Closed-Form Design and Efficient Implementation of Generalized Maximally Flat Half-Band FIR Filters

Soo-Chang Pei and Peng-Hua Wang

Abstract—In this letter, a closed-form expression for the impulse response of the generalized half-band (HB) maximally flat (MF) FIR filters is obtained by solving the linear equations of the MF conditions. Based on the resultant impulse responses, an efficient implementation structure is derived. The dynamic range of the multipliers of the new structure is shown to be greatly reduced in comparison to the one of the direct form impulse response.

Index Terms—Half-band filter, maximal flatness.

I. INTRODUCTION

HALF-BAND (HB) filters are often used for decimation, interpolation, and multirate systems. An Nth order linear phase FIR filter is called HB if its impulse response has the following property

$$h(n) = 0 \quad n = L \pm 2k \quad \text{and } h(L) = c$$

where the desired group delay is $L = N/2$, and c is a constant. In other words, one of the polyphase components of the HB FIR filter is just a delay cz^{-L}. In [1], the definition of the HB FIR filters is recently generalized. According to the generalized definition, the desired group delay L is not restricted to be a half of the filter order, and the resulting HB filter can have nonsymmetric impulse response (i.e., nonlinear phase response).

In [2], the authors summarized the design methods of the HB filters and the Hilbert transformers with linear phase. A closed-form design of the nonlinear phase HB maximally flat (MF) FIR filters was recently proposed in [3] using the Chebyshev polynomial. Although the final impulse response was not expressed for direct form structure, the transfer function represented as the Bernstein polynomial form was analytically obtained. In [4], a closed-form expression for the transfer function of linear phase HB MF FIR filter was derived. Two implementations were presented. It is interesting that one of the two structures needs no multipliers. That is, a multiplierless structure for the linear phase MF HB FIR filter was obtained in [4].

In this letter, a new closed-form expression for the impulse response of the generalized HB MF FIR filters is proposed. The impulse response is solved directly from the linear equations of the MF conditions. An efficient new implementation structure is derived based on the resultant impulse responses. The dynamic range of the multipliers of the new structure is shown to be greatly reduced in comparison with the dynamic range of

the direct form impulse response by an example. The multipliers used are also fewer in the proposed structure.

II. CLOSED-FORM IMPULSE RESPONSE OF GENERALIZED HB MF FIR FILTERS

Suppose the Nth order $(N = 2M)$ generalized HB FIR filter is characterized by the transfer function of

$$H(z) = h(L)z^{-L} + \sum_{m=0}^{M} h(2m)z^{-2m} \tag{1}$$

where the odd integer L is the symmetry center. The desired frequency response is defined by

$$H_d(\omega) = \begin{cases} e^{-j\omega L}, & \text{for } \omega \in \Omega_p \\ 0, & \text{for } \omega \in \Omega_s \end{cases} \tag{2}$$

where Ω_p and Ω_s denote the passband and the stopband, respectively. The filter is said to be maximally flat if the frequency response $H(e^{j\omega})$ satisfies the following properties

$$\left. \frac{\partial^k H(e^{j\omega})}{\partial \omega^k} \right|_{\omega=0}, = \left. \frac{\partial^k H_d(e^{j\omega})}{\partial \omega^k} \right|_{\omega=0}, \quad \text{for } k = 0, 1, \ldots, P \tag{3}$$

and

$$\left. \frac{\partial^k H(e^{j\omega})}{\partial \omega^k} \right|_{\omega=\pi} = \left. \frac{\partial^k H_d(e^{j\omega})}{\partial \omega^k} \right|_{\omega=\pi}, \quad \text{for } k = 0, 1, \ldots, Q \tag{4}$$

where P and Q represent the degree of flatness at $\omega = 0$ and $\omega = \pi$, respectively. Since the desired HB frequency response is symmetric about $\omega = \pi/2$, P is assigned to be equal to Q in this paper.

