM} + \mathbf{BBMB^{-1}} + \cdots\cdots\cdots \\ &\quad + \mathbf{BB^{p-2}MB^{-p+2}} \\ &= \mathbf{B}(\mathbf{A} - \mathbf{B^{-1}MB} + \mathbf{B^{p-1}MB^{-p+1}}).\end{aligned}$$

Since $\mathbf{B}^p = C \cdot \mathbf{I}$, $\mathbf{B^{p-1}MB^{-p+1}} = \mathbf{B}^p\mathbf{B^{-1}MBB}^{-p} = C\mathbf{B^{-1}MB}C^{-1} = \mathbf{B^{-1}MB}$, $\mathbf{AB} = \mathbf{BA}$. #

Theorem 3: Furthermore, if an $N \times N$ matrix $\mathbf{B}$ satisfies

$$(\mathbf{B} + \sigma\mathbf{I})^p = C \cdot \mathbf{I}, \quad \text{where } C \text{ and } \sigma \text{ are some constants} \tag{19}$$

we call it an **offset quasi-periodic matrix**. In this case, we can first set $\mathbf{B_1} = \mathbf{B} + \sigma\mathbf{I}$. Because

$$\mathbf{B}_1^p = C \cdot \mathbf{I} \quad \text{where } \mathbf{B_1} = \mathbf{B} + \sigma\mathbf{I} \tag{20}$$

we can apply Theorem 2 to find the commuting matrix of $\mathbf{B_1}$. Then, since $\mathbf{B} = \mathbf{B_1} - \sigma\mathbf{I}$, if $\mathbf{AB_1} = \mathbf{B_1A}$

$$\mathbf{AB} = \mathbf{AB_1} - \sigma\mathbf{A} = \mathbf{B_1A} - \sigma\mathbf{A} = (\mathbf{B_1} - \sigma\mathbf{I})\mathbf{A} = \mathbf{BA}. \tag{21}$$

That is, the commuting matrix of $\mathbf{B_1}$ is also the commuting matrix of $\mathbf{B}$. Therefore, in (18), we can substitute $\mathbf{B}$ by $\mathbf{B_1} = \mathbf{B} + \sigma\mathbf{I}$. Then, the commuting matrix for $\mathbf{B}$ is

$$\begin{aligned}\mathbf{A} &= \mathbf{M} + (\mathbf{B}+\sigma\mathbf{I})\mathbf{M}(\mathbf{B}+\sigma\mathbf{I})^{-1} + (\mathbf{B}+\sigma\mathbf{I})^2\mathbf{M}(\mathbf{B}+\sigma\mathbf{I})^{-2} \\ &\quad + \cdots\cdots\cdots + (\mathbf{B}+\sigma\mathbf{I})^{\mathbf{p}-1}\mathbf{M}(\mathbf{B}+\sigma\mathbf{I})^{-\mathbf{p}+1}\end{aligned} \tag{22}$$

where $\mathbf{M}$ is an arbitrary $N \times N$ matrix. The matrix $\mathbf{A}$ satisfies $\mathbf{AB} = \mathbf{BA}$.

Corollary 1: For example, suppose that an $N \times N$ matrix $\mathbf{B}$ has **only two eigenspaces**, which correspond to the eigenvalues of λ_1 and λ_2. Since the eigenvalues of $\mathbf{B} - (\lambda_1 + \lambda_2) \cdot \mathbf{I}/2$ are $\pm(\lambda_1 - \lambda_2)/2$ and the eigenvalues of $[\mathbf{B} - (\lambda_1 + \lambda_2) \cdot \mathbf{I}/2]^2$ are $[(\lambda_1 - \lambda_2)/2]^2$, we have

$$\left(\mathbf{B} - \frac{\lambda_1 + \lambda_2}{2}\mathbf{I}\right)^2 = \left(\frac{\lambda_1 - \lambda_2}{2}\right)^2 \mathbf{I}. \tag{23}$$

That is, $\mathbf{B}$ is an **offset quasi-periodic** operation with period 2. Then, from Theorem 3, the commuting matrix of $\mathbf{B}$ is

$$\mathbf{A} = \mathbf{M} + (\mathbf{B} - (\lambda_1 + \lambda_2) \cdot \mathbf{I}/2)\,\mathbf{M}\,(\mathbf{B} - (\lambda_1 + \lambda_2) \cdot \mathbf{I}/2)^{-1} \tag{24}$$

where $\mathbf{M}$ is an arbitrary $N \times N$ matrix. For example, the projection operation (see Section IV) is an offset quasi-periodic operation with period 2.

C. Properties of Commuting Matrices

Theorem 4: If $\mathbf{A}$ commutes with $\mathbf{B}$ and all the eigenvalues of $\mathbf{A}$ are distinct, then **the eigenvectors of $\mathbf{A}$ are also the eigenvectors of $\mathbf{B}$**. The proof of Theorem 4 can be seen from [1].

Theorem 5: If $\mathbf{A_1}, \mathbf{A_2}, \ldots\ldots$, and $\mathbf{A_K}$ commute with $\mathbf{B}$, then it is easy to prove that

(i) A linear combination of $\mathbf{A_1}, \mathbf{A_2}, \ldots\ldots$, and $\mathbf{A_K}$ also commutes with $\mathbf{B}$.

Proof: If $\mathbf{L} = \sum_{k=1}^{K} \tau_k \mathbf{A_k}$

$$\mathbf{LB} = \sum_{k=1}^{K} \tau_k \mathbf{A_k B} = \sum_{k=1}^{K} \tau_k \mathbf{BA_k} = \mathbf{B}\sum_{k=1}^{K} \tau_k \mathbf{A_k} = \mathbf{BL}. \tag{25}$$

(ii) The product of $\mathbf{A_1} \cdot \mathbf{A_2} \cdot \cdots\cdots \cdot \mathbf{A_K}$ also commutes with $\mathbf{B}$.

Proof: $\mathbf{B}(\mathbf{A_1A_2} \cdots\cdots \mathbf{A_K}) = \mathbf{A_1BA_2} \cdots\cdots \mathbf{A_K} = \mathbf{A_1A_2BA_3} \cdots\cdots \mathbf{A_K} = \cdots\cdots = (\mathbf{A_1A_2} \cdots\cdots \mathbf{A_K})\mathbf{B}$. #

Theorem 6: In Theorems 1 and 2, (12) and (18) can be modified as

$$\mathbf{A} = \sum_{\tau=0}^{p-1} \mathbf{B}^{q+\tau}\mathbf{M}\mathbf{B}^{s-\tau}. \tag{26}$$

Proof: Note that (26) is just (12) [or (18)] multiplied by $\mathbf{B}^q$ and $\mathbf{B}^s$. Since $\mathbf{B}^q$, the matrix $\mathbf{A}$ in (12) [or (18)], and $\mathbf{B}^s$ all commute with $\mathbf{B}$, from Theorem 5, their product also commute with $\mathbf{B}$. #

Similarly, in Theorem 3, (22) can be modified as

$$\mathbf{A} = \sum_{\tau=0}^{\mathbf{p}-1} (\mathbf{B} + \sigma\mathbf{I})^{q+\tau}\mathbf{M}(\mathbf{B} + \sigma\mathbf{I})^{s-\tau}. \tag{27}$$

Theorem 7: If $\mathbf{A}$ commutes with $\mathbf{B}$ and $\det(\mathbf{A}) \neq 0$, then $\mathbf{A^{-1}}$ also commutes with $\mathbf{B}$. In this case

$$\mathbf{A^{-1}A} = \mathbf{I},\ \mathbf{A^{-1}AB} = \mathbf{BA^{-1}A},\ \mathbf{A^{-1}BA} = \mathbf{BA^{-1}A}. \tag{28}$$

Multiplying both sides by $\mathbf{A^{-1}}$, we obtain $\mathbf{A^{-1}B} = \mathbf{BA^{-1}}$.

Corollary 2: If $\mathbf{A}$ commutes with $\mathbf{B}$ and $\det(\mathbf{A}) \neq 0$, then $\mathbf{A}^k$ also commutes with $\mathbf{B}$, where k can be any integer (even a negative integer).

Theorem 8: If $\mathbf{B}$ is real or $\mathrm{conj}(\mathbf{B}) = \mathbf{B^{-1}}$, when $\mathbf{A}$ commutes with $\mathbf{B}$, then

$$\mathrm{Re}(\mathbf{A}),\ \mathrm{Im}(\mathbf{A}),\ \text{and}\ \mathrm{conj}(\mathbf{A}) \tag{29}$$

also commute with $\mathbf{B}$.

Proof: Since $\mathbf{AB} = \mathbf{BA}$, if $\mathbf{B}$ is real, then $\mathrm{Re}(\mathbf{A})\mathbf{B} = \mathrm{Re}(\mathbf{AB}) = \mathrm{Re}(\mathbf{BA}) = \mathbf{B}\mathrm{Re}(\mathbf{A})$.

If $\mathrm{conj}(\mathbf{B}) = \mathbf{B^{-1}}$, then, from $\mathbf{AB} = \mathbf{BA}$, $\mathbf{B^{-1}A} = \mathbf{AB^{-1}}$, we have

$$\begin{aligned} \mathbf{A}\mathrm{conj}(\mathbf{B}) &= \mathrm{conj}(\mathbf{B})\mathbf{A} \\ \mathrm{conj}(\mathbf{A})\mathbf{B} &= \mathbf{B}\mathrm{conj}(\mathbf{A}) \\ \mathrm{conj}(\mathbf{A})\mathbf{B} + \mathbf{AB} &= \mathbf{B}\mathrm{conj}(\mathbf{A}) + \mathbf{BA} \\ \mathrm{Re}(\mathbf{A})\mathbf{B} &= \mathbf{B}\mathrm{Re}(\mathbf{A}). \end{aligned} \tag{30}$$

The fact that $\mathrm{Im}(\mathbf{A})$ and $\mathrm{conj}(\mathbf{A})$ also commute with $\mathbf{B}$ can be proven in similar ways. #

D. Generating the Real and Orthogonal Eigenvector Set

The matrix $\mathbf{M}$ used for generating the commuting matrix can be any matrix. However, if we want to derive a real and orthogonal eigenvector set, we should choose $\mathbf{M}$ to satisfy some constraints.

