Eigenvalues and Singular Value Decompositions of Reduced Biquaternion Matrices

Soo-Chang Pei, *Fellow, IEEE*, Ja-Han Chang, Jian-Jiun Ding, and Ming-Yang Chen

***Abstract*—In this paper, the algorithms for calculating the eigenvalues, the eigenvectors, and the singular value decompositions (SVD) of a reduced biquaternion (RB) matrix are developed. We use the SVD to approximate an RB matrix in the least square sense and define the pseudoinverse matrix of an RB matrix. Moreover, the RB SVD is employed to implement the SVD of a color image. The computational complexity for the SVD of an RB matrix is only one-fourth of that for the SVD of conventional quaternion matrices. Therefore, many useful image-processing methods using the SVD can be extended to a color image without separating the color image into three channels. The numbers of the eigenvalues of an $n \times n$ RB matrix, the n^{th} roots of an RB, and the zeros of an RB polynomial with degree n are all finite and equal to n^2, not infinite as those of conventional quaternions.**

***Index Terms*—Quaternion, reduced biquaternion (RB), singular value decomposition (SVD) and eigenvalue of reduced biquaternion (RB) matrix.**

I. Introduction

THE well-known concept of the quaternion was first introduced by Hamilton in 1843 [1]. The quaternion is a generalization of the complex number. It has four components, i.e.

$$q = q_r + q_i i + q_j j + q_k k \tag{1}$$

where q_r, q_i, q_j, and q_k are real and i, j, and k satisfy

$$\begin{aligned} i^2 &= j^2 = k^2 = -1, \quad ij = -ji = k \\ jk &= -kj = i, \quad ki = -ik = j. \end{aligned} \tag{2}$$

From (2), the multiplication of quaternions is not commutative. Owning to this, many operations, such as Fourier transforms [47] and convolutions, are different from those of the complex algebra [25] and the eigenvalues of a quaternion matrix boil down to two categories, left and right eigenvalues [5]

$$\begin{aligned} Q_{(q)}\vec{x}_{(q)} &= \lambda_l \cdot \vec{x}_{(q)} \quad \text{(Left)} \\ Q_{(q)}\vec{z}_{(q)} &= \vec{z}_{(q)} \cdot \lambda_r \quad \text{(Right)}. \end{aligned} \tag{3}$$

In (3), λ_l and λ_r can be quaternion numbers and λ_l may not be equal to λ_r. Moreover, the eigenvalues of a quaternion matrix are infinite. If λ is an eigenvalue of a quaternion matrix $Q_{(q)}$, then every element of the set $\Gamma \equiv \{q\lambda q^{-1} : q \text{ is any unit quaternion with } |q| = 1\}$ is also an eigenvalue of $Q_{(q)}$ [5].

On the other hand, the concept of reduced biquaternions (RBs) was first introduced by Schütte and Wenzel [2]. The major difference between RBs and quaternions is the multiplication rules, which are commutative for RBs. Thus, many operations of RBs are simpler than those of quaternions. Moreover, both the quaternion and RB matrices can be employed to represent color images. The SVD of a conventional quaternion matrix was proposed in [36], [37]. In this paper, we propose the SVD of an RB matrix. Each of these two SVDs can be utilized to decompose color images. Compared to that of the quaternion matrix SVD, the complexity of the RB matrix SVD is reduced to a smaller factor of one-fourth.

In [3], we discussed the definitions and properties of RBs and developed their fast Fourier transform for signal and image processing. A brief review is given as follows.

Definition of RBs:

$$q = q_r + q_i i + q_j j + q_k k,$$

where

$$\begin{aligned} ij &= ji = k, \quad jk = kj = i \\ ik &= ki = -j, \; i^2 = k^2 = -1 \\ j^2 &= 1, \text{ and } q_r, q_i, q_j, \text{ and } q_k \text{ are real.} \end{aligned} \tag{4}$$

This setting produces two special numbers, e_1 and e_2, where

$$\begin{aligned} e_1 &= (1+j)/2, \quad e_2 = (1-j)/2, \text{ and } e_1 e_2 = 0 \\ e_1^n &= e_1^{n-1} = \ldots\ldots = e_1^2 = e_1 \\ e_2^n &= e_2^{n-1} = \ldots\ldots = e_2^2 = e_2. \end{aligned} \tag{5}$$

Therefore, e_1 and e_2 are both idempotent elements ($e_1^2 = e_1$, $e_2^2 = e_2$) and divisors of zero. Any RB with the form $c_1 e_1$ or $c_2 e_2$ is also a divisor of zero and does not have a multiplicative inverse (where c_1 and c_2 are any complex numbers). Thus, for RBs, there is no solution of the variable x in the following equation:

$$ux = 1, \qquad \text{if } u = c_1 e_1 \text{ or } c_2 e_2 \tag{6}$$

and there are infinite solutions to the following equation:

$$ux = 0, \qquad \text{if } u = c_1 e_1 \text{ or } c_2 e_2. \tag{7}$$

Hence, the RB system is not a complete division system.

Three Useful Representations of RBs: We introduced three useful representations of RBs in [3]. These three representations are (a) $e_1 - e_2$ forms, (b) matrix representations, and (c) polar

Manuscript received February 08, 2006; revised July 25, 2007. First published March 07, 2008; current version published October 29, 2008. This work was supported by the National Science Council, R. O. C., under Grants NSC 91-2219-E-002-044 and NSC 93-2752-E-002-006-PAE. This paper was recommended by Associate Editor T. Chen.

The authors are with the Department of Electrical Engineering, National Taiwan University, Taipei, Taiwan (e-mail: pei@cc.ee.ntu.edu.tw; jahan_chang@mtk.com.tw; djj1@ms63.hinet.net; roc3.chen@msa.hinet.net).

Digital Object Identifier 10.1109/TCSI.2008.920068

forms. Each representation has its advantages. The complexity of many operations can be reduced when applying $e_1 - e_2$ forms. The matrix representations are helpful in defining the norm and conjugation. We can understand the geometric meaning of RBs from polar forms. We give a brief review of these representations as follows.

a) $e_1 - e_2$ forms: A RB number q is often represented in the following form:

$$\begin{aligned} q &= (q_{\mathrm{r}} + iq_{\mathrm{i}}) + j(q_{\mathrm{j}} + iq_{\mathrm{k}}) = q_a + jq_b \\ &= q_{(c),1}e_1 + q_{(c),2}e_2 \end{aligned} \tag{8}$$

where $q_a = q_r + iq_{\mathrm{i}}$, $q_b = q_j + iq_k$, $q_{(c),1} = q_a + q_b$, and $q_{(c),2} = q_a - q_b$, are all complex numbers.

b) Matrix representations: The matrix representation of 1, i, j, and k are

$$\begin{aligned} 1 \to I_4 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ i \to M_i &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ j \to M_j &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ k \to M_k &= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{9}$$

where $M_j^2 = I_4$, $M_i^2 = M_k^2 = -I_4$, $M_iM_j = M_jM_i = M_k$, $M_jM_k = M_kM_j = M_i$, and $M_iM_k = M_kM_i = -M_j$. Therefore, the matrix representation of an RB $q = q_r + iq_i + jq_j + kq_k$ is

$$q = q_r + q_i i + q_j j + q_k k \to M_q \equiv \begin{bmatrix} q_r & -q_i & q_j & -q_k \\ q_i & q_r & q_k & q_j \\ q_j & -q_k & q_r & -q_i \\ q_k & q_j & q_i & q_r \end{bmatrix}. \tag{10}$$

The four eigenvalues of M_q are

$$\begin{aligned} \varepsilon &= (q_r+q_j)+i(q_i+q_k), \quad \varepsilon^* = (q_r+q_j)-i(q_i+q_k) \\ \eta &= (q_r-q_j)+i(q_i-q_k), \quad \eta^* = (q_r-q_j)-i(q_i-q_k). \end{aligned} \tag{11}$$

Moreover, the determinant of M_q is the product of the above four eigenvalues.

$$\begin{aligned} \delta &\equiv \begin{vmatrix} q_r & -q_i & q_j & -q_k \\ q_i & q_r & q_k & q_j \\ q_j & -q_k & q_r & -q_i \\ q_k & q_j & q_i & q_r \end{vmatrix} \\ &= \left((q_r+q_j)^2 + (q_i+q_k)^2\right)\left((q_r-q_j)^2 + (q_i-q_k)^2\right) \\ &= \left(q_r^2 + q_i^2 + q_j^2 + q_k^2\right)^2 - 4(q_rq_j + q_iq_k)^2 \geq 0. \end{aligned} \tag{12}$$

c) Polar forms: An RB can be uniquely represented by a polar form if $\delta \neq 0$

$$q = a + ib + jc + kd = Ae^{i\theta_i}e^{k\theta_k}e^{j\theta_j} \tag{13}$$

where δ is defined as in (12), and

$$\begin{aligned} &A = |q| = \sqrt[4]{\delta} \geq 0, \quad \theta_i = \tan^{-1}(b_1/a_1) \in (-\pi, \pi] \\ &\theta_k = \tan^{-1}(d_1/a_1) \in (-\pi/2, \pi/2], \quad \theta_j = \tanh^{-1}(f/e) \\ &e = \sqrt{\frac{1 + (a^2+b^2+c^2+d^2)/\sqrt{\delta}}{2}}, \quad f = \frac{(ac+bd)}{e\sqrt{\delta}} \\ &a_1 = ae - cf, \quad b_1 = be - df, \quad c_1 = ce - af, \quad d_1 = de - bf. \end{aligned}$$

The proof of (13) can be found in [3].

An interesting thing is

$$e^{j\theta_2} = \cosh\theta_2 + j\sinh\theta_2, \quad (j^2 = 1) \tag{14}$$

where $\cosh(\ )$ and $\sinh(\ )$ are the hyperbolic cosine and sine functions, respectively.

RB Matrix: Similar to the RB number, the RB matrix has four components [45] and it is often represented by the linear composition of two complex matrices using $e_1 - e_2$ forms

$$Q_{(\mathrm{RB})} = A_{(c)} + jB_{(c)} = Q_{(c),1}e_1 + Q_{(c),2}e_2. \tag{15}$$

where

$$\begin{aligned} &A_{(c)} = Q_{(q),r} + iQ_{(q),i}, \quad B_{(c)} = Q_{(q),j} + iQ_{(q),k}. \\ &Q_{(c),1} = A_{(c)} + jB_{(c)} \\ &\qquad = Q_{(RB),r} + Q_{(RB),j} + i\left(Q_{(RB),i} + Q_{(RB),k}\right) \\ &Q_{(c),2} = A_{(c)} - jB_{(c)} \\ &\qquad = Q_{(RB),r} - Q_{(RB),j} + i\left(Q_{(RB),i} - Q_{(RB),k}\right). \end{aligned} \tag{16}$$

and $Q_{(RB),r}$, $Q_{(RB),i}$, $Q_{(RB),j}$, and $Q_{(RB),k}$ are the real, i-, j-, and k-parts of an RB matrix, respectively. By this $e_1 - e_2$ form representation, the addition and multiplication of RB matrices can be easily calculated by two additions and two multiplications of complex matrices. Moreover, the transpose $^{\mathrm{T}}$, conjugation and Hermitian transpose ^{H} of an RB matrix can be defined as

$$Q^{\square}_{(\mathrm{RB})} = Q^{\square}_{(c),1}e_1 + Q^{\square}_{(c),2}e_2 \tag{17}$$

where $^{\square}$ can be $^{\mathrm{T}}$, $^{-}$, or $^{\mathrm{H}}$. The conjugation used here is different from the definition in [3], but this conjugation is more suitable for calculating the SVD of an RB matrix. Some algebraic operations of the RB matrices are listed as follows.

(a) $(Q_{1(\mathrm{RB})}Q_{2(\mathrm{RB})})^H = Q^H_{2(\mathrm{RB})}Q_{1(\mathrm{RB})}P^H$.

(b) $\overline{(Q_{1(\mathrm{RB})}PQ_{2(\mathrm{RB})}P)^T} = \overline{Q_{2(\mathrm{RB})}}P^T\overline{Q_{1(\mathrm{RB})}}P^T$.

(c) $\overline{(Q_{1(\mathrm{RB})}PQ_{2(\mathrm{RB})}P)} = \overline{Q_{1(\mathrm{RB})}P}\ \overline{Q_{2(\mathrm{RB})}P}$.