Substituting (1) and (2) for (3) and (4) and simplifying the equations, we obtain the following equations:

$$h(L)L^k + \sum_{m=0}^{M} h(2m)(2m)^k = L^k \tag{5}$$

and

$$h(L)L^k - \sum_{m=0}^{M} h(2m)(2m)^k = 0 \tag{6}$$

for $k = 0, 1, \ldots, P$. It is easy to show that
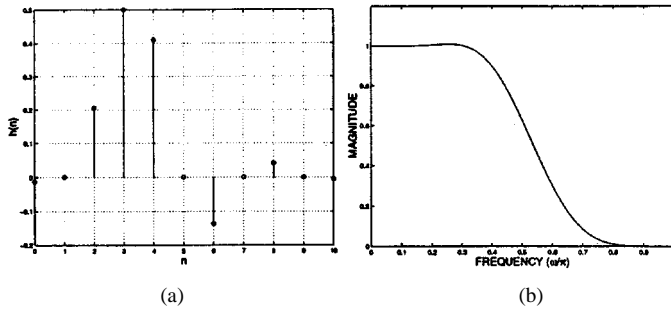
$$h(L) = \frac{1}{2} \tag{7}$$

(a)　　　　　　　　　　(b)

Fig. 1. Design of the tenth-order maximally flat HB FIR filter with the desired group delay $L = 3$. (a) Impulse response. (b) Magnitude response.
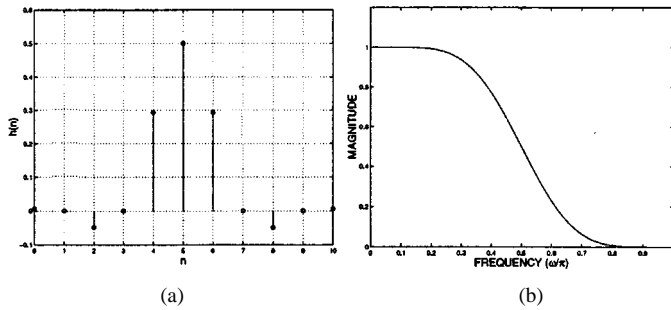

(a)　　　　　　　　　　(b)

Fig. 2. Design of the tenth-order maximally flat HB FIR filter with the desired group delay $L = 5$. (a) Impulse response. (b) Magnitude response.

and (5) and (6) are reduced as

$$\sum_{m=0}^{M} h(2m)(2m)^k = \frac{1}{2}L^k \qquad (8)$$

for $k = 0, 1, \ldots, P$. Since (8) is a Vandermonde system [5], it is of full rank. We conclude that the quantities of P, Q, and M are related as

$$P = Q = M. \qquad (9)$$

Equation (8) can be solved analytically by carrying out the Cramer's rule [6]. The resultant closed-form impulse responses are obtained and expressed by

$$h(2m) = \frac{(-1)^m \prod_{i=0}^{M}(i - L/2)}{(2m - L)m!(M - m)!} \qquad (10)$$

for $m = 0, 1, \ldots, M$.

Fig. 1(a) and (b) shows the impulse responses and the magnitude response of the tenth-order MF HB FIR filter with $L = 3$. The impulse response is not symmetric and consequently, the filter does not have linear phase response. Fig. 2(a) and (b) shows the impulse responses and the magnitude response of the tenth-order MF HB FIR filter with $L = 5$. The impulse response is symmetric about the desired group delay $L = 5$. Thus, the phase response of the designed filter is linear.

III. EFFICIENT IMPLEMENTATION OF GENERALIZED HB MF FIR FILTERS

Although the filter can be implemented in direct form with the impulse response solved in Section II, a new efficient struc-
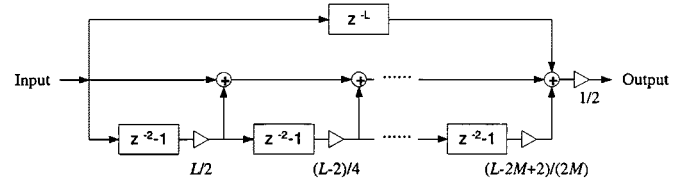

Fig. 3. Efficient implementation structure for the Nth-order generalized MF HB FIR filter, where $M = N/2$, and L is the desired group delay.

ture can be derived. Since $h(L) = 1/2$, the transfer function expressed by (1) can be written as

$$H(z) = \frac{1}{2}(z^{-L} + E(z)) \qquad (11)$$

where

$$E(z) = \sum_{m=0}^{M} \tilde{h}(m)z^{-2m}$$

and $\tilde{h}(m) = 2h(2m)$.