Theorem 9: If the periodic or quasi-periodic matrix $\mathbf{B}$ is **Hermitian symmetric** or **asymmetric**

$$\mathbf{B^H} = \pm\mathbf{B} \tag{31}$$

where the Hermitian operation $^{\mathbf{H}}$ means transpose and conjugation ($\mathbf{B^H} = \mathrm{conj}(\mathbf{B^T})$), then we can also choose $\mathbf{M}$ as a Hermitian symmetric matrix

$$\mathbf{M^H} = \mathbf{M}. \tag{32}$$

In this condition, from (22)

$$\begin{aligned} \mathbf{A^H} &= \mathbf{M^H} + (\mathbf{B}^{-1})^{\mathbf{H}}\mathbf{M^H}\mathbf{B^H} + (\mathbf{B}^{-2})^{\mathbf{H}}\mathbf{M^H}\left((\mathbf{B}^2)^{\mathbf{H}}\right. \\ &\quad + \cdots\cdots\cdots + (\mathbf{B^{-p+1}})^{\mathbf{H}}\mathbf{M^H}(\mathbf{B^{p-1}})^{\mathbf{H}} \\ &= \mathbf{M} + (\pm\mathbf{B})^{-\mathbf{1}}\mathbf{M}(\pm\mathbf{B}) + (\pm\mathbf{B})^{-\mathbf{2}}\mathbf{M}(\pm\mathbf{B})^{\mathbf{2}} + \\ &\quad \cdots\cdots\cdots + (\pm\mathbf{B})^{-\mathbf{p+1}}\mathbf{M}(\pm\mathbf{B})^{\mathbf{p-1}} \\ &= \mathbf{M} + (\pm C^{-1}\mathbf{B^{p-1}})\mathbf{M}(\pm C\mathbf{B^{-p+1}}) \\ &\quad + (\pm C^{-1}\mathbf{B^{p-2}})\mathbf{M}(\pm C\mathbf{B^{-p+2}}) + \cdots\cdots\cdots \\ &\quad + (\pm C^{-1}\mathbf{B})\mathbf{M}(\pm C\mathbf{B}^{-1}) \\ &= \mathbf{M} + \mathbf{B^{p-1}MB^{-p+1}} + \mathbf{B^{p-2}MB^{-p+2}} + \cdots\cdots\cdots \\ &\quad + \mathbf{BMB^{-1}} \\ \mathbf{A^H} &= \mathbf{A}. \end{aligned} \tag{33}$$

That is, the commuting matrix $\mathbf{A}$ is also a Hermitian symmetric matrix. Note that the eigenvectors of a Hermitian symmetric matrix are always orthogonal if all of its eigenvalues are distinct. In fact, even if there is an eigenvalue multiplicity, we can select an orthogonal eigenvector set from the eigenspace.

Corollary 3: For the case where $\mathbf{B}$ satisfies (19) (i.e., offset quasi-periodic), (31) is modified as

$$(\mathbf{B} + \sigma\mathbf{I})^{\mathbf{H}} = \pm(\mathbf{B} + \sigma\mathbf{I}). \tag{34}$$

In this case, if we set $\mathbf{M^H} = \mathbf{M}$, then from the process similar to (33), we can also prove that $\mathbf{A^H} = \mathbf{A}$ and the eigenvectors of $\mathbf{A}$ form a complete-orthogonal set if all of its eigenvalues are distinct.

{Process to find the complete-orthogonal eigenvector set}

If an $N \times N$ matrix $\mathbf{B}$ is a Hermitian symmetric or asymmetric matrix that satisfies (31) (when $\mathbf{B}$ is periodic or quasi-periodic) or (34) (when $\mathbf{B}$ is offset quasi-periodic), to find the complete-orthogonal eigenvector set of $\mathbf{B}$, we can follow the following steps:

Step (1) First, choose an $N \times N$ matrix $\mathbf{M}$. The only constraint for $\mathbf{M}$ is $\mathbf{M} = \mathbf{M^H}$.
Step (2) Then, generate $\mathbf{A}$ from $\mathbf{M}$ by (12), (18), or (22).
Step (3) Find the eigenvalues and eigenvectors of $\mathbf{A}$.

$$\begin{array}{l} \text{If all the eigenvalues of } \mathbf{A} \text{ are distinct, which is} \\ \text{usually the case, then the eigenvectors of } \mathbf{A} \text{ form a} \\ \text{complete and orthogonal eigenvector set of } \mathbf{B}. \\ \text{If } \mathbf{A} \text{ has repeated eigenvalues, we may return to} \\ \text{Step 1 or choose eigenvectors from the eigenspace} \\ \text{properly carefully. For example, if the multiplicity is 2} \\ \text{we can try to select two eigenvectors, one which is even} \\ \text{the other odd, from the eigenspace. Then, sometimes} \\ \text{both the even and the odd eigenvectors} \\ \text{are eigenvectorsof } \mathbf{B}, \text{ such as the case} \\ \text{of the offset DFT.} \end{array} \tag{35}$$

Theorem 10: If a periodic or quasi-periodic matrix $\mathbf{B}$ is orthogonal

$$\mathbf{B^H B} = \rho \cdot \mathbf{I} \quad \text{where } \rho \text{ is some constant} \tag{36}$$

we can choose $\mathbf{M}$ to satisfy $\mathbf{M} = \mathbf{M^H}$. It is not difficult to prove that the commuting matrix $\mathbf{A}$ derived from $\mathbf{M}$ satisfies

$$\mathbf{A} = \mathbf{A^H} \tag{37}$$

and we can also apply the algorithm in (35) to derive the complete-orthogonal eigenvector set of $\mathbf{B}$ from $\mathbf{A}$. When $\mathbf{B}$ is offset quasi-periodic, the theorem can also be applied, but (36) is modified as

$$(\mathbf{B} + \sigma\mathbf{I})^{\mathbf{H}}(\mathbf{B} + \sigma\mathbf{I}) = \rho \cdot \mathbf{I} \quad \text{where } \rho \text{ is some constant.} \tag{38}$$

Theorem 11: When a periodic or quasi-periodic $\mathbf{B}$ satisfies

$$\mathbf{B} = \pm\mathbf{B^T} \tag{39}$$

if we choose $\mathbf{M} = \mathbf{M^T}$, then it is not difficult to prove that $\mathbf{A} = \mathbf{A^T}$. In this case, the eigenvectors of $\mathbf{B}$ derived from $\mathbf{A}$ may not satisfy the orthogonality property

$$\mathbf{E^H E} = \mathbf{D} \quad \text{(may be not satisfied)} \tag{40}$$

where $\mathbf{D}$ is some diagonal matrix and each column of $\mathbf{E}$ is an eigenvector derived from $\mathbf{A}$. However, if all the eigenvalues of $\mathbf{A}$ are distinct, then the eigenvectors derived from $\mathbf{A}$ satisfy

$$\mathbf{E^T E} = \mathbf{D_1} \quad \text{(satisfied)} \tag{41}$$

where $\mathbf{D_1}$ is a diagonal matrix. If $\mathbf{B}$ is offset quasi-periodic, (39) is modified as

$$(\mathbf{B} + \sigma\mathbf{I})^{\mathbf{T}} = \pm(\mathbf{B} + \sigma\mathbf{I}). \tag{42}$$

Theorem 12: When $\mathbf{B}$ is periodic or quasi-periodic, if

$$\text{(i) } \mathbf{B} \text{ is a real matrix, or} \tag{43}$$

$$\text{(ii) } \mathrm{conj}(\mathbf{B}) = \eta \cdot \mathbf{B}^{-1} \text{ where } \eta \text{ is some constant} \tag{44}$$

we can choose $\mathbf{M}$ as a real matrix. Then, the commuting matrix $\mathbf{A}$ generated from $\mathbf{M}$ is also a real matrix. Case (i) is very easy to prove. Case (ii) is proved as follows (note that for the DFT and the offset DFT, when $a = b = L/2$, where L is an integer, the transform matrix $\mathbf{F}$ is periodic and $\mathrm{conj}(\mathbf{F}) = \mathbf{F^{-1}}$).