(d) $(Q_{1(\mathrm{RB})}PQ_{2(\mathrm{RB})}P)^{-1} = Q_{2(\mathrm{RB})}P^{-1}Q_{1(\mathrm{RB})}P^{-1}$ if $Q_{1(\mathrm{RB})}P$ and $Q_{2(\mathrm{RB})}P$ are invertible.

(e) $(Q_{1(\mathrm{RB})}P^H)^{-1} = (Q_{1(\mathrm{RB})}P^{-1})^H$.

(f) $(Q_{1(\mathrm{RB})}P^T)^{-1} = (Q_{1(\mathrm{RB})}P^{-1})^T$.

(g) $\overline{(Q_{1(\mathrm{RB})}P)}^{-1} = \overline{(Q_{1(\mathrm{RB})}P^{-1})}$.

These properties can be easily proved by (17) and the corresponding properties of the complex matrices. In general, (b), (c), (f), and (g) are not satisfied by quaternion matrices [5].

In Section II, we introduce the ways to derive the eigenvalues, eigenvectors, SVD, and inverse of an RB matrix. One application of the RB matrix eigenvalues for calculating the zeros of an RB polynomial is discussed in Section III. Three applications of the RB SVD for the least square error problem, the pseudoinverse of an RB matrix, and color image processing are given in Section IV.

Notation and Definition List: Throughout this paper, we use the following notations.

- $\boldsymbol{q_r}$, $\boldsymbol{q_i}$, $\boldsymbol{q_j}$, $\boldsymbol{q_k}$: the real part, i-part, j-part, and k-part of a quaternion or an RB number $q = q_r + iq_i + jq_j + kq_k$.
- $\boldsymbol{Q_{(c)}}, \boldsymbol{Q_{(q)}}, \boldsymbol{Q_{(RB)}}$: complex, quaternion, and RB matrices.
- $\boldsymbol{Q_{(\bullet),r}}, \boldsymbol{Q_{(\bullet),i}}, \boldsymbol{Q_{(\bullet),j}}, \boldsymbol{Q_{(\bullet),k}}$: the real, i-, j-, and k-parts of a quaternion (or an RB) matrix. $Q_{(\bullet)} = Q_{(\bullet),r} + iQ_{(\bullet),i} + jQ_{(\bullet),j} + kQ_{(\bullet),k}$, where $_{(\bullet)}$ can be $_{(q)}$ or $_{(\mathrm{RB})}P$.
- $\vec{\boldsymbol{x}}_{(c)}, \vec{\boldsymbol{x}}_{(q)}, \vec{\boldsymbol{x}}_{(\mathrm{RB})}P$: complex, quaternion, and RB vectors.
- (**conjugation**): For both the quaternion and the RB, $\overline{q} = q_r - iq_i - jq_j - kq_k$.
- $^{\mathbf{H}}$ (**Hermitian**): conjugation + transpose of a matrix or vector.

$$Q_{(c)}^H = \left(\overline{Q_{(c)}}\right)^T, \quad Q_{(q)}^H = \left(\overline{Q_{(q)}}\right)^T, \quad Q_{(\mathrm{RB})}P^H = \left(\overline{Q_{(\mathrm{RB})}P}\right)^T$$

- || (**Norms**): In this paper, for both the quaternion and the RB

$$|q| = \left(q_r^2 + q_i^2 + q_j^2 + q_k^2\right)^{1/2}.$$

- \+ (**addition**): $p + q = (p_r + q_r) + i(p_i + q_i) + j(p_j + q_j) + k(p_k + q_k)$
- **Multiplication for quaternions**:

$$pq = (p_rq_r - p_iq_i - p_jq_j - p_kq_k) + i(p_rq_i + p_iq_r + p_jq_k - p_kq_j) + j(p_rq_j - p_iq_k + p_jq_r + p_kq_i) + k(p_rq_k + p_iq_j - p_jq_i + p_kq_r).$$

- **Multiplication for RBs**:

$$pq = (p_rq_r - p_iq_i + p_jq_j - p_kq_k) + i(p_rq_i + p_iq_r + p_jq_k + p_kq_j) + j(p_rq_j - p_iq_k + p_jq_r - p_kq_i) + k(p_rq_k + p_iq_j + p_jq_i + p_kq_r).$$

- **Eigenvectors and Eigenvalues**: For quaternions, $Q_{(q)}\vec{x}_{(q)} = \lambda_l \cdot \vec{x}_{(q)}$ (left form) and $Q_{(q)}\vec{z}_{(q)} = \vec{z}_{(q)}\lambda_r$ (right form). For RBs, $Q_{(\mathrm{RB})}P\vec{x}_{(\mathrm{RB})}P = \lambda\vec{x}_{(\mathrm{RB})}P = \vec{x}_{(\mathrm{RB})}P\lambda$.
- $\boldsymbol{e_1}, \boldsymbol{e_2}$: idempotent elements of RBs.

$$e_1 = (1+j)/2 \text{ and } e_2 = (1-j)/2.$$

- $\hat{\boldsymbol{Q}}_{\boldsymbol{(c)}}$: the equivalent complex matrix of $Q_{(q)}$.
- $\hat{\boldsymbol{Q}}_{\boldsymbol{(c)(RB)}}$: the equivalent complex matrix of an RB matrix $Q_{(\mathrm{RB})}P$.
- $\boldsymbol{A_{(c)}}$, $\boldsymbol{B_{(c)}}$: the "real + i parts" and the "i + k parts" of a quaternion (or an RB) matrix, respectively. $A_{(c)} = Q_{(\bullet),r} + iQ_{(\bullet),i}$ and $B_{(c)} = jQ_{(\bullet),j} + kQ_{(\bullet),k}$. where $_{(\bullet)}$ can be $_{(q)}$ or $_{(\mathrm{RB})}P$.
- $\boldsymbol{Q_{(c)}}_{,1}$, $\boldsymbol{Q_{(c)}}_{,2}$: For RBs, $Q_{(c),1} = A_{(c)} + jB_{(c)}$ and $Q_{(c),2} = A_{(c)} - jB_{(c)}$, i.e., $Q_{(\mathrm{RB})}P = Q_{(c),1}e_1 + Q_{(c),2}e_2$.

II. Eigenvalues, Eigenvectors, and SVD of an RB Matrix

A. Algebraic Structures of RB

In this subsection, the inherent group structure of RBs is investigated in a way for giving itself a character similar to the relationship between quaternions and the rotation group $SO(3)$. Note that for every RB $q = a + bi + cj + dk$, we have

$$\begin{aligned} q &= (a+bi) + (c+di)j \\ &= (a+bi)\left(\frac{1+j}{2} + \frac{1-j}{2}\right) + (c+di)\left(\frac{1+j}{2} + \frac{1-j}{2}\right) \\ &= (a+bi)(e_1+e_2) + (c+di)(e_1-e_2) \\ &= ((a+c)+(b+d)i)\,e_1 + ((a-c)+(b-d)i)\,e_2. \end{aligned}$$

Moreover, if there are two complex numbers c_1 and c_2 such that $c_1e_1 + c_2e_2 = 0$, then by $e_1e_2 = 0$, $e_1^2 = e_1$, and $e_2^2 = e_2$ multiplying the equation with e_1 gives $c_1e_1 = 0$ and multiplying it with e_2 gives $c_2e_2 = 0$. Thus, $c_1 = c_2 = 0$ and consequently e_1 and e_2 are linearly independent over complex numbers. In other words, the set of RBs can be denoted as the following direct sum:

$$Ce_1 \oplus Ce_2$$

where C stands for the complex number field. Let Q^{x} be the group consisting of all RBs with unique multiplicative inverse. In order to set a group representation of Q^{x}, we recall a simple representation of C as follows:

$$C \to M_2(R), \quad a + bi \mapsto \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

where $\mathbf{R}$ stands for the real number field. Therefore, we get an alternative representation of any RB element $q = a + bi + cj + dk \in Q^{\mathrm{x}}$ as follows:

$$\begin{aligned} &Q^X \to M_4(R) \\ &a+bi+cj+dk \mapsto \begin{bmatrix} a+c & b+d & 0 & 0 \\ -(b+d) & a+c & 0 & 0 \\ 0 & 0 & a-c & b-d \\ 0 & 0 & -(b-d) & a-c \end{bmatrix}. \end{aligned}$$

Let

$$\lambda_1 = \sqrt{(a+c)^2 + (b+d)^2}, \quad \lambda_2 = \sqrt{(a-c)^2 + (b-d)^2}$$
$$\theta_1 = \tan^{-1}\left(\frac{b+d}{a+c}\right), \text{ and } \theta_2 = \tan^{-1}\left(\frac{b-d}{a-c}\right)$$

the above formula can then be further simplified to

$$a+bi+cj+dk \mapsto \begin{bmatrix} \lambda_1\cos\theta_1 & \lambda_1\sin\theta_1 & 0 & 0 \\ -\lambda_1\sin\theta_1 & \lambda_1\cos\theta_1 & 0 & 0 \\ 0 & 0 & \lambda_2\cos\theta_2 & \lambda_2\sin\theta_2 \\ 0 & 0 & -\lambda_2\sin\theta_2 & \lambda_2\cos\theta_2 \end{bmatrix}.$$

One can easily see that this mapping is a **group homomorphism**, because an RB $q = a + bi + cj + dk$ has its unique multiplicative inverse $q^{-1} = x + yi + zj + wk$ if and only if the following linear system is nonsingular:

$$\begin{bmatrix} a & -b & c & -d \\ b & a & d & c \\ c & -d & a & -b \\ d & c & b & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{18}$$

i.e.,

$$\det\left(\begin{bmatrix} a & -b & c & -d \\ b & a & d & c \\ c & -d & a & -b \\ d & c & b & a \end{bmatrix}\right) = \left((a+c)^2 + (b+d)^2\right)\left((a-c)^2 + (b-d)^2\right) > 0$$

which guarantees that $\lambda_1, \lambda_2 > 0$, i.e., the matrix

$$\begin{bmatrix} \lambda_1\cos\theta_1 & \lambda_1\sin\theta_1 & 0 & 0 \\ -\lambda_1\sin\theta_1 & \lambda_1\cos\theta_1 & 0 & 0 \\ 0 & 0 & \lambda_2\cos\theta_2 & \lambda_2\sin\theta_2 \\ 0 & 0 & -\lambda_2\sin\theta_2 & \lambda_2\cos\theta_2 \end{bmatrix}$$

is of full rank. In summary, this representation tells us that the geometric meaning of Q^{x} is a decomposition of the four-dimensional Euclidean space into two independent planes, and each of their coordinates is transformed by a rotation plus a contraction or expansion $SO(2,2)$.

B. Eigenvalues and Eigenvectors of a Quaternion Matrix

The eigenvalues and eigenvectors of a quaternion matrix were discussed in [4], [5]. Let $Q_{(q)}$ be an $n \times n$ quaternion matrix and

$$Q_{(q)} = A_{(c)} + B_{(c)}j \tag{19}$$

where $A_{(c)} = Q_{(q),r} + iQ_{(q),i}$ and $B_{(c)} = Q_{(q),j} + iQ_{(q),k}$.

Note that $A_{(c)}$ and $B_{(c)}$ are two complex matrices. Then the eigenvalues and eigenvectors of $Q_{(q)}$ can be calculated by the eigenvalues and eigenvectors of the corresponding **equivalent complex matrix** [4], [5]

$$\hat{Q}_{(c)} = \begin{bmatrix} A_{(c)} & B_{(c)} \\ -\overline{B}_{(c)} & \overline{A}_{(c)} \end{bmatrix}_{2n\times 2n}. \tag{20}$$

If λ is an eigenvalue of $\hat{Q}_{(c)}$, then $\overline{\lambda}$ will be another one. Both of these two numbers are the eigenvalues of the quaternion matrix $Q_{(q)}$. There are $2n$ eigenvalues of the complex matrix $\hat{Q}_{(c)}$. Thus, we can get n complex eigenvalues with nonnegative imaginary part of $Q_{(q)}$ [4]. However, the quaternion eigenvalues of a quaternion matrix are infinite. If λ is an eigenvalue of a quaternion matrix, then $\overline{q}\lambda q$ is an eigenvalue, too (q is any quaternion that satisfies $|q| = 1$). This proof can be found in [5].