Proof:

$$\begin{aligned}
\mathrm{conj}(\mathbf{A}) &= \mathbf{M} + \mathrm{conj}(\mathbf{B})\mathbf{M}\mathrm{conj}(\mathbf{B^{-1}}) \\
&\quad + \mathrm{conj}(\mathbf{B^2})\mathbf{M}\mathrm{conj}(\mathbf{B^{-2}}) + \ldots\ldots \\
&\quad + \mathrm{conj}(\mathbf{B^{p-1}})\mathbf{M}\mathrm{conj}(\mathbf{B^{-p+1}}) \\
&= \mathbf{M} + \mathbf{B^{-1}MB} + \mathbf{B^{-2}MB^2} + \ldots\ldots \\
&\quad + \mathbf{B^{-p+1}MB^{p-1}} \\
&= \mathbf{M} + C^{-1}\mathbf{B^{p-1}MB^{-p+1}}C \\
&\quad + C^{-1}\mathbf{B^{p-2}MB^{-p+2}}C + \ldots\ldots \\
&\quad + C^{-1}\mathbf{B^1MB^{-1}}C \quad (\text{where } \mathbf{B^p} = C\mathbf{I}) \\
&= \mathbf{M} + \mathbf{B^{p-1}MB^{-p+1}} + \mathbf{B^{p-2}MB^{-p+2}} + \\
&\quad \ldots\ldots + \mathbf{B^1MB^{-1}} = \mathbf{A}.
\end{aligned}$$

#

Corollary 4: When $\mathbf{B}$ is offset quasi-periodic, Theorem 12 can also be applied, but (43) and (44) are modified as

$$\text{(i) } \mathbf{B} + \sigma\mathbf{I} \text{ is a real matrix, or} \tag{45}$$

$$\text{(ii) } \mathrm{conj}(\mathbf{B} + \sigma\mathbf{I}) = \eta \cdot (\mathbf{B} + \sigma\mathbf{I})^{-1} \text{ where } \eta \text{ is some constant.} \tag{46}$$

Theorem 13: If a periodic or quasi-periodic matrix $\mathbf{B}$ is real and $\mathbf{B} = \pm\mathbf{B^T}$, then, from Theorem 9 (or Theorem 11) and Theorem 12, we can choose $\mathbf{M}$ as a real matrix such that $\mathbf{M} = \mathbf{M^T}$. Then, the commuting matrix $\mathbf{A}$ generated from (12) and (18) satisfies

$$\mathbf{A} \text{ is real} \quad \text{and} \quad \mathbf{A} = \mathbf{A^T}. \tag{47}$$

From the theorem in linear algebra, if all the eigenvalues of $\mathbf{A}$ are distinct, the eigenvectors of $\mathbf{A}$ are not only **complete-orthogonal** but also **real**.

Theorem 14: Generally speaking, if (**I**) one of the constraints in (31), (34), (36), and (38) is satisfied and (**II**) one of the constraints in (43), (44), (45), and (46) is satisfied, then we can set

$$\mathbf{M} \text{ is real} \quad \text{and} \quad \mathbf{M} = \mathbf{M^T}. \tag{48}$$

Then the commuting matrix $\mathbf{A}$ generated from $\mathbf{M}$ satisfies

$$\mathbf{A} \text{ is real} \quad \text{and} \quad \mathbf{A} = \mathbf{A^T}. \tag{49}$$

Thus, under these conditions, we can also follow the process in (35) to find the **real**, **complete**, and **orthogonal** eigenvector set of $\mathbf{B}$. The only difference is that, in Step 1, $\mathbf{A}$ should be real and $\mathbf{M} = \mathbf{M^T}$.

For example, for the case of the DFT and the offset DFT when $a = b = L/2$, where L is some integer, since the transform matrix $\mathbf{F}$ is periodic, $\mathbf{F^H F} = \mathbf{I}$ [satisfies (36)], and $\mathrm{conj}(\mathbf{F}) = \mathbf{F^{-1}}$ [satisfies (44)], we can choose $\mathbf{M}$ to be a real matrix and satisfy $\mathbf{M} = \mathbf{M^T}$ and use it to generate the commuting matrix $\mathbf{A}$. Then, the eigenvector set of $\mathbf{F}$ derived from $\mathbf{A}$ is real, complete, and orthogonal.

In Table I, we summarize the theorems stated in this subsection and show the methods of choosing $\mathbf{M}$ if we want to derive the real, complete, and orthogonal eigenvector set of $\mathbf{B}$.

III. Commuting Matrices for Offset DFT

The **offset DFT** [10] is defined as

$$X_{a,b}[m] = \sum_{n=0}^{N-1} F_{a,b}[m,n]x[n]$$
$$\text{where} \quad F_{a,b}[m,n] = \sqrt{N^{-1}} e^{-j\frac{2\pi}{N}(m-a)(n-b)}. \tag{50}$$

This is a generalization of the DFT and is useful for filter design, fast algorithm design, and signal representation. In [7], the eigenvectors of the offset DFT when $a = b = 1/2$ were derived. In [8], we found that, when $a+b$ is an integer, the following matrix will commute with the offset DFT:

$$\mathbf{S_{a,b}} = \begin{bmatrix} D_0 & C_2 & 0 & 0 & \cdots & 0 & C_3 \\ C_1 & D_1 & C_2 & 0 & \cdots & 0 & 0 \\ 0 & C_1 & D_2 & C_2 & \ddots & \vdots & \vdots \\ 0 & 0 & C_1 & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & C_2 & 0 \\ 0 & \vdots & \vdots & \cdots & C_1 & D_{N-2} & C_2 \\ C_4 & 0 & 0 & \cdots & 0 & C_1 & D_{N-1} \end{bmatrix}$$
$$\mathbf{S_{a,b}F_{a,b}} = \mathbf{F_{a,b}S_{a,b}} \tag{51}$$
$$\begin{aligned} \text{where } D_m &= 2\cos\left[2\pi\left(m - (a+b)/2\right)/N\right] \\ C_1 &= e^{j\frac{\pi}{N}(b-a)},\ C_2 = e^{j\frac{\pi}{N}(a-b)} \\ C_3 &= e^{j\frac{\pi}{N}(b-a)}e^{-j2\pi b},\ C_4 = e^{j\frac{\pi}{N}(a-b)}e^{-j2\pi a}. \end{aligned} \tag{52}$$

In fact, this previous work can be much generalized by Theorem 2 described in Section II-B.

A. General Commuting Matrix of the Offset DFT

Note that when $a+b$ is an integer, the offset DFT satisfies

$$(\mathbf{F_{a,b}})^4 = \exp\left[j2\pi(a-b)^2/N\right] \cdot \mathbf{I}. \tag{53}$$

From (17), the offset DFT is a quasi-periodic operation with period 4. Therefore, from (18), if

$$\mathbf{A} = \mathbf{M} + \mathbf{F_{a,b}MF^H_{a,b}} + \mathbf{F^2_{a,b}M}\left(\mathbf{F^2_{a,b}}\right)^{\mathbf{H}} + \mathbf{F^3_{a,b}M}\left(\mathbf{F^3_{a,b}}\right)^{\mathbf{H}}$$
$$\text{where } \mathbf{M} \text{ is an arbitrary } N \times N \text{ matrix} \tag{54}$$

then matrix $\mathbf{A}$ commutes with the offset DFT $\mathbf{F_{a,b}}$

$$\mathbf{A}\mathbf{F_{a,b}} = \mathbf{F_{a,b}}\mathbf{A}. \tag{55}$$

Moreover, since $\mathbf{F^H_{a,b}}\mathbf{F_{a,b}} = \mathbf{I}$, i.e., the second condition in Table I is satisfied, in (54), we can choose $\mathbf{M}$ as an $N \times N$ matrix that satisfies

$$\mathbf{M} = \mathbf{M^H}. \tag{56}$$

Then, since $\mathbf{A} = \mathbf{A^H}$, if all the eigenvalues of $\mathbf{A}$ are distinct, we can use $\mathbf{A}$ to derive the **complete-orthogonal** eigenvector set of the offset DFT $\mathbf{F_{a,b}}$.

Specifically, when $a = b = L/2$, where L is some integer, then the seventh condition in Table I is satisfied. In this case, we can set $\mathbf{M} = \mathbf{M^T}$ and $\mathbf{M}$ is real. Then, we can use the commuting matrix $\mathbf{A}$ generated from $\mathbf{M}$ to derive the **real**, **complete**, and **orthogonal** eigenvector set of the offset DFT.

From (54), we can much generalize the work in [7] and [8] to find many commuting matrices of the offset DFT and derive a variety of orthogonal eigenvector sets of the offset DFT.

B. *Generalized* $\mathbf{S}$ *Matrix for Offset DFTs*

In (54), we can choose $\mathbf{M}$ as the following diagonal matrix:

$$\begin{aligned} M[n,n] &= \cos\left(2\pi k\left[n-(a+b)/2\right]/N\right) \\ &\quad \text{for } n = 0, 1, \ldots, N-1 \\ M[m,n] &= 0 \text{ otherwise.} \end{aligned} \tag{57}$$

Then, the resultant commuting matrix is shown in (58)–(61), at the bottom of the page. Note that the $\mathbf{S_{a,b}}$ matrix proposed in [8] is a special case of (58) when $k = 1$. As $\mathbf{S_{a,b,}}$, the new $\mathbf{S_{a,b,k}}$ matrix is also a matrix whose entries are nonzero only in three oblique lines. The difference is that *the distances between the two off-diagonal lines and the main diagonal line are* k, *not 1.*

In addition to the conditions where N is even and $N/2+a+b$ is even, all the eigenvalues of $\mathbf{S_{a,b,k}}$ are distinct. Therefore, the eigenvectors of $\mathbf{S_{a,b,k}}$ are also the eigenvectors of the offset DFT. Even when N is even and $N/2+a+b$ is even, although the multiplicity of $\lambda = 0$ is two, we can divide the eigenspace into the "even part" and the "odd part." Then the obtained vectors are also the eigenvectors of the offset DFT. Moreover, since

$$\mathbf{S_{a,b,k}} = \mathbf{S^H_{a,b,k}} \tag{62}$$

we can use $\mathbf{S_{a,b,k}}$ to derive the complete-orthogonal eigenvector set of the offset DFT.

C. *Generalized* $\mathbf{T}$ *Matrix for Offset DFTs*

In [4], we modify Grünbaum's work [2] and propose the $\mathbf{T}$ matrix [see (5)], which can commute with the original nonoffset DFT. However, until now, there has been no similar work for the case of the offset DFT. Here, we find that, if in (54) we choose $\mathbf{M}$ as of the following form:

$$\begin{aligned} M[n,n+k] &= 4e^{j\frac{\pi}{N}k(a-b)} \\ &\quad \times \cos\!\Big(2\pi\left\{k\left[n-(a+b)/2\right]+k^2/2\right\}/N\Big) \\ M[n+k,n] &= M^*[n,n+k] \;\text{ for } 0 \le n \le N-k+1 \end{aligned} \tag{63}$$

$$\begin{aligned} M[r-k+n,r] &= 4e^{j\frac{\pi}{N}k(a-b)-j2\pi a} \\ &\quad \times \cos\!\Big(2\pi\left\{r\left[n-(a+b)/2\right]-k^2/2\right\}/N\Big) \\ M[r,r-k+n] &= M^*[r-k+n,r] \;\text{ for } 0 \le r \le k \le 1 \end{aligned} \tag{64}$$

$$\mathbf{S_{a,b,k}} = \zeta \begin{bmatrix} D_0 & 0 & 0 & \cdots & C_2 & 0 & 0 & \cdots & C_3 & 0 & 0 & \cdots & 0 \\ 0 & D_1 & 0 & \cdots & 0 & C_2 & 0 & \cdots & 0 & C_3 & 0 & \cdots & 0 \\ 0 & 0 & D_2 & \cdots & 0 & 0 & C_2 & \cdots & 0 & 0 & C_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ C_1 & 0 & 0 & \cdots & D_k & 0 & 0 & \cdots & C_2 & 0 & 0 & \cdots & C_3 \\ 0 & C_1 & 0 & \cdots & 0 & D_{k+1} & 0 & \cdots & 0 & C_2 & 0 & \cdots & 0 \\ 0 & 0 & C_1 & \cdots & 0 & 0 & D_{k+2} & \cdots & 0 & 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ C_4 & 0 & 0 & \cdots & C_1 & 0 & 0 & \cdots & D_{N-k} & 0 & 0 & \cdots & C_2 \\ 0 & C_4 & 0 & \cdots & 0 & C_1 & 0 & \cdots & 0 & D_{N-k+1} & 0 & \cdots & 0 \\ 0 & 0 & C_4 & \cdots & 0 & 0 & C_1 & \cdots & 0 & 0 & D_{N-k+2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & C_4 & 0 & 0 & \cdots & C_1 & 0 & 0 & \cdots & D_{N-1} \end{bmatrix} \tag{58}$$

$$\begin{aligned} \text{where } D_m &= 2\cos\left[2\pi k\left(m-(a+b)/2\right)/N\right] \\ C_1 &= e^{j\pi k(b-a)/N},\; C_2 = C_1^* \end{aligned} \tag{59}$$

$$\begin{aligned} C_3 &= e^{j\frac{\pi}{N}k(b-a)}e^{j2\pi a} \\ C_4 &= e^{j\frac{\pi}{N}k(a-b)}e^{-j2\pi a},\; \zeta \text{ is some constant} \end{aligned} \tag{60}$$

$$\mathbf{S_{a,b,k}}\mathbf{F_{a,b}} = \mathbf{F_{a,b}}\mathbf{S_{a,b,k}}. \tag{61}$$

and $M[m,n]=0$ otherwise, then the commuting matrix generated from $\mathbf{M}$ (denoted by $\mathbf{T_{a,b,k}}$) is

$$\begin{aligned}
&T_{a,b,k}[n,n+k]\\
&\quad= e^{j\frac{\pi}{N}k(a-b)}\cos\left(2\pi\left\{k\left[n-(a+b)/2\right]+k^2/2\right\}/N\right)\\
&T_{a,b,k}[n+k,n]\\
&\quad= T^*_{a,b,k}[n,n+k] \quad \text{for } n=0,1,\ldots,N-k-1\\
&T_{a,b,k}[n,n+k-N]\\
&\quad= e^{j\frac{\pi}{N}k(a-b)}e^{-j2\pi a}\cos\left(\frac{2\pi}{N}\left\{k\left[n-\frac{a+b}{2}\right]+k^2/2\right\}\right)\\
&T_{a,b,k}[n+k-N,n]=T^*_{a,b,k}[n,n+k-N]\\
&\qquad \text{for } n=N-k,N-k+1,\ldots,N-1\\
&T_{a,b,k}[m,n]=0 \text{ otherwise.}
\end{aligned}\tag{65}$$

This satisfies the commuting property

$$\mathbf{T_{a,b,k}F_{a,b}}=\mathbf{F_{a,b}T_{a,b,k}} \quad \text{when } a+b \text{ is an integer.}\tag{66}$$

When $N+a+b$ is odd, all the eigenvalues of $\mathbf{T_{a,b,k}}$ are distinct and $\mathbf{T_{a,b,k}}$ commutes with $\mathbf{F_{a,b}}$, the eigenvectors of $\mathbf{T_{a,b,k}}$ are also the eigenvectors of $\mathbf{F_{a,b}}$. When $N+a+b$ is even, the eigenspace corresponding to $\lambda=0$ has a multiplicity of 2. In this case, if $\mathbf{e_1}$ is an eigenvector of $\mathbf{T_{a,b,k}}$ with $\lambda=0$, we can divide $\mathbf{e_1}$ into the even and the odd parts. Then, the resultant vectors are also the eigenvectors of the offset DFT. Moreover, since

$$\mathbf{T_{a,b,k}}=\mathbf{T^H_{a,b,k}}\tag{67}$$

the eigenvectors of the offset DFT derived by $\mathbf{T_{a,b,k}}$ will be orthogonal.

Note that the $\mathbf{T}$ matrix in (5) can be expressed as a linear combination of $\mathbf{T_{0,0,0}}$, $\mathbf{S}$, and $\mathbf{I}$

$$\mathbf{T}=\mathbf{S}+\mathbf{T_{0,0,0}}/\cos(\pi/N)+2\mathbf{I}.\tag{68}$$

D. Shifted n^2 Matrix for the Offset DFT and HGF-Like Eigenvectors

In [19], Santhanam proposed a new matrix that can commute with the central DFT, which is a special case of the offset DFT where

$$a=b=(N-1)/2.\tag{69}$$

Here, we modify this work and propose a matrix that can commute with the offset DFT.

For the offset DFT, we can use the following matrix to derive the commuting matrix and hence the eigenvectors of the offset DFT:

$$\begin{aligned}
M[n,n]&=(n-(a+b)/2)^2\\
&\qquad \text{for } 0\le n<(a+b)/2+N/2\\
M[n,n]&=(n-N-(a+b)/2)^2\\
&\qquad \text{for } (a+b)/2+N/2\le n<N\\
M[m,n]&=0 \text{ otherwise.}
\end{aligned}\tag{70}$$

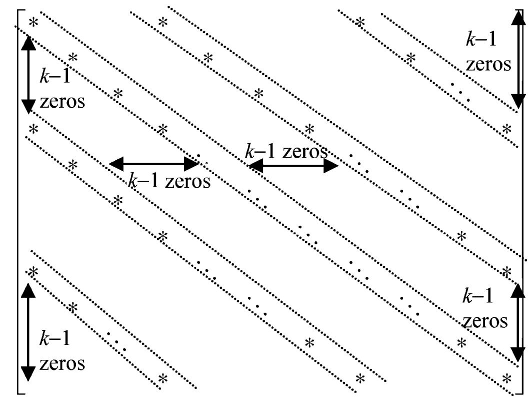

Fig. 1. Form of $\mathbf{S_{a,b,k}}$ in Section III-B and $\mathbf{T_{a,b,k}}$ in Section III-C ($*$ means the nonzero entry).

Then, we can substitute (70) into (54) and obtain the commuting matrix of the offset DFT. We call the resultant commuting matrix the **offset n^2 matrix**. Since the shifted n^2 matrix $\mathbf{A}$ satisfies $\mathbf{A}=\mathbf{A^H}$, the offset DFT eigenvector set derived from the shifted n^2 matrix is complete and orthogonal.

Compared with the $\mathbf{S_{a,b,k}}$ matrix and the $\mathbf{T_{a,b,k}}$ matrix, the offset DFT eigenvectors derived from the shifted n^2 matrix are nearer to the samples of the eigenfunctions the continuous offset FT. The eigenfunctions of the continuous offset FT [16] is

$$\begin{aligned}
\Phi_p(x)=&\exp\left[j(\eta-\tau)x/2-(2x-\eta-\tau)^2/8\right]\\
&\times H_p\left(x-(\eta+\tau)/2\right)\\
&\text{where } H_p(x) \text{ is the } p^{\text{th}}-\text{order}\\
&\text{Hermite function.}
\end{aligned}\tag{71}$$

It is the shifting and modulation version of the Hermite-Gaussian function (HGF). Since the offset DFT with parameters $\{a,b\}$ corresponds to the continuous offset FT with the following parameters:

$$n=x\sqrt{N/2\pi},\ m=\omega\sqrt{N/2\pi},\ a=\tau\sqrt{N/2\pi},\ b=\eta\sqrt{N/2\pi}\tag{72}$$

therefore, the eigenvectors of the offset DFT should be of the following form:

$$\begin{aligned}
e_p(n)\approx&\exp\left[jn\frac{b-a}{2}\sqrt{2\pi/N}-(2n-a-b)^2\pi/4N\right]\\
&\times H_p\left(\left[n-\frac{a+b}{2}\right]\sqrt{2\pi/N}\right)\\
&\text{when } 0\le n\le(N+a+b)/2
\end{aligned}\tag{73}$$

$$\begin{aligned}
e_p(n)\approx&\, s_1\exp\left[jn_1\frac{b-a}{2}\sqrt{2\pi/N}-(2n_1-a-b)^2\pi/4N\right]\\
&\times H_p\left(\left[n_1-\frac{a+b}{2}\right]\sqrt{2\pi/N}\right)\\
&\text{when } (N+a+b)/2<n\le N-1\\
&\text{where } n_1=n-N.
\end{aligned}\tag{74}$$

$s_1=1$ when $a+b$ is even and $s_1=-1$ when $a+b$ is odd. We then performed several experiments to investigate whether the eigenvectors of the offset DFT derived from $\mathbf{S_{a,b,k}}$, $\mathbf{T_{a,b,k}}$, and the offset-n^2 matrix approximate the samples of the shifted and modulated HGFs, i.e., satisfy (73) and (74).