Furthermore, if $\vec{x}_{(c)} = [\vec{x}^T_{1(c)}, \vec{x}^T_{2(c)}]^{\mathrm{T}}$ is an eigenvector of the complex matrix for $\hat{Q}_{(c)}$, then $\vec{x}_{(q)} = [\vec{x}_{1(c)}] - [\vec{x}^*_{2(c)}] \cdot j$ is an eigenvector of the quaternion matrix $Q_{(q)}$ for λ, where the superscript * represents the complex conjugation, $\vec{x}_{1(c)}$ and $\vec{x}_{2(c)}$ are two $n \times 1$ complex column vectors, $\vec{x}_{(c)}$ is a $2n \times 1$ complex column vector, and $\vec{x}_{(q)}$ is an $n \times 1$ **quaternion** column vector.

C. Eigenvalues and Eigenvectors of a RB Matrix

For RBs, the multiplication is commutative. Thus, $\overline{q}\lambda q = \lambda\overline{q}q = \lambda$ for $|q| = 1(\overline{q}q = \overline{q}q = |q|^2 = 1)$. Consequently, the eigenvalues and eigenvectors of an RB matrix are finite. Here, we illustrate two different ideas to compute the eigenvalues and eigenvectors. The results of these two methods are identical.

Method 1: Using the Equivalent Complex Matrix of an RB Matrix: Similar to the quaternion matrix, an RB matrix $Q_{(\mathrm{RB})}P = A_{(c)} + jB_{(c)}$ has its equivalent complex matrix, too. If

$$\begin{aligned} Q_{(\mathrm{RB})}P &= A_{(c)} + B_{(c)}j \text{ where } A_{(c)} = Q_{(RB),r} + iQ_{(RB),i} \\ \text{and } B_{(c)} &= Q_{(RB),j} + iQ_{(RB),k} \end{aligned} \tag{21}$$

the equivalent complex matrix of the RB matrix $Q_{(\mathrm{RB})}P$ is

$$\hat{Q}_{(c)RB} = \begin{bmatrix} Q_{(c),1} & 0 \\ 0 & Q_{(c),2} \end{bmatrix}_{2n\times 2n} \tag{22}$$

where $Q_{(c),1}$ and $Q_{(c),2}$ are defined in (16). The relations between $Q_{(\mathrm{RB})}P$ and $Q_{(c),\mathrm{RB}}$ are shown as follows:

(a) If $Q_{(\mathrm{RB})}P = I_n$ where I_n is an $n \times n$ identity matrix, then $\hat{Q}_{(c),RB} = I_{2n}$.

(b) If $Q_{(\mathrm{RB})}P = Q_{1(\mathrm{RB})}P \square Q_{2(\mathrm{RB})}P$, then $Q_{(c),RB} = \hat{Q}_{1(c),RB} \square \hat{Q}_{2(c),RB}$, where $\square = +$ or $\times$.

(c) $(\hat{Q}_{(c),RB})^{-1}$ is the equivalent complex matrix of $(Q_{(\mathrm{RB})}P)^{-1}$.

(d) If $\lambda = \lambda_a + j\lambda_b$ is an eigenvalue of $Q_{(\mathrm{RB})}P$ and $\vec{x} = \vec{x}_a + j\vec{x}_b$ are the corresponding eigenvector, then

$$\lambda_{a+b} = \lambda_a + \lambda_b \quad \text{and} \quad \lambda_{a-b} = \lambda_a - \lambda_b$$

are the eigenvalue of the complex matrix $Q_{(c),1}$ and $Q_{(c),2}$, respectively, and $(\vec{x}_a + \vec{x}_b)$ and $(\vec{x}_a - \vec{x}_b)$ are the corresponding eigenvectors.

Moreover, the converse is also true. If λ_{a+b} and λ_{a-b} ($\vec{x}_{a+b}$ and $\vec{x}_{a-b}$) are the eigenvalues (eigenvectors) of the $Q_{(c),1}$ and $Q_{(c),2}$, respectively, then $[\lambda_{a+b} + \lambda_{a-b} + j(\lambda_{a+b} - \lambda_{a-b})]/2$ is the eigenvalue of the RB matrix $Q_{(\mathrm{RB})}P = A_{(c)} + jB_{(c)}$ and $\vec{x}_{a+b} + \vec{x}_{a-b} + j(\vec{x}_{a+b} - \vec{x}_{a-b})$ is the corresponding eigenvector.

The proof of (a)–(c) is very easy by (22). Here, we only give the proof of (d).

(Proof of (d)): Assume $Q_{(\mathrm{RB})}P = A_{(c)} + jB_{(c)}$ to be an $n \times n$ RB matrix, and $\lambda = \lambda_a + \lambda_b$ is an eigenvalue of $Q_{(\mathrm{RB})}P$ where λ_a and λ_b are two complex numbers and $\vec{x} = \vec{x}_a + j\vec{x}_b$ is the corresponding eigenvector of λ with $\vec{x}_a$ and $\vec{x}_b$ being two complex vectors. Then

$$\begin{aligned} Q_{(\mathrm{RB})}P\vec{x} &= \lambda\vec{x} \\ &\Rightarrow \left(A_{(c)}\vec{x}_a + B_{(c)}\vec{x}_b\right) + j\left(A_{(c)}\vec{x}_b + B_{(c)}\vec{x}_a\right) \\ &= (\lambda_a\vec{x}_a + \lambda_b\vec{x}_b) + j(\lambda_a\vec{x}_b + \lambda_b\vec{x}_a). \end{aligned} \tag{23}$$

We have

$$\begin{gathered} \begin{cases} A_{(c)}\vec{x}_a + B_{(c)}\vec{x}_b = \lambda_a\vec{x}_a + \lambda_b\vec{x}_b & \text{(a)} \\ A_{(c)}\vec{x}_b + B_{(c)}\vec{x}_a = \lambda_a\vec{x}_b + \lambda_b\vec{x}_a & \text{(b)} \end{cases} \\ \Rightarrow \begin{bmatrix} A_{(c)} & B_{(c)} \\ B_{(c)} & A_{(c)} \end{bmatrix} \begin{bmatrix} \vec{x}_a \\ \vec{x}_b \end{bmatrix} = \begin{bmatrix} \lambda_a & \lambda_b \\ \lambda_b & \lambda_a \end{bmatrix} \begin{bmatrix} \vec{x}_a \\ \vec{x}_b \end{bmatrix}. \end{gathered} \tag{24}$$

In addition

$$\begin{aligned}(a)+(b) \Rightarrow &\left(A_{(c)}+B_{(c)}\right)(\vec{x}_a+\vec{x}_b)\\ &=(\lambda_a+\lambda_b)(\vec{x}_a+\vec{x}_b)\\ (a)-(b) \Rightarrow &\left(A_{(c)}-B_{(c)}\right)(\vec{x}_a-\vec{x}_b)\\ &=(\lambda_a-\lambda_b)(\vec{x}_a-\vec{x}_b) \qquad (25)\\ &\Rightarrow \begin{bmatrix} Q_{(c),1} & 0\\ 0 & Q_{(c),2}\end{bmatrix}\begin{bmatrix}\vec{x}_a+\vec{x}_b\\ \vec{x}_a-\vec{x}_b\end{bmatrix}\\ &=\begin{bmatrix}\lambda_a+\lambda_b & 0\\ 0 & \lambda_a-\lambda_b\end{bmatrix}\begin{bmatrix}\vec{x}_a+\vec{x}_b\\ \vec{x}_a-\vec{x}_b\end{bmatrix}. \qquad (26)\end{aligned}$$

Therefore, $(\lambda_a+\lambda_b)$ and $(\lambda_a-\lambda_b)$ are the eigenvalues of the complex matrix $Q_{(c),1}$ and $Q_{(c),2}$, respectively, and $(\vec{x}_{a+b}+\vec{x}_{a-b})$ and $(\vec{x}_{a+b}-\vec{x}_{a-b})$ are the corresponding eigenvectors. We can calculate the eigenvalues and eigenvectors of an RB matrix by calculating the eigenvalues and eigenvectors of the two complex matrices $Q_{(c),1}$ and $Q_{(c),2}$.

Moreover, if

$$Q_{(c),1}\vec{x}_{a+b}=\lambda_{a+b}\vec{x}_{a+b},\quad Q_{(c),2}\vec{x}_{a-b}=\lambda_{a-b}\vec{x}_{a-b}$$

and $Q_{(\mathrm{RB})}P=A_{(c)}+jB_{(c)}$, then

$$\begin{aligned}&Q_{(\mathrm{RB})}P\left[\vec{x}_{a+b}+\vec{x}_{a-b}+j(\vec{x}_{a+b}-\vec{x}_{a-b})\right]\\ &=\left[Q_{(c),1}e_1+Q_{(c),2}e_2\right]\left[2\vec{x}_{a+b}e_1+2\vec{x}_{a-b}e_2\right]\\ &=2\vec{x}_{a+b}\lambda_{a+b}e_1+2\vec{x}_{a-b}\lambda_{a-b}e_2\\ &=(\lambda_{a+b}e_1+\lambda_{a-b}e_2)[2\vec{x}_{a+b}+2\vec{x}_{a-b}]\\ &=\frac{\lambda_{a+b}+\lambda_{a-b}+j(\lambda_{a+b}-\lambda_{a-b})}{2}\\ &\quad\times\left[\vec{x}_{a+b}+\vec{x}_{a-b}+j(\vec{x}_{a+b}-\vec{x}_{a-b})\right] \qquad (27)\end{aligned}$$

and the converse of (d) is also true. **Q.E.D.**

From (d), because there are n eigenvalues (eigenvectors) of a complex matrix, there are n^2 eigenvalues (eigenvectors) of an RB matrix.

Method 2: Using e_1-e_2 Forms of RBs: Alternatively, we can represent an RB matrix using the two idempotent elements e_1 and e_2 as in (8)

$$Q_{(\mathrm{RB})}P=A_{(c)}+jB_{(c)}=Q_{(c),1}e_1+Q_{(c),2}e_2. \qquad (28)$$

Therefore, if λ_{a+b} and λ_{a-b} ($\vec{x}_{a+b}$ and $\vec{x}_{a-b}$) are the eigenvalues (eigenvectors) of the $Q_{(c),1}$ and $Q_{(c),2}$, respectively, then $(\lambda_{a+b}e_1+\lambda_{a-b}e_2)$ will be the eigenvalue of the RBs matrix and $(\vec{x}_{a+b}e_1+\vec{x}_{a-b}e_2)$ will be the corresponding eigenvector. The proof is shown as follows.

$$\begin{aligned}&Q_{(\mathrm{RB})}P(\vec{x}_{a+b}\cdot e_1+\vec{x}_{a-b}\cdot e_2)\\ &=\left[Q_{(c),1}e_1+Q_{(c),2}e_2\right](\vec{x}_{a+b}\cdot e_1+\vec{x}_{a-b}\cdot e_2)\\ &=Q_{(c),1}\vec{x}_{a+b}\cdot e_1+Q_{(c),2}\vec{x}_{a-b}\cdot e_2\\ &=\lambda_{a+b}\vec{x}_{a+b}\cdot e_1+\lambda_{a-b}\vec{x}_{a-b}\cdot e_2\\ &=(\lambda_{a+b}e_1+\lambda_{a-b}e_2)(\vec{x}_{a+b}e_1+\vec{x}_{a-b}e_2). \qquad (29)\end{aligned}$$

The results obtained by method 1 and method 2 are identical. Again, using e_1-e_2 forms can simplify the analysis of RB matrices. A comparison between the eigenvalues and eigenvectors of a quaternion matrix and an RB matrix is shown in Table I.

TABLE I
COMPARISONS BETWEEN THE EIGENVALUES AND EIGENVECTORS OF A QUATERNION AND RB MATRIX

	$n\times n$ quaternion matrix	$n\times n$ RB matrix
Numbers of eigenvalues	Infinite	n^2
Calculation method (Equivalent complex matrix)	Calculate the eigenvalues and eigenvectors of a $2n\times 2n$ complex matrices $\hat{Q}_{(c)}=\begin{bmatrix}A_{(c)} & B_{(c)}\\ -\overline{B}_{(c)} & \overline{A}_{(c)}\end{bmatrix}_{2n\times 2n}$	Calculate the eigenvalues and eigenvectors of two $n\times n$ complex matrices $Q_{(c),1}$ and $Q_{(c),2}$ (defined in (16))

The complexity of computing the eigenvalues of an RB matrix is much lower than that of the conventional quaternion matrix.