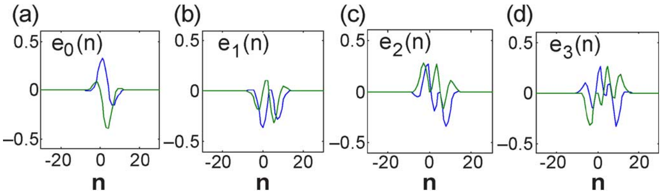

Fig. 2. The first four eigenvectors of the offset DFT derived by the offset-n^2 matrix when $N = 61$, $a = 0.3$, and $b = 1.7$. The two lines are the real part and the imaginary part, respectively.

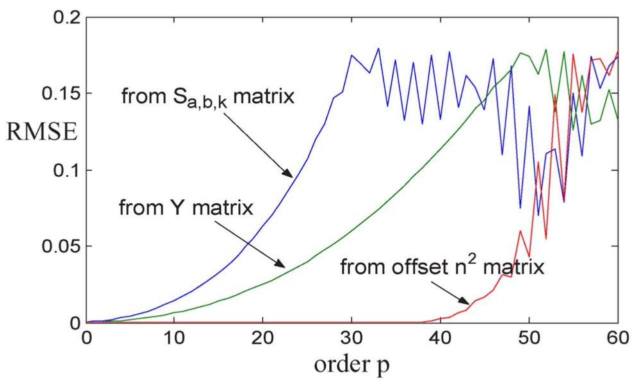

Fig. 3. RMSEs of the offset DFT eigenvectors derived from the $\mathbf{S_{a,b,k}}$ matrix, the $\mathbf{Y}$ matrix [defined in (75)], and the offset-n^2 matrix for $N = 61, a = 0.3$, and $b = 1.7$.

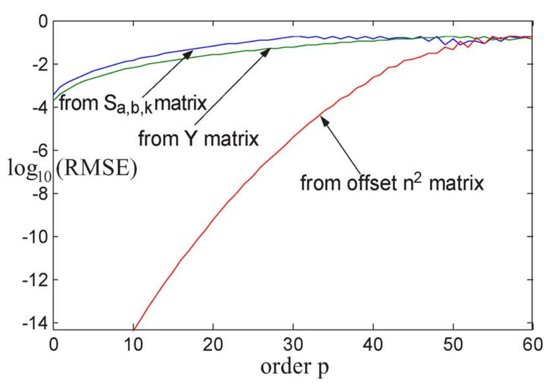

Fig. 4. $\mathrm{Log}_{10}(\mathrm{RMSEs})$ of the offset DFT eigenvectors where $N = 61, a = 0.3$, and $b = 1.7$.

In Fig. 2, we plot the offset DFT eigenvectors ($N = 61$, $a = 0.3$, and $b = 1.7$) derived from $\mathbf{M}$, which we call the offset-n^2 matrix. Then, in Figs. 3 and 4, we compare these with those derived from the $\mathbf{S_{a,b,1}}$ matrix and the $\mathbf{Y}$ matrix, where

$$\mathbf{Y} = \mathbf{S_{a,b,1}} + \mathbf{T_{a,b,1}}/\cos(\pi/N) + 2\mathbf{I}. \tag{75}$$

We can find that the offset DFT eigenvectors derived from the offset-n^2 matrix are indeed closer to the samples of (71) than those derived from other methods, especially when the order p is small.

E. Summary

In this section, we use Theorem 2 to find the commuting matrices of the offset DFT, such as the $\mathbf{S_{a,b}}$ matrix, the $\mathbf{T_{a,b}}$ matrix, the offset-n^2 matrix, and their combinations. We then use these to find the eigenvectors of the offset DFT, which are very close to the samples of the shifting and modulating version of Hermite-Gaussian functions.

IV. Commuting Matrices and Eigenvectors for Other Periodic and Offset Quasi-Periodic Operations

We have stated that, if an operation is periodic, quasi-periodic, or offset quasi-periodic, we can use Theorem 1, 2, or 3 to derive the commuting matrices and hence the complete-orthogonal eigenvector set. An example of the offset DFT (a quasi-periodic operation with period 4) was given in Section III. In this section, we show other examples using the generalized commuting matrix to derive the eigenvectors of discrete sinusoid transforms, the Walsh transform, and the projection operation.

Example 1: Discrete Sinusoid Transforms and Walsh Transforms: The **discrete cosine transform** (DCT), the **discrete sine transform** (DST) [17], and the **discrete Hartley transform** (DHT) [18] have a variety of definitions. Some of them are periodic operations. For example, the DCT of type 1 (denoted by DCT-I) is defined as

$$\text{DCT-I}: \; B[m,n] = \sqrt{\frac{2}{N-1}} k_m k_n \cos\left(\frac{\pi m n}{N-1}\right),$$
$$k_n = 1/\sqrt{2} \text{ when } n=0 \text{ or } N-1, \; k_n = 1 \text{ otherwise}. \tag{76}$$

It has a period of two and satisfies

$$\mathbf{B^2} = \mathbf{I}. \tag{77}$$

Moreover, the DCT of types 4, 5, and 8, the DST of types 1, 4, 5, and 8, and the DHT of types 1 and 4 (definitions can be seen from [8], [17], and [18]) also have a period of 2 and satisfy (77). Therefore, for **any** $N \times N$ matrix $\mathbf{M}$, if

$$\mathbf{A} = \mathbf{M} + \mathbf{BMB}, \quad (\text{note that if } \mathbf{B}^2 = \mathbf{I}, \text{ then } \mathbf{B^{-1}} = \mathbf{B}) \tag{78}$$

where $\mathbf{B}$ is the corresponding transform matrix of DCTs of types 1, 4, 5, or 8, DSTs of types 1, 4, 5, or 8, or DHT of types 1 or 4, then $\mathbf{A}$ will commute with $\mathbf{B}$. If all the eigenvalues of $\mathbf{A}$ are distinct, the eigenvectors of $\mathbf{A}$ are also the eigenvectors of $\mathbf{B}$, and we can use $\mathbf{A}$ to derive the eigenvectors of the DCT, DST, or DHT. Moreover, if we choose $\mathbf{M}$ as a symmetric matrix, then $\mathbf{A}$ is also symmetric

$$\mathbf{A} = \mathbf{A^H} \quad \text{if } \mathbf{M} = \mathbf{M^H}. \tag{79}$$

The eigenvector sets of the DCT, DST, or DHT derived from the above commuting matrix $\mathbf{A}$ are complete and orthogonal. The recent tridiagonal commuting matrices of the DCT-I, DCT-IV, DST-I, and DST-IV derived by Pei and Hsue [13] are special cases of the general commuting matrix $\mathbf{A}$ in (78).

For the **Walsh transform** (also called the **Hadamard transform**) matrix $\mathbf{W}$ [14], since

$$\mathbf{W}^2 = N\mathbf{I} \tag{80}$$

from (18), the following matrix commutes with $\mathbf{W}$:

$$\mathbf{A} = \mathbf{M} + \mathbf{WMW}/N \tag{81}$$

$$\mathbf{AW} = \mathbf{WA} \tag{82}$$

where $\mathbf{M}$ is an arbitrary $N \times N$ matrix. Since $\mathbf{W} = \mathbf{W^T}$ and $\mathbf{W}$ is real, we can choose $\mathbf{M}$ as a real matrix that satisfies $\mathbf{M} = \mathbf{M^H}$. If the eigenvalues of $\mathbf{A}$ generated from $\mathbf{M}$ are distinct, we can use $\mathbf{A}$ to find the real, complete, orthogonal eigenvector set of the Walsh (Hadamard) transform.

Example 2: Projection Operations: The operation $\mathbf{P}$ that projects an N-dimensional space into a K-dimensional space is the discrete linear operation with two eigenvalues, 0 and 1. The eigenspace corresponding to the eigenvalue of 1 has the dimension of K and the eigenspace corresponding to 0 has the dimension of $N - K$. Note that

$$(\mathbf{P} - \mathbf{I}/2)^2 = \mathbf{I}/4. \tag{83}$$

This is an **offset quasi-periodic** operation that satisfies (19). Therefore, from (22), the following matrix will commute with the permuting matrix $\mathbf{P}$:

$$\begin{aligned}\mathbf{A} &= \mathbf{M} + (\mathbf{P} - \mathbf{I}/2)\mathbf{M}(\mathbf{P} - \mathbf{I}/2)^{-1} \\ &= \mathbf{M} + (\mathbf{P} - \mathbf{I}/2)\mathbf{M}(4\mathbf{P} - 2\mathbf{I}) \\ &\quad \big(\text{from } (\mathbf{P} - \mathbf{I}/2)^{-1} = 4(\mathbf{P} - \mathbf{I}/2)\big) \\ &\quad \text{where } \mathbf{M} \text{ is an arbitrary} \\ &\quad N \times N \text{matrix} \end{aligned} \tag{84}$$

$$\mathbf{A}(\mathbf{P} - \mathbf{I}/2) = (\mathbf{P} - \mathbf{I}/2)\mathbf{A}. \tag{85}$$

Thus, from Theorem 4, if all the eigenvalues of $\mathbf{A}$ are distinct, the eigenvectors of $\mathbf{A}$ are also the eigenvectors of the permuting operation $\mathbf{P}$. Moreover, if $\mathbf{P}$ is a Hermitian symmetric real matrix and the matrix $\mathbf{M}$ we choose is real and satisfies $\mathbf{M^H} = \mathbf{M}$, then

$$\mathbf{A^H} = \mathbf{A} \quad \text{and} \quad \mathbf{A} \text{ is real} \tag{86}$$

and the eigenvectors of $\mathbf{A}$ form a real, complete, and orthogonal eigenvector set of $\mathbf{P}$.