D. SVD of a RB Matrix

The algorithm for calculating the SVD of a quaternion matrix using its equivalent complex matrix was developed in [36], [37]. We can obtain the SVD of an RB matrix using steps similar to those of quaternions and the equivalent complex matrix of an RB matrix. However, using the e_1-e_2 form representation can simplify the steps. Thus, we only discuss the method of e_1-e_2 form representation. Assume that the RB matrix $Q_{(\mathrm{RB})}P$ is

$$Q_{(\mathrm{RB})}P=A_{(c)}+jB_{(c)}\equiv Q_{(c),1}e_1+Q_{(c),2}e_2 \qquad (30)$$

where $Q_{(c),1}$ and $Q_{(c),2}$ are defined in (16). The SVDs of $Q_{(c),1}$ and $Q_{(c),2}$ are in fact the SVDs of two complex matrices. Suppose that

$$Q_{(c),1}=U_1[\Lambda_1][V_1]^H \text{ and } Q_{(c),2}=U_2[\Lambda_2][V_2]^H \qquad (31)$$

where

$$U_1[U_1]^H=V_1[V_1]^H=U_2[U_2]^H=V_2[V_2]^H=I_n \qquad (32)$$

$[\Lambda_1]$ and $[\Lambda_2]$ are two diagonal matrices with real elements $\sigma_{1,i}$ and $\sigma_{2,i}$, respectively, $(i=1,2,\ldots,n)$ and the superscript ^{H} means the Hermitian transpose. Then the SVD of an RB matrix is

$$\begin{aligned}Q_{(\mathrm{RB})}P=&\left(U_1[\Lambda_1][V_1]^H\right)e_1+\left(U_2[\Lambda_2][V_2]^H\right)e_2\\ =&(U_1e_1+U_2e_2)(\Lambda_1e_1+\Lambda_2e_2)\\ &\times\left([V_1]^He_1+[V_2]^He_2\right)\\ =&U_{(\mathrm{RB})}P\Lambda_{(\mathrm{RB})}P\left[V_{(\mathrm{RB})}P\right]^H\\ &(\because e_1e_2=0, e_1^2=e_1, e_2^2=e_2) \qquad (33)\end{aligned}$$

$$\begin{aligned}&\text{where } \Lambda_{(\mathrm{RB})}P=(\Lambda_1)e_1+(\Lambda_2)e_2,\ U_{(\mathrm{RB})}P=U_1e_1+U_2e_2\\ &\text{and } V_{(\mathrm{RB})}P=V_1e_1+V_2e_2. \qquad (34)\end{aligned}$$

By (32), $U_{(\mathrm{RB})}$ and $V_{(\mathrm{RB})}$ are unitary matrices, too.

$$\begin{aligned}U_{(\mathrm{RB})}PU_{(\mathrm{RB})}P^H&=U_{(\mathrm{RB})}^HU_{(\mathrm{RB})}=V_{(\mathrm{RB})}V_{(\mathrm{RB})}^H=V_{(\mathrm{RB})}^HV_{(\mathrm{RB})}\\ &=I_ne_1+I_ne_2=I_n. \qquad (35)\end{aligned}$$

Note that diagonal matrix $\Lambda_{(\mathrm{RB})}$ is not a real matrix unless $(\Lambda_{x(c)})=(\Lambda_{y(c)})$. Usually, it has real and j parts.

TABLE II
COMPARISONS BETWEEN THE SVDS OF A QUATERNION MATRIX AND AN RB MATRIX

	$n{\times}n$ quaternion matrix	$n{\times}n$ RB matrix
SVD	$Q_{(q)} = U_{(q)}\Lambda_{(q)}V_{(q)}^{H}$	$Q_{(RB)} = U_{(RB)}\Lambda_{(RB)}V_{(RB)}^{H}$
Unitary matrix	$U_{(q)}$ and $V_{(q)}$	$U_{(RB)}$ and $V_{(RB)}$
Diagonal matrix	$\Lambda_{(c)}$ is a real diagonal matrix.	$\Lambda_{(RB)}$ is a non-real diagonal matrix
Calculation method	Calculate the SVD of a $2n{\times}2n$ complex matrices $\hat{Q}_{(c)} = \begin{bmatrix} A_{(c)} & B_{(c)} \\ -\overline{B}_{(c)} & \overline{A}_{(c)} \end{bmatrix}_{2n\times 2n}$	Calculate the SVD of two $n{\times}n$ complex matrices $Q_{(c),1}$ and $Q_{(c),2}$
Complexity of SVD	$(2n)^3 = \mathbf{8}n^3$	$\mathbf{2}n^3$
No. of **real multiplications** for reconstructing the original color image	$\mathbf{8}n^3$	$\mathbf{6}n^3$

Therefore, using the two elements e_1 and e_2, we can calculate the SVD of an RB matrix by the SVD of two complex matrices without developing a new algorithm. The complexity of the SVD of an RB matrix is one-fourth of that of the SVD of a quaternion matrix. Moreover, the original matrix is usually reconstructed by the sum of the outer products

$$Q_{(x)} = U_{(x)}\left[\Lambda_{(x)}\right]\left[V_{(x)}\right]^{H} = \sum_{i=1}^{n}\lambda_{i(x)}u_{i(x)}v_{i(x)}^{H}$$

where $x = q$ or RB. (36)

The complexity of (36) using the RB matrix will be only three-fourth of that of using the quaternion matrix, because six (eight) real multiplications are necessary and sufficient to compute the product of two RBs (quaternions) [38]–[42]. Consequently, using RB matrices for the SVD of a color image is more efficient than using quaternion matrices.

The comparison between the SVDs of a quaternion matrix and an RB matrix is illustrated in Table II.

E. Inverse of a RB Matrix

For an RB matrix $Q_{(\mathrm{RB})} = A_{(c)} + jB_{(c)} \equiv Q_{(c),1}e_1 + Q_{(c),2}e_2$ the inverse of $Q_{(\mathrm{RB})}$ exists, if and only if the inverses of $Q_{(c),1}$ and $Q_{(c),2}$ exist. Moreover, the inverse of $Q_{(\mathrm{RB})}$ is $[Q_{(c),1}]^{-1}e_1 + [Q_{(c),2}]^{-1}e_2$. The deviations are as follows.

(a) $\rightarrow$: Suppose that the inverse of $Q_{(\mathrm{RB})}$ exists and is denoted as $[Q_{(\mathrm{RB})}]^{-1} = P_1e_1 + P_2e_2$. Then

$$\begin{aligned} Q_{(\mathrm{RB})}\left[Q_{(\mathrm{RB})}\right]^{-1} &= \left[Q_{(c),1}P_1\right]e_1 + \left[Q_{(c),2}P_2\right]e_2 = I_n \\ \Rightarrow Q_{(c),1}P_1 &= Q_{(c),2}P_2 = I_n \end{aligned} \quad (37)$$

Thus, the inverses of $Q_{(c),1}$ and $Q_{(c),2}$ exist and they are

$$\left[Q_{(c),1}\right]^{-1} = P_1 \quad \text{and} \quad \left[Q_{(c),2}\right]^{-1} = P_2.$$

(b) $\leftarrow$: Suppose that the inverses of $Q_{(c),1}$ and $Q_{(c),2}$ exist and are denoted as $[Q_{(c),1}]^{-1}$ and $[Q_{(c),2}]^{-1}$. Then

$$\begin{aligned} &Q_{(\mathrm{RB})}\left(\left[Q_{(c),1}\right]^{-1}e_1 + \left[Q_{(c),2}\right]^{-1}e_2\right) \\ &\quad = Q_{(c),1}\left[Q_{(c),1}\right]^{-1}e_1 + Q_{(c),2}\left[Q_{(c),2}\right]^{-1} = I_n. \end{aligned} \quad (38)$$

Therefore, the inverse of $Q_{(\mathrm{RB})}$ exists and it is $[Q_{(c),1}]^{-1}e_1 + [Q_{(c),2}]^{-1}e_2$.

III. APPLICATIONS OF EIGENVALUES OF AN RB MATRIX FOR FINDING ZEROS OF RB POLYNOMIAL

We can use the eigenvalues of an RB matrix to calculate the zeros of an RB polynomial. Before discussing the zeros of an RB polynomial, we first review the n^{th} roots of a conventional quaternion and then discuss the n^{th} roots of an RB.

A. The n^{th} Roots of a Conventional Quaternion: $x^n = q$

To find the n^{th} roots of a quaternion [8], [9], it is useful to represent quaternions in the polar form as follows.

$$q = a + bi + cj + dk \equiv |q|e^{\mu\theta} \quad (39)$$

where $|q| = (a^2 + b^2 + c^2 + d^2)^{1/2}$, $\cos\theta = a/|q|$, $\sin\theta = \pm(b^2 + c^2 + d^2)^{1/2}/|q|$, and if $b^2 + c^2 + d^2 \neq 0$, then

$$\mu = \pm\frac{bi + cj + dk}{(b^2 + c^2 + d^2)^{1/2}}$$

is a pure unit quaternion and $\mu^2 = -1$. The n^{th} roots of a quaternion $q = |q|e^{\mu\theta}$ are

$$q_k = |q|^{1/n}e^{\frac{\theta + 2\pi k}{n}\mu} \quad (k = 0, 1, 2, \ldots\ldots, n-1). \quad (40)$$

Therefore, for a nonreal general quaternion, the number of n^{th} roots is n. However, a positive real quaternion has just **two**

TABLE III
COMPARISON BETWEEN THE n^{th} ROOTS OF A QUATERNION AND AN RB

	A quaternion	A reduced biquaternion
Polar form	$q = \vert q \vert e^{\mu\theta}$	$q = Ae^{i\theta_1}e^{j\theta_2}e^{k\theta_3}$
Number of n^{th} roots	(a) n, for non-real quaternion (b) Infinite, for real quaternion except for $n = 2$	n^2, for any RBs

square roots but **infinite** n^{th} **roots** for $n > 2$. In addition, a negative real quaternion has **infinite** n^{th} **roots**, both for $n = 2$ and $n > 2$. The real quaternions have infinite roots because the choice of the number μ of a real quaternion can be arbitrary pure unit quaternions [8], [9].

B. The n^{th} Roots of a RB $x^n = q$:

For an RB number, the nth roots can be computed by two different methods. The first method is using polar forms of an RB and the other is using the $e_1 - e_2$ forms.