(Example 3) The Matrix With More Than Two Eigenspaces: In Corollary 1 and Example 2, we have showed how to use commuting matrix to find the orthogonal eigenvector set of a matrix with two eigenspaces. Here, we show that if a matrix has H eigenspace where $H > 2$, we can also use Theorem 3 to find its orthogonal eigenvector set.

Suppose that a matrix $\mathbf{B}(\mathbf{B} = \mathbf{B^H})$ has only **three eigenvalues** λ_1, λ_2, and λ_3. First, we set $\mathbf{B_2}$ as

$$\mathbf{B_2} = [\mathbf{B} - (\lambda_2 + \lambda_3)/2]^2. \tag{87}$$

It is not difficult to prove that $\mathbf{B_2}$ has two eigenvalues, $(\lambda_2 - \lambda_3)^2/4$ and $[\lambda_1 - (\lambda_2 + \lambda_3)/2]^2$

$$\mathbf{B_2x} = \frac{(\lambda_2 - \lambda_3)^2}{4}\mathbf{x} \text{ if } \mathbf{Bx} = \lambda_2\mathbf{x} \text{ or } \mathbf{Bx} = \lambda_3\mathbf{x} \tag{88}$$

$$\mathbf{B_2x} = \left(\lambda_1 - \frac{\lambda_2 + \lambda_3}{2}\right)^2 \mathbf{x} \text{ if } \mathbf{Bx} = \lambda_1\mathbf{x}. \tag{89}$$

Therefore, $\mathbf{B_2}$ has two eigenspaces and we can use the method in corollary 1 to find its commuting matrix. Since $\mathbf{B} = \mathbf{B^H}$, λ_k should be real, $\mathbf{B_2} = \mathbf{B_2^H}$ is also satisfied. Therefore, we can choose $\mathbf{M}$ to satisfy $\mathbf{M} = \mathbf{M^H}$ and the eigenvectors of $\mathbf{B_2}$ derived from the commuting matrix generated from $\mathbf{M}$ form a complete and orthogonal eigenvector set. We denote the derived eigenvector set of $\mathbf{B_2}$ by $\mathbf{E}$

$$\begin{aligned}\mathbf{E} &= [\mathbf{e_{1,1}} \ \mathbf{e_{1,2}} \ \cdots \ \mathbf{e_{1,N_1}} | \mathbf{e_{2,1}} \ \mathbf{e_{2,2}} \ \cdots \ \mathbf{e_{2,N_3}}] \\ \mathbf{B_1e_{1,n}} &= \left(\lambda_1 - \frac{\lambda_2 + \lambda_3}{2}\right)^2 \mathbf{e_{1,n}} \\ \mathbf{B_1e_{2,n}} &= \frac{(\lambda_2 - \lambda_3)^2}{4}\mathbf{e_{2,n}}, \ N_1 + N_2 = N, \ \text{size}(\mathbf{B}) = N \times N.\end{aligned} \tag{90}$$

Then, from (89), $\mathbf{e_{1,n}}$ is also an eigenvector of $\mathbf{B}$ and the corresponding eigenvalue is λ_1. Although $\mathbf{e_{2,n}}$ may not be an eigenvector of $\mathbf{B}$, $\mathbf{e_{2,n}} \in \Omega_2 \cup \Omega_3$ where Ω_2 and Ω_3 are the eigenspaces of $\mathbf{B}$ corresponding to the eigenvalues of λ_2 and λ_3, respectively. In fact, if $\mathbf{y} \in \Omega_2 \cup \Omega_3$, then $\mathbf{y}$ can be expressed as a linear combination of $\mathbf{e_{2,n}}$, where $n = 1, 2, \ldots, N_1$. Therefore

$$\mathbf{E_2E_2^Hy} = \mathbf{y}, \quad \text{where} \quad \mathbf{E_2} = [\mathbf{e_{2,1}} \quad \mathbf{e_{2,2}} \quad \cdots \quad \mathbf{e_{2,N_2}}]. \tag{91}$$

If

$$\left(\mathbf{E_2^HBE_2}\right)\mathbf{z} = \tau\mathbf{z} \text{ and } \mathbf{z_2} = \mathbf{E_2z} \ \text{(note that } \mathbf{z_2} \in \Omega_2 \cup \Omega_3) \tag{92}$$

then

$$\begin{aligned}\mathbf{E_2^HBz_2} &= \tau\mathbf{z} \\ \mathbf{E_2E_2^HBz_2} &= \tau\mathbf{E_2z} \ \text{(note that } \mathbf{Bz_2} \in \Omega_2 \cup \Omega_3) \end{aligned} \tag{93}$$

$$\begin{aligned}\mathbf{Bz_2} &= \tau\mathbf{z_2} \ (\text{since } \mathbf{z_2} \in \Omega_2 \cup \Omega_3 \\ &\qquad \tau = \lambda_2 \text{ or } \lambda_3).\end{aligned} \tag{94}$$

That is, if $(\mathbf{E_2^HBE_2})\mathbf{z} = \tau\mathbf{z}$, then $\mathbf{z_2} = \mathbf{E_2z}$ is an eigenvector of $\mathbf{B}$ corresponding to the eigenvalues of λ_2 or λ_3. Thus, to find the orthogonal eigenvector set of $\mathbf{B}(\mathbf{B} = \mathbf{B^H})$ with three eigenspaces, we can

(i) First, find the eigenvectors of $\mathbf{B_2}$ defined in (87), which has only two eigenspaces and Corollary 1 can be applied. The eigenvectors correspond to λ_1 can be derived in this step.

(ii) Then, find the eigenvectors of $\mathbf{E_2^HBE_2}$. It also has only two eigenspaces and Corollary 1 can also be applied. The eigenvectors correspond to λ_2 and λ_3 can be derived in this step.

When $\mathbf{B}$ have **four** eigenvalues, λ_1, λ_2, λ_3, and λ_4, then we can first convert it into $\mathbf{B_2}$ where

$$\mathbf{B_3} = [\mathbf{B} - (\lambda_3 + \lambda_4)/2]^2. \tag{95}$$

Then, we can prove that $\mathbf{B_3}$ has three eigenvalues: $[\lambda_1-(\lambda_3+\lambda_4)/2]^2$, $[\lambda_2-(\lambda_3+\lambda_4)/2]^2$, and $(\lambda_3-\lambda_4)^2/4$. Therefore, we can follow the process described above to find the eigenvectors of $\mathbf{B_3}$. Then

$$\begin{aligned}
&\text{if } \mathbf{B_3e_{1,n}} = [\lambda_1-(\lambda_3+\lambda_4)/2]^2\,\mathbf{e_{1,n}},\ \mathbf{Be_{1,n}}=\lambda_1\mathbf{e_{1,n}}\\
&\text{if } \mathbf{B_3e_{2,n}} = [\lambda_1-(\lambda_3+\lambda_4)/2]^2\,\mathbf{e_{2,n}},\ \mathbf{Be_{2,n}}=\lambda_2\mathbf{e_{2,n}}\\
&\text{if } \mathbf{B_3e_{3,n}} = (\lambda_3-\lambda_4)^2/4\mathbf{e_{3,n}},\ \mathbf{e_{3,n}}\in\Omega_3\cup\Omega_4
\end{aligned}\tag{96}$$

where Ω_3 and Ω_4 are the eigenspaces of $\mathbf{B}$ corresponding to λ_3 and λ_4, respectively. We can use the process similar to (92)–(94) to further derive the eigenvectors of $\mathbf{B}$ corresponding to λ_3 and λ_4 from $\mathbf{e_{3,n}}$.

We can further generalize the above method to find the orthogonal eigenvector set of a Hermitian symmetric matrix $\mathbf{B}$ if $\mathbf{B}$ has **more than four** eigenspaces.

V. Further Exploration of DFT Eigenvectors

We can also use the proposed theorems to further extend the previous works about DFT eigenvectors. The commuting matrices of the DFT that have been found are as follows:

(a) the $\mathbf{S}$ matrix in (3);
(b) the $\mathbf{T}$ matrix in (5);
(c) Candan's matrix in [11]; and
(d) the matrix proposed by Santhanam (commutes with the central-form of the DFT) [19].

In this paper, we find that the matrices described in Sections V-A, V-B, and V-C can also commute with the DFT:

A. *Multiple Off-Diagonal* $\mathbf{S}$ *and* $\mathbf{T}$ *Matrices*

In (58) and (65), if we set $a=b=0$, then the matrices $\mathbf{S_{0,0,k}}$ and $\mathbf{T_{0,0,k}}$ can commute with the non-offset DFT. They have the multiple diagonal forms as in Fig. 1 and are the generalizations of the $\mathbf{S}$ and $\mathbf{T}$ matrices in (3) and (5).