Method 1: By Means of the Polar Form of an RB: For an RB, its polar form is

$$q = a + bi + cj + dk \equiv Ae^{i\theta_1}e^{j\theta_2}e^{k\theta_3}.$$

The parameters A, θ_1, θ_2, and θ_3 can be calculated by (13). We can calculate the n^{th} roots of an RB using this polar form. Let $q = Ae^{i\theta_1}e^{j\theta_2}e^{k\theta_3}$, then the n^{th} roots of q are

$$A^{1/n}e^{i\cdot\frac{(\theta_1+2\pi\cdot l_1)}{n}}e^{j\cdot\frac{\theta_2}{n}}e^{k\cdot\frac{(\theta_3+2\pi\cdot l_2)}{n}}j^{l_3} \tag{41}$$

where $l_1, l_2 \in \{0, 1, \ldots, n-1\}$, and if n is odd, $l_3 = 0$, else $l_3 = 0$ or 1. Therefore, there are n^2 n^{th} roots of an RB if n is odd; and there are $2n^2$ n^{th} roots of an RB if n is even. However, there are n^2 duplicate roots when n is even due to the following property

$$e^{i\theta_1}e^{k\theta_3} = e^{i(\theta_1\pm\pi)}e^{k(\theta_3\pm\pi)} = e^{i\left(\theta_1\pm\frac{2\pi\frac{n}{2}}{n}\right)}e^{k\left(\theta_3\pm\frac{2\pi\frac{n}{2}}{n}\right)}. \tag{42}$$

Therefore, if n is even, (l_1, l_2), $(l_1+n/2, l_2+n/2)$, would give the same duplicate roots. If we set the range of the value l_2 as following equation,

$$l_2 \in \begin{cases} \{0, 1, \ldots\ldots, n/2-1\} & \text{if } n \text{ is even} \\ \{0, 1, \ldots\ldots, n/2-1, \ldots\ldots, n\} & \text{if } n \text{ is odd} \end{cases} \tag{43}$$

there are n^2 n^{th} roots of an RB for any value of n. However, for $n = 4$ m, each root still has two duplicate roots because

$$\begin{aligned} j &= -ik = -e^{i\pi/2}e^{k\pi/2} = -e^{i\left(\frac{2\pi n/4}{n}\right)}e^{k\left(\frac{2\pi n/4}{n}\right)} \\ &= e^{i\left(\frac{2\pi 3n/4}{n}\right)}e^{k\left(\frac{2\pi 3n/4}{n}\right)} = e^{i\left(\frac{2\pi n/4}{n}\right)}e^{k\left(\frac{2\pi 3n/4}{n}\right)}. \end{aligned} \tag{44}$$

Method 2: By Means of the $e_1 - e_2$ Form Representation of an RB: On the other hand, we can represent an RB using the two numbers e_1 and e_2 as in (8)

$$q = (q_r + iq_i) + j(q_j + iq_k) = q_{a+b}e_1 + q_{a-b}e_2. \tag{45}$$

By (5) and (45), the solutions of $q^n = Q_{(\mathrm{RB})}$, i.e., the nth roots of $Q_{(RB)}$, can be obtained from the following:

$$\begin{aligned} q^n &= q_{a+b}^n e_1 + q_{a-b}^n e_2 = Q_{(c),1}e_1 + Q_{(c),2}e_2 \\ &\Rightarrow \begin{cases} q_{a+b}^n = Q_{(c),1} \\ q_{a-b}^n = Q_{(c),2} \end{cases} \end{aligned} \tag{46}$$

where $Q_{(c),1}$ and $Q_{(c),2}$ are defined in (16). Note that $q_{a+b}^n = Q_{a+b}$ and $q_{a-b}^n = Q_{a-b}$ are problems of the n^{th} roots of a complex number. Thus, there are n n^{th} complex roots of each equation. Therefore, there are totally n^2 n^{th} roots of an RB. This result is the same as that obtained from the polar form of an RB.

The comparison between the nth roots of a quaternion and the nth roots of an RB is shown in Table III. The following two examples are given to demonstrate the correctness of our results.

Example 1: Calculating the **square roots** of any RB number q

$$q = q_r + iq_i + jq_j + kq_k \equiv Ae^{i\theta_1}e^{j\theta_2}e^{k\theta_3}.$$

In [2], the authors showed that the square roots of an RB q are

$$\frac{\sigma}{2}\left\{(1+j)\sqrt{a_{11}+ia_{12}} + (1-j)\sqrt{a_{11}-ia_{12}}\right\} \tag{47}$$

where $\sigma = \{1, -1, j, -j\}$, $a_{11} = q_r + iq_i$, and $ia_{12} = q_j + iq_k$. In fact, these solutions are the same as ours in (41) or (46). We can show that (47) and (41) are equivalent,

$$\begin{aligned} a_{11} &= q_r + iq_i \\ &= (Ae^{i\theta_1}e^{j\theta_2}e^{k\theta_3} + Ae^{i\theta_1}e^{-j\theta_2}e^{-k\theta_3})/2 \end{aligned} \tag{48}$$

$$\begin{aligned} ia_{12} &= q_j + iq_k \\ &= j(Ae^{i\theta_1}e^{j\theta_2}e^{k\theta_3} - Ae^{i\theta_1}e^{-j\theta_2}e^{-k\theta_3})/2 \end{aligned} \tag{49}$$

$$\begin{aligned} a_{11}+ia_{12} = \big(&(1+j)Ae^{i\theta_1}e^{j\theta_2}e^{k\theta_3} \\ &+(1-j)Ae^{i\theta_1}e^{-j\theta_2}e^{-k\theta_3}\big)/2, \end{aligned} \tag{50}$$

$$\begin{aligned} a_{11}-ia_{12} = \big(&(1-j)Ae^{i\theta_1}e^{j\theta_2}e^{k\theta_3} \\ &+(1+j)Ae^{i\theta_1}e^{-j\theta_2}e^{-k\theta_3}\big)/2. \end{aligned} \tag{51}$$

Assume that

$$\sqrt{Ae^{i\theta_1}e^{j\theta_2}e^{k\theta_3}} = A^{1/2}e^{i\theta_1/2}e^{j\theta_2/2}e^{k\theta_3/2} \tag{52}$$

and use the fact that $(1+j)(1-j)=0$, then

$$\begin{aligned}&(1+j)\sqrt{a_{11}+ia_{12}}\\&=\sqrt{(1+j)^3Ae^{i\theta_1}e^{j\theta_2}e^{k\theta_3}/2}\\&=(1+j)A^{1/2}e^{i\theta_1/2}e^{j\theta_2/2}e^{k\theta_3/2}\end{aligned}\tag{53}$$

$$\begin{aligned}&(1-j)\sqrt{a_{11}-ia_{12}}\\&=\sqrt{(1-j)^3Ae^{i\theta_1}e^{j\theta_2}e^{k\theta_3}/2}\\&=(1-j)A^{1/2}e^{i\theta_1/2}e^{j\theta_2/2}e^{k\theta_3/2}\end{aligned}\tag{54}$$

$$\begin{aligned}&\therefore\left[(1+j)\sqrt{a_{11}+ia_{12}}+(1-j)\sqrt{a_{11}-ia_{12}}\right]/2\\&=A^{1/2}e^{i\theta_1/2}e^{j\theta_2/2}e^{k\theta_3/2}.\end{aligned}\tag{55}$$

C. The Zeros of a RB Polynomial

Definition 1—A RB Polynomial: Given a series of RB coefficients $q_0, q_1 \ldots, q_{n-1}$, a monic RB polynomial of degree n is expressed as the function of the with RB variable x

$$p(x)=x^n+q_{m-1}x^{n-1}+\ldots\ldots+q_1x+q_0.\tag{56}$$

Definition 2—A Zero of an RB Polynomial: Given an RB polynomial $p(x)$, we say that λ is a zero if $p(\lambda)=0$. For quaternion polynomials, the fundamental theorem of algebra was first considered by Eilenberg and Niven [10], [11]. They prove that every quaternion polynomial has at least one zero. Niven's algorithm can be found in [11]. A simpler method modified from Niven's algorithm was developed in [12] for computing the zeros of a quaternion polynomial. A companion matrix associated with the polynomial is proposed herein for calculating the information about the trace and the norm of the zero. For a quaternion polynomial with degree n, the number of zeros may be n or infinite.

On the other hand, for RB polynomials, we will develop two methods for calculating the zeros. The first method is using the companion matrix which is similar to the biquaternion one. The second one is using the e_1-e_2 form representation as in (8) that divides an RB polynomial into two complex polynomials.

Method 1: By Means of Companion Matrices:

Definition 3—Companion Matrix: Given an RB polynomial as (56), the matrix

$$C_p=\begin{bmatrix}0&1&&0\\\vdots&&\ddots&\\0&0&&1\\-q_0&-q_1&\cdots&-q_{n-1}\end{bmatrix}_{n\times n}\tag{57}$$

is called the companion matrix associated with the RB polynomial $p(x)$.

Theorem 1: If λ is an eigenvalue of the companion matrix C_p, then (a) λ is a zero of $p(x)$, and (b) $v=[1,\lambda,\lambda^2,\ldots,\lambda^{n-1}]^T$ is an associated eigenvector.

Proof:

TABLE IV
COMPARISONS BETWEEN THE ZEROS OF A QUATERNION AND RB POLYNOMIALS

	A quaternion polynomial of degree *n*	A reduced biquaternion polynomial of degree *n*
Number of zeros	*n* or infinite	n^2
Calculation method	Companion matrix and Niven's algorithm	Companion matrix or the e_1-e_2 form

(a) Assume that $v=[v_1,v_2,\ldots,v_n]^T$ is the associated eigenvector of λ, then

$$\begin{bmatrix}0&1&&0\\\vdots&&\ddots&\\0&0&&1\\-q_0&-q_1&\cdots&-q_{n-1}\end{bmatrix}\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\lambda\cdot\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}.\tag{58}$$

Multiplying both sides yields the following equations

$$v_i=\lambda v_{i-1}\Rightarrow v_i=\lambda^{i-1}v_1\quad(i=2,3,\ldots,n)\tag{59}$$
$$-q_0v_1-q_1v_2-\cdots-q_{n-1}v_n=\lambda v_n.\tag{60}$$

Substituting (59) into (60), we obtain

$$\begin{aligned}-q_0\nu_1-q_1\lambda\nu_1-\ldots-q_{n-1}\lambda^{n-1}\nu_1=\lambda\lambda^{n-1}\nu_1\Leftrightarrow\\(\lambda^n+q_{n-1}\lambda^{n-1}+\ldots\ldots+q_1\lambda+q_0)\nu_1.\end{aligned}\tag{61}$$

Since $v_1\neq 0$, as v cannot be the zero vector, we conclude

$$\lambda^n+q_{n-1}\lambda^{n-1}+\ldots\ldots+q_1\lambda+q_0=0\tag{62}$$

and the eigenvalue λ is a zero of the polynomial $p(x)$.

(b) If we choose $v_1=1$, then by (59) we obtain $v=[1,\lambda,\lambda^2,\lambda,\lambda^{n-1}]^T$ and we conclude that v is an eigenvector associated with the eigenvalue λ. **Q.E.D.**

In Section II, we know that an $n\times n$ RB matrix has exactly n^2 eigenvalues. Consequently, there are exactly n^2 zeros of an RB polynomial with degree n.

Method 2: By Means of e_1-e_2 Forms: Given an RB polynomial $p(x)=x_n+q_{n-1}x^{n-1}+\ldots+q_1x+q_0$, we can divide this RB polynomial into two complex polynomials $p_1(x_+)$ and $p_2(x_-)$ as

$$p(x)=p_1(x_+)e_1+p_2(x_-)e_2\tag{63}$$
$$\text{where } p_1(x_+)=x_+^n+c_{n-1}x_+^{n-1}+\ldots\ldots+c_1x_++c_0$$
$$p_2(x_-)=x_-^n+d_{n-1}x_-^{n-1}+\ldots\ldots+d_1x_-+d_0$$
$$x=x_+e_1+x_-e_2\tag{64}$$
$$\text{and } q_i=c_ie_1+d_ie_2\quad(i=0,1,\ldots,n-1).\tag{65}$$

By the complex algebra, we know that both $p_1(x_+)$ and $p_2(x_-)$ have exactly n complex zeros. Therefore, there are n^2 zeros of an RB polynomial. This result is the same as that obtained by method 1. The comparison between the zeros of a quaternion polynomial and an RB polynomial is shown in

Table IV. The following example is employed to demonstrate the correctness of our methods.

Example 2: Then we try to calculate the zeros of the following two RB polynomials.

(a)

$$p_1(x)=x^3+(-3-j+k)x^2+(-2+5i-15j)x+(4-2i-5k) \quad (66)$$

(b)

$$p_2(x)=x^6+(i+j+k)x^5+(1+j-k)x^4+(1+i-j)x^3 +(i+j-k)x^2+(-i+j)x+(1-k). \quad (67)$$

(a) The companion matrix [see (57)] corresponding to (66) is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4+2i+5k & 2-5i+15j & 3+j-k \end{bmatrix} \equiv A_{(c)}+jB_{(c)}. \quad (68)$$

(b) The eigenvalues of the companion matrix can be calculated by the following steps.

1) First, we divide the companion matrix into two complex matrices $(A_{(c)}+B_{(c)})$ and $(A_{(c)}-B_{(c)})$ and calculate their eigenvalues. The eigenvalues of $(A_{(c)}+B_{(c)})$ are $6.5188-1.1713i$, $0.3194-0.2783i$, and $-2.8381+0.4496i$, and the eigenvalues of $(A_{(c)}-B_{(c)})$ are $1.7599-3.0575i$, $-0.3277-0.0863i$, and $0.5678+4.1438i$.

2) The nine eigenvalues of the companion matrix, that is the nine zeros of the RB polynomial, are shown in Table V.

(c) The companion matrix associated with $p_2(x)$ in (67) is shown in (69) at the bottom of the page.

We follow the steps as (a) to calculate the eigenvalues of the companion matrix. Here, we only show the six eigenvalues of $(A_{(c)}+B_{(c)})$ and $(A_{(c)}-B_{(c)})$ in Table VI. The 36 zeros of $p_2(x)$ in (67) can be computed from $(\lambda_{a+b}e_1+\lambda_{a-b}e_2)$.

TABLE V
ZEROS OF THE POLYNOMIAL OF EXAMPLE 2(A), $(\lambda_{a+b}e_1+\lambda_{a-b}e_2)$, WHERE λ_{a+b} AND λ_{a-b} ARE THE EIGENVALUES OF $(A_{(c)}+B_{(c)})$ AND $(A_{(c)}-B_{(c)})$, RESPECTIVELY

1	4.1393 – 2.1144*i* + 2.3794*j* + 0.9431*k*
2	3.0955 – 0.6288*i* + 3.4232*j* – 0.5425*k*
3	3.5433 + 1.4863*i* + 2.9755*j* – 2.6575*k*
4	1.0396 – 1.6679*i* – 0.7203*j* + 1.3896*k*
5	–0.0042 – 0.1823*i* + 0.3235*j* – 0.0960*k*
6	0.4436 + 1.9327*i* – 0.1242*j* – 2.2110*k*
7	–0.5391 – 1.3040*i* – 2.2990*j* + 1.7536*k*
8	–1.5829 + 0.1817*i* – 1.2552*j* + 0.2679*k*
9	–1.1352 + 2.2967*i* – 1.7030*j* – 1.8471*k*

TABLE VI
EIGENVALUES OF THE TWO COMPLEX MATRICES DERIVED FROM THE COMPANION MATRIX ASSOCIATED WITH THE POLYNOMIAL OF EXAMPLE 2(B)

	Eigenvalues of $(A_{(c)}+B_{(c)})$, λ_{a+b}	Eigenvalues of $(A_{(c)}-B_{(c)})$, λ_{a-b}
1	-1.1695 – 2.6861*i*	1.5415 – 0.9264*i*
2	-0.0229 + 1.1089*i*	0.5220 + 1.4092*i*
3	0.7835 + 0.4689*i*	-1.2538 + 0.4075*i*
4	0.4871 – 0.8832*i*	-0.6431 – 0.5513*i*
5	-0.6065 + 0.4230*i*	0.4258 – 0.7218*i*
6	-0.4715 – 0.4314*i*	0.4076 + 0.3828*i*

IV. APPLICATIONS OF THE SVD OF AN RB MATRIX

A. Pseudoinverse of an RB Matrix

We can use the SVD of an RB matrix $Q_{(\mathrm{RB})}$ to compute its pseudoinverse. Assume that $Q_{(\mathrm{RB})}=U_{(\mathrm{RB})}\Lambda_{(\mathrm{RB})}V_{(\mathrm{RB})}^T$, as in (33), and

$$\Lambda_{(\mathrm{RB})}=\begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}_{n\times n}. \quad (70)$$

Then the pseudoinverse matrix is

$$Q_{(\mathrm{RB})}^{+}=V_{(\mathrm{RB})}\left[\Lambda_{(\mathrm{RB})}\right]^{+}U_{(\mathrm{RB})}^T \quad (71)$$

where

$$\left[\Lambda_{(\mathrm{RB})}\right]^{+}=\begin{bmatrix} \sigma_1^{+} & & 0 \\ & \ddots & \\ 0 & & \sigma_n^{+} \end{bmatrix}_{n\times n} \quad (72)$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1+k & i-j & -i-j+k & -1-i+j & -1-j+k & -i-j-k \end{bmatrix} \equiv A_{(c)}+j\cdot B_{(c)}. \quad (69)$$

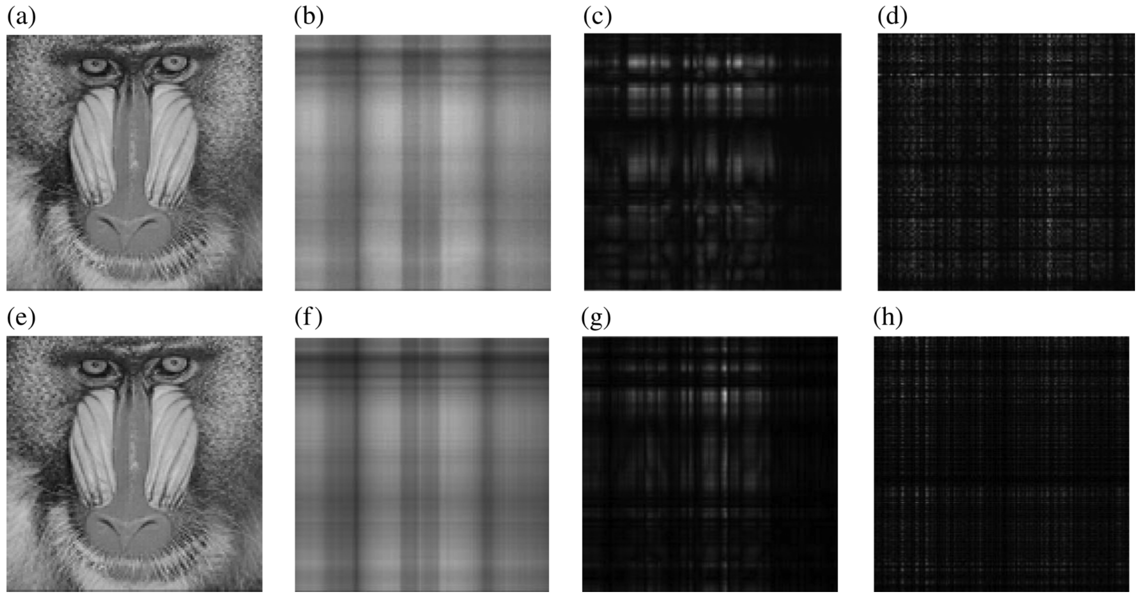


Fig. 1. Selected eigenimages of Mandrill baboon: (a), (b), (c), (d) using RB SVDs. (e), (f), (g), (h) using quaternion SVDs. (a) (e) original image. (b), (f) the first eigenimages. (c), (g) the fifth eigenimages. (d), (h) the twenty-fifth eigenimages.

$$\sigma_i^+ = \begin{cases} 1/\sigma_i = c_{1i}^{-1}e_1 + c_{2i}^{-1}e_2 & \text{if } \sigma_i = c_{1i}e_1 + c_{2i}e_2 \\ & \text{and } c_{1i} \times c_{2i} \neq 0 \\ c_{1i}^{-1}e_1 & \text{if } c_{1i} \neq 0 \text{ but } c_{2i} = 0 \\ c_{2i}^{-1}e_1 & \text{if } c_{1i} = 0 \text{ but } c_{2i} \neq 0 \\ 0 & \text{if } c_{1i} = c_{2i} = 0 \end{cases} \cdot \tag{73}$$

B. Least Square Error Problem for RBs

Suppose that there is an RB matrix $Q_{(\mathrm{RB})}$ and an RB vector $\vec{b}_{(\mathrm{RB})}$. We want to find an RB vector $\vec{x}_{(\mathrm{RB})}$ such that

square error of approximation

$$= \left|\vec{b}_{(\mathrm{RB})} - Q_{(\mathrm{RB})}\vec{x}_{(\mathrm{RB})}\right|_{(\mathrm{RB})}^2 \tag{74}$$

is minimized. Here, the norm of an RB vector is defined as

$$\left|\vec{z}_{(\mathrm{RB})}\right| = \left(\sum_n \vec{z}_r^2[n] + \sum_n \vec{z}_i^2[n] + \sum_n \vec{z}_j^2[n] + \sum_n \vec{z}_k^2[n]\right)^{1/2} \tag{75}$$

where $\vec{z}_r$, $\vec{z}_i$, $\vec{z}_j$, and $\vec{z}_k$ are real and correspond to the real-i-, j-, and k-parts of $\vec{z}$, respectively. The problem to minimize (74) can be solved by the SVD of $Q_{(\mathrm{RB})}$. Assume that $Q_{(\mathrm{RB})} = U_{(\mathrm{RB})}\Lambda_{(\mathrm{RB})}V_{(\mathrm{RB})}^T$, as in (33). Then $\vec{x}_{(\mathrm{RB})}$ can be solved from

$$\vec{x}_{(\mathrm{RB})} = Q_{(\mathrm{RB})}^+\vec{b} = V_{(\mathrm{RB})}\left[\Lambda_{(\mathrm{RB})}\right]^+ U_{(\mathrm{RB})}^T\vec{b}_{(\mathrm{RB})} \tag{76}$$

where the pseudoinverse $[\Lambda_{(\mathrm{RB})}]^+$ is defined in (71) and (72). The RB vector $\vec{x}_{(\mathrm{RB})}$ solved from (76) will be the solution to minimize the square error in (74).

Proof: Suppose that $\vec{z}_{(\mathrm{RB})} = \vec{b}_{(\mathrm{RB})} - Q_{(\mathrm{RB})}\vec{x}_{(\mathrm{RB})}$. Then $\vec{z}_{(\mathrm{RB})}$ can be expressed as

$$\vec{z}_{(\mathrm{RB})} = \left[(\vec{z}_r + \vec{z}_j) + i(\vec{z}_i + \vec{z}_k)\right]e_1 + \left[(\vec{z}_r - \vec{z}_j) + i(\vec{z}_i - \vec{z}_k)\right]e_2$$

where $\vec{z}_r$, $\vec{z}_i$, $\vec{z}_j$, and $\vec{z}_k$ are the real-i-, j-, and k-parts of $\vec{z}_{(\mathrm{RB})}$, respectively. Note that

$$\begin{aligned} &\left|(\vec{z}_r + \vec{z}_j) + i(\vec{z}_i + \vec{z}_k)\right|^2 + \left|(\vec{z}_r - \vec{z}_j) + i(\vec{z}_i - \vec{z}_k)\right|^2 \\ &\quad = \sum_n (\vec{z}_r[n] + \vec{z}_j[n])^2 + \sum_n (\vec{z}_i[n] + \vec{z}_k[n])^2 \\ &\qquad + \sum_n (\vec{z}_r[n] - \vec{z}_j[n])^2 + \sum_n (\vec{z}_i[n] - \vec{z}_k[n])^2 \\ &\quad = \sum_n \big(\vec{z}_r^2[n] + \vec{z}_r[n]\vec{z}_j[n] + \vec{z}_j^2[n] + \vec{z}_i^2[n] + \vec{z}_i[n]\vec{z}_k[n] \\ &\qquad\quad + \vec{z}_k^2[n] + \vec{z}_r^2[n] - \vec{z}_r[n]\vec{z}_j[n] + \vec{z}_j^2[n] + \vec{z}_i^2[n] \\ &\qquad\quad - \vec{z}_i[n]\vec{z}_k[n] + \vec{z}_k^2[n]\big) \\ &\quad = 2\left|\vec{z}_{(\mathrm{RB})}\right|^2 \end{aligned} \tag{77}$$

Thus, if both $|(\vec{z}_r+\vec{z}_j)+i(\vec{z}_i+\vec{z}_k)|^2$ and $|(\vec{z}_r-\vec{z}_j)+i(\vec{z}_i-\vec{z}_k)|^2$ are minimized, then $|\vec{z}_{(\mathrm{RB})}|^2$ can also be minimized. Note that if we use (15) and (16) to decompose $\vec{b}_{(\mathrm{RB})}$, $Q_{(\mathrm{RB})}$ and $\vec{x}_{(\mathrm{RB})}$ as the $e_1 - e_2$ for:

$$\begin{aligned} \vec{b}_{(\mathrm{RB})} &= \vec{b}_{(c),1}e_1 + \vec{b}_{(c),2}e_2, \quad Q_{(\mathrm{RB})} = Q_{(c),1}e_1 + Q_{(c),2}e_2 \\ \vec{x}_{(\mathrm{RB})} &= \vec{x}_{(c),1}e_1 + \vec{x}_{(c),2}e_2 \end{aligned} \tag{78}$$

then

$$\begin{aligned} \vec{z}_r + \vec{z}_j + i(\vec{z}_i + \vec{z}_k) &= \vec{b}_{(c),1} - Q_{(c),1}\vec{x}_{(c),1} \\ \text{and } \vec{z}_r - \vec{z}_j + i(\vec{z}_i - \vec{z}_k) &= \vec{b}_{(c),2} - Q_{(c),2}\vec{x}_{(c),2}. \end{aligned} \tag{79}$$