B. *Improved Form of* $\mathbf{n^2}$ *Matrix*

In (70), if we set $a=b=0$, then

$$\begin{aligned}
M[n,n] &= n^2,\ \text{for } n\le N/2\\
M[n,n] &= (N-n)^2 \text{ for } n> N/2\\
M[m,n] &= 0 \text{ when } m\ne n.
\end{aligned}\tag{97}$$

If we set

$$\mathbf{A}=\mathbf{M}+\mathbf{FMF^{-1}}\tag{98}$$

[note that, since $\mathbf{F^2MF^{-2}}=\mathbf{M}$, (12) can be simplified as (98)], then $\mathbf{A}$ commutes with the original DFT (In comparison, in [19], Santhanam proposed a matrix that commutes with the central-form of the DFT.) We call $\mathbf{A}$ in (98) the $\boldsymbol{n^2}$ **matrix**. Since all the eigenvalues of $\mathbf{A}$ are distinct and $\mathbf{A}$ is a real matrix that satisfies $\mathbf{A}=\mathbf{A^T}$, the eigenvectors of $\mathbf{A}$ are the real and complete-orthogonal eigenvector set of the DFT.

We then compare the DFT eigenvectors obtained from the n^2 matrix and those of the $\mathbf{S}$ matrix, the $\mathbf{T}$ matrix, $\mathbf{S}+15\mathbf{T}$, and

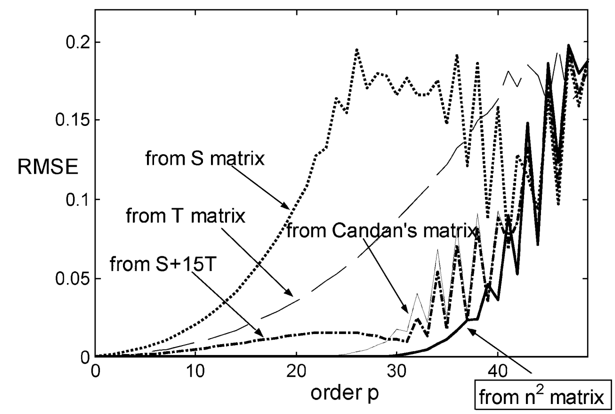

Fig. 5. Comparison of the RMSE of the DFT eigenvectors obtained from the $\mathbf{S}$ matrix, the $\mathbf{T}$ matrix, $\mathbf{S}+15\mathbf{T}$, Candan's matrix, and the n^2 matrix where the number of points $N=50$.

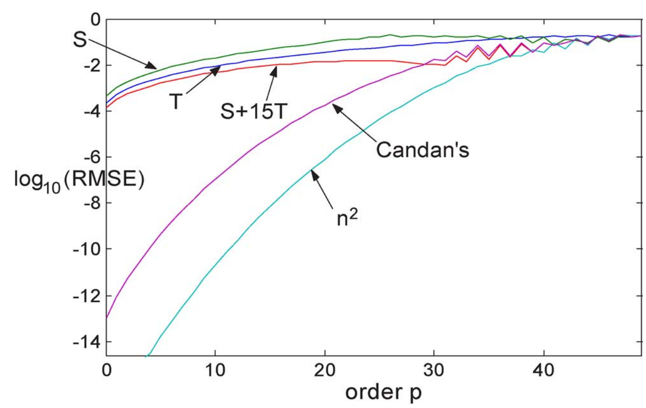

Fig. 6. Comparison of $\log_{10}(\mathrm{RMSEs})$ for the DFT eigenvectors obtained by the n^2 matrix and other methods when $N=50$.

Candan's matrix [11]. We calculate the RMSE between the DFT eigenvector and the Hermite-Gaussian function (HGF)

$$\begin{aligned}
&RMSE_p=\left(\sum_{n=0}^{N-1}\left|h_p(t\sqrt{2\pi/N})-e_p[n]\right|^2\right)^{0.5}\\
&h_p(t): p^{\text{th}} \text{ order HGF}\\
&e_p[n]: p^{\text{th}} \text{ DFT eigenvector(sorted by zero crossings).}
\end{aligned}\tag{99}$$

When $p=0$, the RMSEs of the DFT eigenvectors are

$$\begin{aligned}
\mathbf{S}\text{ matrix}: \mathrm{RMSE} &= 4.81\times10^{-4}\\
\mathbf{T}\text{ matrix}: \mathrm{RMSE} &= 2.35\times10^{-4}\\
\mathbf{S}+15\mathbf{T}: \mathrm{RMSE} &= 3.38\times10^{-6}\\
\text{Candan's matrix}: \mathrm{RMSE} &= 1.15\times10^{-13}\\
n^2\text{ matrix}: \mathrm{RMSE} &= 9.68\times10^{-16}.
\end{aligned}\tag{100}$$

In Fig. 5, we plot the RMSE for all ps for $N=50$. From (100), and Fig. 5, it is apparent that in most of the cases the RMSE of the DFT eigenvectors obtained from the n^2 matrix is much less than those obtained from $\mathbf{S}$, $\mathbf{T}$, $\mathbf{S}+15\mathbf{T}$, and Candan's matrices. We also plot $\log_{10}(\mathrm{RMSE})$ for $N=50$ in Fig. 6 and plot $\log_{10}(\mathrm{RMSE})$ for $N=20$ and 40 in Fig. 7.

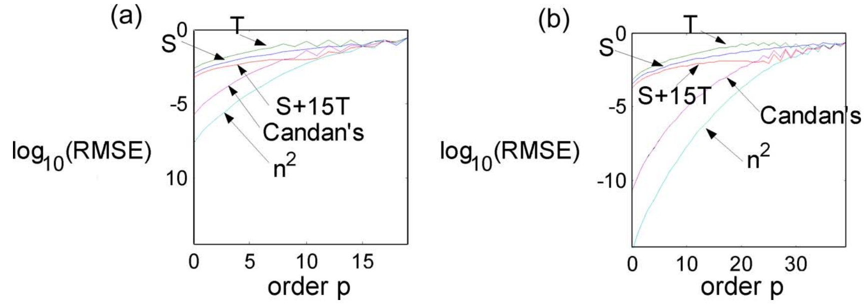

Fig. 7. Comparison of $\log_{10}(\mathrm{RMSEs})$ when (a) $N = 20$ and (b) $N = 40$.

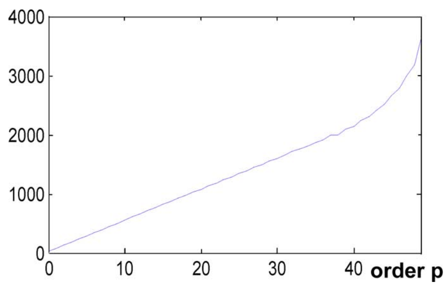

Fig. 8. Relation between the order p and the eigenvalue of the n^2 matrix when $N = 50$.

Note that the approximation error of the DFT eigenvector derived from the n^2 matrix is especially small when the order p is very small. In fact, from a series of experiments, we find that

$$\text{when } p < 0.72N - 8 \quad \text{RMSE} < 10^{-4}. \tag{101}$$

When the order p is near to N, the approximation error of the DFT eigenvector obtained from the n^2 matrix may be as large as those obtained from other methods. There are two ways to illustrate the phenomenon. First, the n^2 matrix can be viewed as the "digital implementation" of (4). To make the digital implementation accurate, the coefficients should not vary fast. However, the term n^2 varies fast when n is larger, and most of the energy of the eigenvector concentrate on the region where n is larger when the order p is higher. Therefore, when p is lager, the n^2 matrix may not be a good way to digitally implement the differential equation in (4).

Moreover, from $\tau = -p-1$ in (4), the eigenvalues of the continuous Hermite-Gaussian functions are linear with p. However, for the n^2 matrix, the eigenvalues is linear with p only when p is not near to N. For example, when $N = 50$, the relation between p and the eigenvalue of the n^2 matrix is plotted in Fig. 8. Therefore, the n^2 matrix cannot well approximate (4) when p is larger.

C. Linear Combination Form

Since the $\mathbf{S}$ matrix, the $\mathbf{T}$ matrix, the $\mathbf{S_{0,0,k}}$ matrix, the $\mathbf{T_{0,0,k}}$ matrix, (i.e., multiple diagonal $\mathbf{S}$ and $\mathbf{T}$ matrices), Candan's matrix, and the n^2 matrix all commute with the DFT, from Theorem 5, their linear combination also commutes with the DFT

$$\begin{aligned} \mathbf{X} &= \eta_0\mathbf{I} + \eta_1\mathbf{A} + \eta_2\mathbf{C} + \sum_k \tau_k\mathbf{S_{0,0,k}} + \sum_k \sigma_k\mathbf{T_{0,0,k}} \\ \mathbf{XF} &= \mathbf{FX} \end{aligned} \tag{102}$$

TABLE II
AVERAGE RMSES OF THE HIGHER-ORDER DFT EIGENVECTORS DERIVED FROM DIFFERENT METHODS

	S matrix	**T** matrix	Candan's	n^2 matrix	**X** matrix
$N=50$	0.156541	0.143401	0.093102	0.082696	0.080501

where $\mathbf{A}$ is the n^2 matrix, $\mathbf{C}$ is Candan's matrix, and η_0, η_1, η_2, τ_k, and σ_k are some constants.