From (76)

$$\begin{aligned} \vec{x}_{(c),1} &= V_{(c),1}\left[\Lambda_{(c),1}\right]^+ U_{(c),1}^T\vec{b}_{(c),1} \\ \vec{x}_{(c),2} &= V_{(c),2}\left[\Lambda_{(c),2}\right]^+ U_{(c),2}^T\vec{b}_{(c),2} \end{aligned} \tag{80}$$

where

$$\begin{aligned} V_{(c)} &= V_{(c),1}e_1 + V_{(c),2}e_2, \quad \Lambda_{(c)} = \Lambda_{(c),1}e_1 + \Lambda_{(c),2}e_2 \\ U_{(c)} &= U_{(c),1}e_1 + U_{(c),2}e_2. \end{aligned} \tag{81}$$

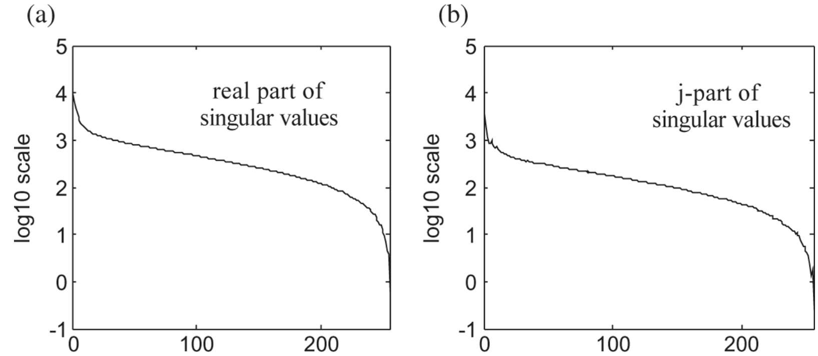


Fig. 2. Singular values of the color image **Baboon**. (a) Real part. (b) j-part.

TABLE VII
PSNRs OF THE APPROXIMATED COLOR IMAGE **Baboon** FOR DIFFERENT PARTIAL SUMS OF K EIGENIMAGES OF THE QUATERNION, RB, AND SEPARABLE SVDs

K	10	20	30	40	50
quaternion	26.14	27.61	28.75	29.81	30.84
RB	**26.24**	**27.73**	**28.91**	**30.00**	**31.06**
separable	26.13	27.60	28.75	29.82	30.85

TABLE VIII
PSNRs OF THE APPROXIMATED COLOR IMAGE **Pepper** FOR DIFFERENT PARTIAL SUMS OF K EIGENIMAGES OF THE QUATERNION, RB, AND SEPARABLE SVDs

K	10	20	30	40	50
quaternion	26.69	30.67	33.35	35.61	37.51
RB	**26.84**	**30.80**	**33.50**	**35.74**	**37.65**
separable	26.79	30.67	33.35	35.56	37.46

From the discussion in Section II-D, $Q_{(c),1} = U_{(c),1}\Lambda_{(c),1}V_{(c),1}^T$ and $Q_{(c),2} = U_{(c),2}\Lambda_{(c),2}V_{(c),2}^T$. Thus, from the conventional SVD theory in complex algebra, $\vec{x}_{(c),1}$ can minimize $|\vec{b}_{(c),1} - Q_{(c),1}\vec{x}_{(c),1}|_{(c)}^2$ and hence minimize $|(\vec{z}_r+\vec{z}_j)+i(\vec{z}_i+\vec{z}_k)|^2$. can minimize $|\vec{b}_{(c),2} - Q_{(c),2}\vec{x}_{(c),2}|_{(c)}^2$ and, hence, minimize $|(\vec{z}_r - \vec{z}_j)+i(\vec{z}_i-\vec{z}_k)|^2$. Since both the two terms are minimize, $x_{(\mathrm{RB})}$ solved from (76) can minimize $|\vec{z}_{(\mathrm{RB})}|^2$ and is the solution to minimize the square problem in (74). **Q.E.D.**

C. Color Image Processing

We can use an RB matrix $f_{(\mathrm{RB})}(m,n)$ to express a color image

$$f_{(\mathrm{RB})}(m,n) = f_R(m,n)i + f_G(m,n)j + f_B(m,n)k \quad (82)$$

where $f_R(m,n)$, $f_G(m,n)$, $f_B(m,n)$ are the R, G, B parts of the color image [15]. Alternatively, we can also place $f_R(m,n)$, $f_G(m,n)$, $f_B(m,n)$ into the j-, k-, and i-parts of $f_B(m,n)$. In fact, there are $3! = 6$ ways to assign the j-, k-, and i-parts of $f_{(\mathrm{RB})}(m,n)$. Then, by (33), we can get the SVD of a color image and represent it as a vector outer product notation.

$$f_{(\mathrm{RB})} = U_{(\mathrm{RB})}\Lambda_{(\mathrm{RB})}V_{(\mathrm{RB})}^H = \sum_{i=1}^{R} \lambda_{i(\mathrm{RB})}u_{i(\mathrm{RB})}v_{i(\mathrm{RB})}^H \quad (83)$$

where $u_{i(\mathrm{RB})}$, $v_{i(\mathrm{RB})}$ are the RB column vectors of $U_{(\mathrm{RB})}$, $V_{(\mathrm{RB})}$, respectively, and $\lambda_{i(\mathrm{RB})}$ is the nonreal diagonal term of $\Lambda_{(\mathrm{RB})}$ with the value $a + cj$. Hence, the color image $f_{(\mathrm{RB})}$ can be considered as the linear combination of R color eigenimages $(u_{i(\mathrm{RB})} \cdot v_{i(\mathrm{RB})}^H)$.

Fig. 1(a)-(d) shows some selected eigenimages of the color images known as “Mandrill baboon” using the RB SVD. For comparison, we show the results of the quaternion SVD in Fig. 1(e)–(h). These figures present normalized absolute-value versions of the first, fifth, and twenty-fifth eigenimages obtained from the SVD decomposition of the original image. Similar to the SVD in complex algebra, the first eigenimages represent the lower-frequency components of the original image, while the latter ones represent the higher-frequency components.

In (83), for the consideration of compression, we may use only K terms $(K < R)$ to approximate the RQ matrix $f_{(\mathrm{RB})}$

$$[f_K]_{(\mathrm{RB})} = \sum_{i=1}^{K} \lambda_{i(\mathrm{RB})}u_{i(\mathrm{RB})}v_{i(\mathrm{RB})}^H. \quad (84)$$

Then the storage requirements drop from N^2 to $K(2N+1)$. As seen in Fig. 2, the singular values decay very fast. Hence, even when using a small K we can provide a good approximation of the color image.

Fig. 3(a)–(d) shows the partial sums of images of the color image “Mandrill baboon” using RB SVD. The results obtained from the quaternion SVD are shown in Fig. 3(e)–(h). Then, we use the peak signal-to-noise ratio (PSNR) [36] to compare the performances of the RB SVD, the quaternion SVD, and the separable method (i.e., doing the SVD for R, G, and B parts of the color image separately) in Tables VII and VIII. As can be seen, the performance of the RB SVD is better than those of the quaternion and the separable method. Thus, we think that the RB SVD is more effective and suitable for the SVD of a color image.

In addition to the above applications, quaternions and RB SVDs can be very useful for vector-sensor signal processing in acoustic, seismic, communications, and electromagnetism [42].

V. CONCLUSION

In this paper, we first introduce the eigenvalues, eigenvectors, SVD, and pseudoinverse of an RB matrix. Then we discuss the n^{th} roots of an RB number and the zeros of an RB polynomial.

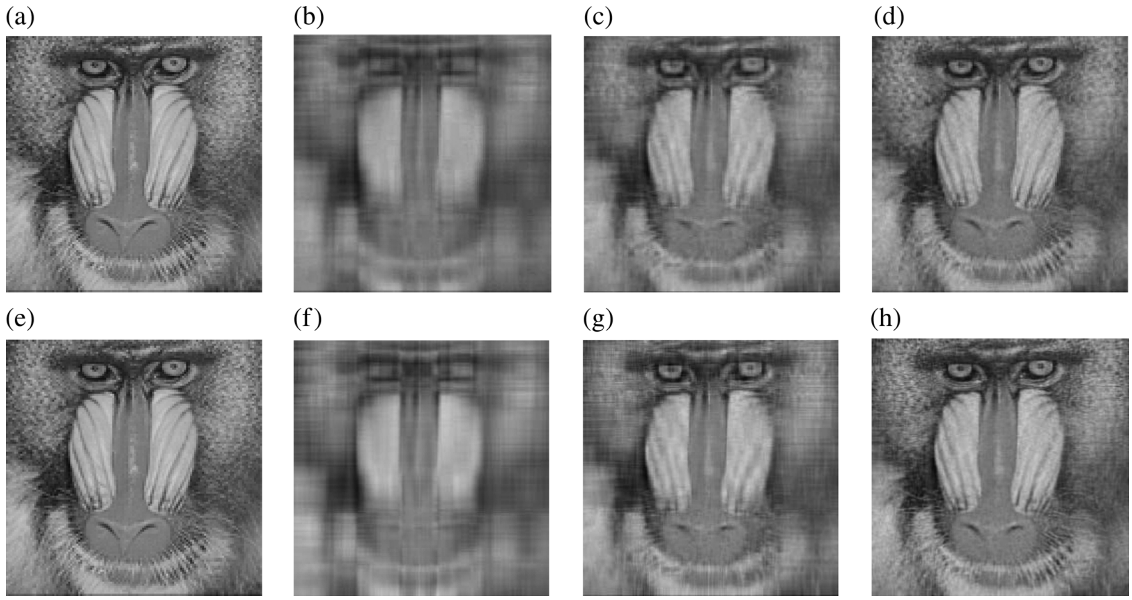


Fig. 3. Selected partial sums of images of Mandrill baboon, (a), (b), (c), (d) using RB SVDs, (e), (f), (g), (h) using quaternion SVDs, (a), (e) original image, (b), (f) $[f_5](x)$, (c), (g) $[f_{15}](x)$, (d), (h) $[f_{35}](x)$. ($x = q$ or RB).

We find the numbers of the eigenvalues of an $n \times n$ RB matrix, the n^{th} roots of an RB number, and the zeros of an RB polynomial with degree n all equal to n^2. Finally, we give some applications using the SVD of an RB matrix.

The $e_1 - e_2$ form and the polar form can be employed to simplify the analysis of the problems related to RBs. Any RB problems can be reduced to two complex problems using the two elements e_1 and e_2.

The SVD of an RB matrix can be utilized to solve many problems, such as the least square problem with RBs, the pseudoinverse of an RB matrix, and color image processing. Many color image-processing applications can be performed by the SVD of the color image.

Moreover, we give the comparisons of these concepts between the quaternion and RBs. In general, the complexity and the calculation of RBs are much simpler than the ones of quaternions, i.e. the complexity of the RB SVD is only one-fourth of that of the quaternion SVD and the complexity of reconstructing the original color image using RB matrices is only three-fourth of that of using quaternion matrices. Thus, we believe that using RBs is better than using quaternions in many cases. The commutative multiplication rule can reduce the complexity of many problems using RBs.

ACKNOWLEDGMENT

The authors thank the anonymous referees for their helpful comments, which significantly improve the presentation of this paper. They also thank Dr. M.-C. Kang, Dr. H. Chu, and Dr. S.-J. Hu for their helpful comments.

REFERENCES

[1] W. R. Hamilton, *Elements of Quaternions*. London, U.K.: Longmans, Green, 1866.

[2] H. D. Schütte and J. Wenzel, "Hypercomplex numbers in digital signal processing," in *Proc. IEEE Int. Symp. Circuits Syst.*, 1990, vol. 2, pp. 1557–1560.

[3] S. C. Pei, J. H. Chang, and J. J. Ding, "Commutative reduced biquaternions and their Fourier transform for signal and image processing," *IEEE Trans. Signal Process.*, vol. 52, no. 7, pp. 2012–2032, Jul. 2004.

[4] H. C. Lee, "Eigenvalues and Canonical forms of matrices with quaternion coefficients," *Proc. R.I.A.*, vol. 52, pp. 253–260, 1949.

[5] F. Zhang, "Quaternions and matrices of quaternions," *Linear Algebra and Its Appl.*, vol. 251, pp. 21–57, Jan. 1997.

[6] H. C. Andrews and C. L. Patterson, "Singular value decomposition and digital image processing," *IEEE Trans. Acoust., Speech, Signal Process.*, pp. 81–108, Feb. 1976.

[7] H. C. Andrews and C. L. Patterson, "Singular value decomposition (SVD) image coding," *IEEE Trans. Commun.*, pp. 72–79, Apr. 1976.

[8] I. Niven, "The roots of a quaternion," *Amer. Math. Month.*, vol. 49, pp. 386–388, 1942.

[9] L. Brand, "The roots of a quaternion," *Amer. Math. Month.*, vol. 49, pp. 519–520, 1942.

[10] S. Eilenberg and I. Niven, "The fundamental theorem of algebra for quaternions," *Bull. Amer. Math. Soc.*, vol. 50, pp. 246–248, 1944.

[11] I. Niven, "Equations in quaternions," *Amer. Math. Month.*, vol. 48, pp. 654–661, 1941.

[12] R. Serodio and E. Pereira, "Computing the zeros of quaternion polynomials," *Comp. Math. Appl.*, vol. 42, pp. 1229–1237, 2001.

[13] S. C. Pei and C. M. Cheng, "A novel block truncation coding of color image using a quaternion-moment-preserving principle," *IEEE Trans. Commun.*, vol. 45, pp. 583–595, 1997.

[14] T. Bülow and G. Sommer, *Multi-Dimensional Signal Processing Using an Algebraically Extended Signal Representation*, G. Sommer and J. J. Koenderink, Eds. New York: Springer-Verlag, 1997, vol. 1315, Lecture Notes in Comp. Sci., pp. 148–163.

[15] S. J. Sangwine, "Colour image edge detector based on quaternion convolution," *Electron. Lett.*, vol. 34, no. 10, pp. 969–971, May 1998.

[16] S. C. Pei and C. M. Cheng, "Color image processing using binary quaternion-moment preserving thresholding technique," *IEEE Trans. Image Process.*, vol. 8, no. 5, pp. 614–628, 1999.

[17] S. J. Sangwine and T. A. Ell, "Hypercomplex auto- and cross-correlation of color images," in *Proc. IEEE International Conference on Image Processing*, 1999, pp. 319–323.

[18] T. Bülow, "Hypercomplex spectral signal representations for the processing and analysis of images," Ph.D. dissertation, Christian-Albrechts Univ. Kiel, , Aug. 1999.

[19] C. J. Evans, S. J. Sangwine, and T. A. Ell, "Hypercomplex color-sensitive smoothing filters," in *Proc. IEEE Int. Conf. Image Process.*, Sep. 2000, vol. 1, pp. 541–544.

[20] C. J. Evans, S. J. Sangwine, and T. A. Ell, "Color-sensitive edge detection using hypercomplex filters," in *Proc. EUSIPCO*, Tampere, Finland, Sep. 2000, vol. 1, pp. 107–110.

[21] S. C. Pei, J. J. Ding, and J. H. Chang, "Color pattern recognition by quaternion correlation," in *Proc. IEEE Int. Conf. Image Process.*, 2000, vol. 1, pp. 847–849.
[22] H. Toyoshima, "Computationally efficient multipliers for DSP," *IEICE Trans. Inf. Syst.*, vol. E81-D, no. 2, pp. 236–238, 1998.
[23] K. Ueda and S. I. Takahashi, "Digital filters with hypercomplex coefficients," in *Proc. IEEE Int. Symp. Circuits Syst.*, May 1993, vol. 1, pp. 479–482.
[24] T. A. Ell, "quaternion-Fourier transforms for analysis of two-dimensional linear time invariant partial differential systems," in *Proc. 32nd Conf. Decision Control*, Dec. 1993, pp. 1830–1841.
[25] S. C. Pei, J. J. Ding, and J. H. Chang, "Efficient implementation of quaternion Fourier transform, convolution, and correlation by 2-D FFT," *IEEE Trans. Signal Process.*, vol. 49, no. 11, pp. 2783–2797, Nov. 2001.
[26] S. J. Sangwine, "The discrete quaternion Fourier transform," in *Proc. 6th Int. Conf. Image Process. Appl.*, Jul. 1997, vol. 2, pp. 14–17.
[27] M. Felsberg, Fast Quaternion Fourier Transform Christian-Albrechts Univ. of Kiel, Cognitive Syst. Group, Inst. of Comp. Sci. Appl. Math., Sep. 1997 [Online]. Available: www.informatik.uni-kiel.de/~tbl/index.html, Tech. Report
[28] C. M. Davenport, "A commutative hypercomplex algebra with associated function theory," in *Clifford Algebra With Numeric and Symbolic Computation*, R. Ablamowicz, Ed. Boston, MA: Birkhauser, 1996, pp. 213–227.
[29] T. A. Ell and S. J. Sangwine, "Decomposition of 2D hypercomplex fourier transforms into pairs of complex Fourier transforms," in *Proc. EUSIPCO*, 2000, pp. 151–154.
[30] T. A. Ell and S. J. Sangwine, "Hypercomplex Wiener-Khintchine theorem with application to color image processing," in *Proc. IEEE Int. Conf. Image Process.*, Sep. 2000, vol. 2, pp. 792–795.
[31] T. A. Ell, "Hypercomplex spectral transform," Ph.D. dissertation, Univ. Minnesota, Minneapolis, 1992.
[32] S. J. Sangwine, "Fourier transforms of colour images using quaternion or hypercomplex numbers," *Electron. Lett.*, vol. 32, no. 21, pp. 1979–1980, Oct. 1996.
[33] S. J. Sangwine and T. A. Ell, "The discrete Fourier transforms of a color image," in *Proc. 2nd IMA Conf. Image Process. Image Process. II: Math. Methods, Algorithms and Appl.*, 2000, pp. 430–441.
[34] S. J. Sangwine and T. A. Ell, "Hypercomplex Fourier transforms of color image," in *Proc. IEEE Int. Conf. Image Process.*, Oct. 2001, vol. 1, pp. 137–140.
[35] G. Baley Price, *An Introduction to Multicomplex Space and Functions*. New York: Marcel Dekker, 1991.
[36] S. C. Pei, J. H. Chang, and J. J. Ding, "Quaternion matrix singular value decomposition and its applications for color image processing," in *Proc. Int. Conf. Image Process.*, Sep. 2003, vol. 1, pp. 805–808.
[37] N. Le Bihan and S. J. Sangwine, "Quaternion principal component analysis of color images," in *Proc. Int. Conf. Image Process.*, Sept. 2003, vol. 1, pp. 809–812.
[38] H. F. de Groote, "On the complexity of quaternion multiplications," *Inf. Process. Lett.*, vol. 3, pp. 177–179, 1975.
[39] S. Winograd, "Some bilinear forms whose multiplicative complexity depends on the field of constants," *Math. Syst. Theory*, vol. 10, pp. 169–180, 1977.
[40] K. Muzikami, H. Toyoshima, and S. Takahashi, "A residue number system for the hypercomplex arithmetic," in *Proc. Joint Tech. Conf. Circuits/Syst., Comput. Commun.*, 1992, pp. 163–167.
[41] V. S. Dimitrov, T. V. Cooklev, and B. D. Donevsky, "On the multiplication of reduced biquaternions and applications," *Inf. Process. Lett.*, vol. 43, pp. 161–164, 1992.
[42] N. L. Bihan and J. Mars, "Singular value decomposition of quaternion matrices: A new tool for vector-sensor signal processing," *Signal Process.*, vol. 84, pp. 1177–1199, Jul. 2004.
[43] D. Choukroun, I. Bar-Itzhack, and Y. Oshman, "Novel quaternion Kalman filter," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 42, no. 1, pp. 174–190, Jan. 2006.
[44] F. Wang, S. Wang, and Y. Wu, "Estimating frequencies of two dimensional harmonics with extended quaternion matrix pencil," in *IEEE Int. Symp. Microw., Antenna, Propag. EMC Technol. Wireless Commun.*, Aug. 2005, vol. 1, pp. 199–203.
[45] N. L. Bihan, S. Miron, and J. I. Mars, "MUSIC algorithm for vector-sensors array using biquaternions," *IEEE Trans. Signal Process.*, vol. 55, no. 9, pp. 4523–4533, Sep. 2007.
[46] T. A. Ell, "Hypercomplex color affine filters," in *IEEE Int. Conf. Image Process.*, Sep. 2007, vol. 5, pp. 249–252.
[47] Z. Lu, Y. Xu, X. Yang, L. Song, and L. Traversoni, "2D quaternion Fourier transform: The spectral properties and its application in color image representation," in *IEEE Int. Conf. Multimedia Expo.*, Jul. 2007, pp. 1715–1718.

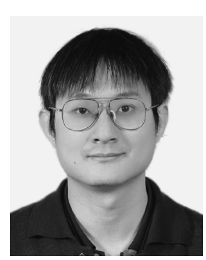
Soo-Chang Pei (SM'89-F'00) was born in Soo-Auo, Taiwan, in 1949. He received the B.S.E.E. degree from the National Taiwan University (NTU), Taipei, Taiwan, in 1970, and the M.S.E.E. and Ph.D. degrees from the University of California, Santa Barbara (UCSB), in 1972 and 1975, respectively.

He was an Engineering Officer in the Chinese Navy Shipyard from 1970 to 1971. From 1971 to 1975, he was a Research Assistant with the UCSB. He was the Professor and Chairman of the Electrical Engineering Department, Tatung Institute of Technology, and NTU, from 1981 to 1983 and 1995 to 1998, respectively. Presently, he is the Dean of Electrical Engineering and Computer Science College and a Professor with the Electrical Engineering Department, NTU. His research interests include digital signal processing, image processing, optical information processing, and laser holography.

Dr. Pei is a member of Eta Kappa Nu and the Optical Society of America. He received the National Sun Yet-Sen Academic Achievement Award in Engineering in 1984, the Distinguished Research Award from the National Science Council from 1990 to 1998, the outstanding Electrical Engineering Professor Award from the Chinese Institute of Electrical Engineering in 1998, the Academic Achievement Award in Engineering from the Ministry of Education in 1998, the Pan Wen-Yuan Distinguished Research Award in 2002, and the National Chair Professor Award from the Ministry of Education in 2002. He was President of the Chinese Image Processing and Pattern Recognition Society in Taiwan from 1996 to 1998. He became an IEEE Fellow in 2000 for contributions to the development of digital eigenfilter design, color image coding and signal compression, and electrical engineering education in Taiwan.

Ja-Han Chang was born in Taipei, Taiwan, in 1977. He received the B.S. and Ph.D. degrees, both in electrical engineering, from National Taiwan University (NTU) , Taipei, in 1999 and 2004, respectively.

He is currently a senior engineer working in the Display Processing Division, MediaTek Corporation. His current research areas include image watermarking, image processing, pattern recognition, quaternion and quaternion Fourier transforms, reduced biquaternions, and reduced biquaternion Fourier transforms.



Jian-Jiun Ding was born in 1973 in Taiwan. He received the B.S. degree in 1995, the M.S. degree in 1997, and the Ph.D. degree in 2001, all in electrical engineering from the National Taiwan University (NTU), Taipei.

During 2001–2005, he was a Postdoctoral Researcher with the Department of Electrical Engineering, NTU. He is currently an Assistant Professor with the Department of Electrical Engineering, NTU. His current research areas include time-frequency analysis, fractional Fourier transforms, linear canonical transforms, image processing, orthogonal polynomials, fast algorithms, quaternion algebra, pattern recognition, and filter design.

Ming-Yang Chen was born in Taipei, Taiwan, in 1981. He received the B.S.E.E. and M.S. degrees from National Taiwan University (NTU), Taipei, in 2003 and 2005, respectively.

He is currently working toward the Ph.D. degree with the Department of Electrical Engineering, Stanford University, Stanford, CA. His current research interests include multiple-input-multiple-output (MIMO) systems, Ramsey graphs, nonlinear systems biology dynamics, and bioinformatics.