For example, when $N = 50$, we can choose

$$\begin{aligned} \mathbf{X} = \mathbf{A} + 0.05\mathbf{C} + 0.15\mathbf{S_{0,0,1}} + 1.35\mathbf{T_{0,0,1}} \\ -1.5\mathbf{T_{0,0,2}} + 10\mathbf{T_{0,0,2}}. \end{aligned} \tag{103}$$

[The coefficients in (103) are obtained by the sequential optimization from iterative computer simulations.] Then, the DFT eigenvectors derived from it have lower RMSEs when the order p is higher. In Table II, the average RMSEs of the higher-order DFT eigenvectors is measured by

$$\begin{aligned} &\left[\sum_{p=N_1}^{N-2}\left(\sum_{n=0}^{N-1}\left|h_p(t\sqrt{2\pi/N}) - e_p[n]\right|^2\right)^{0.5}\right. \\ &\left.\quad + \sum_{n=0}^{N-1}\left|h_q(t\sqrt{2\pi/N}) - e_q[n]\right|^2\Bigg)^{0.5}\right] / (N - N_1) \end{aligned} \tag{104}$$

where $N_1 = \mathrm{round}(2N/3)$ and $q = N - \mathrm{mod}(N, 2)$. The RMSE in (104) is similar to that in Wikipedia. The difference is that we consider only the higher order case. The second term in (104) comes from that the highest order DFT eigenvector is similar to the $N - \mathrm{mod}(N, 2)^{\mathrm{th}}$ order HGF.

D. Discrete Fractional Fourier Transform

As in [3]–[6] and [10], one can use the eigenvectors of the DFT (denoted by $\mathbf{e_k}$) to define the **discrete fractional Fourier transform (DFRFT)**

$$\text{DFRFT}: \quad \mathbf{F}^a = \sum_{k=0}^{N-2} e^{-j\pi ka/2}\mathbf{e}_k\mathbf{e}_k^T + e^{-j\pi Ma/2}\mathbf{e}_M\mathbf{e}_M^T \tag{105}$$

where $M = N-1$ when N is odd and $M = N$ when N is even. If we define the DFRFT based on eigenvectors generated from the n^2 matrix, this will work in a similar way to the continuous FRFT.

VI. Extension to the Continuous Case

A. General Commuting Operations for Continuous FTs

We can also extend the results in Section III into the case of the continuous FT. The differential equation in (4) is known to be commutative with the continuous FT (denoted by $\Im$) [3]

$$\Im\left[\frac{d^2}{dt^2}h_k(t)-4\pi^2t^2h_k(t)\right]=\left[\frac{d^2}{dt^2}h_k(t)-4\pi^2t^2h_k(t)\right]\Im. \tag{106}$$

Therefore, the eigenfunctions of (4), which are Hermite-Gaussian functions, are also the eigenfunctions of the FT. Now, as we found that for the DFT there are many commuting matrices other than the $\mathbf{S}$ matrix in (3) and the $\mathbf{T}$ matrix in (5), we also ask whether there are other operators commuting with the continuous FT. In fact, (see Theorem 15).

Theorem 15: For any continuous operation O_M, if

$$O_A=O_M+\Im O_M\Im^{-1}+\Im^2O_M\Im^{-2}+\Im^3O_M\Im^{-3} \tag{107}$$

then O_A will be the operator commuting with the continuous FT. If all the eigenvalues of O_A are distinct, the eigenfunctions of O_A are also the eigenfunctions of the continuous FT. Using Theorem 15, we can derive a variety of operators that commute with the continuous FT.

B. General Commuting Differential Equations for Periodic or Quasi-Periodic Continuous Linear Operations

The general theorem for commuting matrices in Sections II-A and II-B can also be extended to the continuous case. For a continuous linear operation O_B

$$O_B\left[g(t)\right]=\int K(s,t)g(t)dt. \tag{108}$$

We define O_B^m as performing the operation $O_B m$ times

$$O_B^m\left[g(t)\right]=\int K(s,t_m)\cdots\cdots\int K(t_2,t_1)\times\int K(t_1,t)g(t)dtdt_1\cdots\cdots dt_m. \tag{109}$$

Theorem 16: If the continuous operation O_B satisfies

$$O_{B_1}^p\left[g(t)\right]=C\cdot g(t),\ \text{where}\ O_{B_1}\left[g(t)\right]=O_B\left[g(t)\right]+\sigma g(t) \tag{110}$$

σ and C are some constants, i.e., O_B is a continuous **offset quasi-periodic** operation, then the eigenfunctions of O_{B1} are also the eigenfunctions of O_B. In this case, we can first use the following way generate the commuting operation O_A:

$$\begin{aligned}O_A=&O_M+O_{B_1}O_MO_{B_1}^{-1}+O_{B_1}^2O_MO_{B_1}^{-2}+\cdots\cdots\\&+O_{B_1}^{p-1}O_MO_{B_1}^{-p+1}\end{aligned} \tag{111}$$

$$O_AO_B=O_BO_A \tag{112}$$

where O_M is any continuous operation. If all the eigenvalues of O_A are distinct, then the eigenfunctions of O_A are also the eigenfunctions of O_{B1} and, hence, O_B.

Proof:

$$\begin{aligned}O_AO_B=&O_MO_B+O_BO_M+O_B^2O_MO_B^{-1}+O_B^3O_MO_B^{-2}\\&+\cdots\cdots+O_B^{p-1}O_MO_B^{-p+2}\\=&O_BO_M+O_B^2O_MO_B^{-1}+O_B^3O_MO_B^{-2}+\cdots\cdots\\&+O_B^{p-1}O_MO_B^{-p+2}+O_MO_B\\=&O_B\left(O_M+O_BO_MO_B^{-1}+O_B^2O_MO_B^{-2}\right.\\&\left.+O_B^3O_MO_B^{-3}+\cdots\cdots+O_B^{p-1}O_MO_B^{-p+1}\right)\\=&O_BO_A.\end{aligned} \tag{113}$$

#

Note that Theorem 16 is analogous to Theorem 3. Specially, in (110), when $\sigma=0$ the continuous operation O_B is said to be **quasi-periodic**. When $\sigma=0$ and $C=1$, O_B is said to be **periodic**.

$$\begin{aligned}&\text{periodic}: \quad O_B^p\left[g(t)\right]=g(t)\\&\text{quasi-periodic}: \quad O_B^p\left[g(t)\right]=C\cdot g(t).\end{aligned} \tag{114}$$

In these cases Theorem 16 can also be applied.

Moreover, as the discrete case, we can choose O_M is a symmetric linear operation

$$O_M\left[g(t)\right]=\int K_M(s,t)g(t)dt\ \text{and}\ K_M(s,t)=K_M^*(t,s) \tag{115}$$

then O_A is also symmetric and we can use O_A to derive the orthogonal eigenfunction set of O_B.

Many continuous operations are periodic, quasi-periodic [satisfy (114)], or offset quasi-periodic [satisfy (110)]. For example, the continuous cosine transform, the continuous sine transform, and the continuous Hartley transform have periods of 2. The rotation operation by the angle of $2\pi Q/P$ has a period of P. The Hilbert transform has a period of 2. The reflection operation has a period of 2. Moreover, sometimes the twisting operation in geometry and some operations in the optics system are also periodic. For these operations, we can use (111) to find their corresponding commuting operations O_A and use O_A to find the eigenfunctions of these operations.

For example, as the discrete case, the projection operation in the continuous case (denoted by O_p) is offset quasi-periodic. Since O_p has only two eigenvalues, 0 and 1

$$O_{B_1}^2\left[g(t)\right]=g(t)/4\quad\text{where}\ O_{B_1}\left[g(t)\right]=O_P\left[g(t)\right]-g(t)/2. \tag{116}$$

From Theorem 16 and (111), the following operation will commute with O_{B1}:

$$O_A=O_M+O_{B1}O_MO_{B1}^{-1}. \tag{117}$$

If all the eigenvalues of O_A are distinct, the eigenfunctions of O_A are also the eigenfunctions of O_{B1} and hence the eigenfunctions of the projection operation O_p.

Moreover, with some modifications, Theorems 4 to 14, Corollaries 1 to 4, and Table I in Section II can all be applied in the continuous case.

VII. CONCLUSION

Candan previously found the general method to derive the commuting matrix of the DFT [11], and in this paper we have shown that, for any periodic ($\mathbf{B}^p = \mathbf{I}$), quasi-periodic ($\mathbf{B}^p = C \cdot \mathbf{I}$), and offset quasi-periodic transforms ($(\mathbf{B} + \sigma\mathbf{I})^p = C \cdot \mathbf{I}$), we can use Theorems 1, 2, and 3 to find the commuting matrix from an arbitrary matrix $\mathbf{M}$ and hence derive the complete eigenvector set of these operations. The proper ways to choose $\mathbf{M}$ for generating the commuting matrices are summarized in Table I. We have shown the examples that use the proposed theorems to derive the eigenvectors of the offset DFT (quasi-periodic, $p = 4$), the DFT (periodic, $p = 4$), the DCT, the DST, the DHT (periodic, $p = 2$), the Walsh (Hadamard) transform (quasi-periodic, $p = 2$), the projection operation (offset quasi-periodic, $p = 2$), and the matrix with H eigenspaces (can be decomposed into $H - 1$ offset quasi-periodic matrices with $p = 2$.) For example, for the offset DFT, we found that the multiple-diagonal $\mathbf{S}$ matrix, the multiple-diagonal $\mathbf{T}$ matrix, the offset-n^2 matrix, and their combinations can commute with the offset DFT, and the eigenvectors derived from these matrices are orthogonal and very similar to the samples of the shifted and modulated version of HGFs. Moreover, we also extended our results into the continuous case and found general ways of generating the commuting operators for any periodic, quasi-periodic, and offset quasi-periodic continuous operation.

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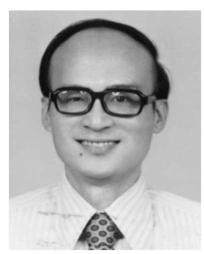

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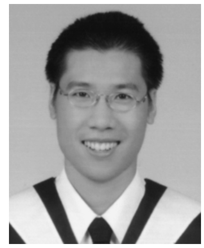

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