If we express $E(z)$ in the following form:

$$E(z) = \sum_{n=0}^{M} b(n)(z^{-2} - 1)^n$$

for $m = 0, 1, \ldots, M$. It is easy to show that

$$b(n) = \sum_{m=n}^{M} \tilde{h}(m) \binom{m}{n}. \qquad (12)$$

Substituting $\tilde{h}(m)$ for (13), the weighting coefficient $b(n)$ is simplified as

$$b(n) = \frac{\prod_{i=0}^{n-1}(L/2 - i)}{n!} \qquad (13)$$

for $n = 1, 2, \ldots, M$ and $b(0) = 1$. Based on the property that $b(n)$ can be expressed by the following cumulative product:

$$b(n) = \frac{L}{2} \cdot \frac{L-2}{4} \cdots \frac{L-2n+2}{2n}$$

then we can derive an efficient implementation structure as shown in Fig. 3.

The main advantage of the new structure is that the dynamic range of the weighting coefficients is greatly reduced. For example, according to (10), the impulse response of the tenth-order MF HB FIR filter with $L = 3$ is

$$\{-7/512, 0, 105/512, 1/2, 105/256, 0,$$
$$- 35/256, 0, 21/512, 0, -3/512\}$$

where the ratio of the largest amplitudes of coefficients to the least one is $(1/2)/(3/512) \approx 85$. If the filter is implemented by an FIR lattice structure, the reflection coefficients are

$$\{0.6565, 0.6906, 1.1565, -0.2667, 6.0000, 1.0000,$$
$$-1.1538, 4.2000, 0, 0.4286\}$$

where the ratio of the largest amplitudes of coefficients to the least one is $6/0.2667 \approx 22.5$. However, the weighting coefficients of the proposed structure are

$$\{1, 3/2, 1/4, -1/6, -3/8, -1/2\}$$

and the ratio of the largest amplitudes of multiplier to the least one is $(3/2)/(1/6) = 9$. Another advantage of the structure in Fig. 3 is there is one multiplier less than the direct form implementation. There are $N/2 + 1$ multipliers for the direct form structure, and $N/2$ is needed for the proposed one.

IV. Conclusion

In this paper, a closed-form expression for the impulse response of the generalized HB MF FIR filters is derived. We solve the impulse response directly from the linear equations of the MF conditions by using the Carmer's rule. The impulse response form is rather simple. Design examples are presented.

A new efficient implementation structure is obtained based on the derived impulse response. The dynamic range of the multipliers of the new structure can be greatly reduced in comparison to the one of the direct form impulse response.

References

[1] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*. Englewood Cliffs, NJ: Prentice-Hall, 1993.

[2] H. W. Schüßler and P. Steffen, "Halfband filters and Hilbert transformers," *Circuits Syst. Signal Process.*, vol. 17, pp. 137–164, Oct. 1998.

[3] S. Samadi, A. Nishihara, and H. Iwakura, "Generalized half-band maximally flat FIR filters," in *Proc. IEEE Int. Symp. Circuits and Systems*, Orlando, FL, May 1999.

[4] S. Samadi, H. Iwakura, and A. Nishihara, "Mutiplierless and hierarchical structure for maximally flat half-band FIR filters," *IEEE Trans. Circuits Syst. II*, vol. 46, pp. 1225–1230, Sept. 1999.

[5] G. H. Golub and C. F. Van Loan, *Matrix Computing*. Baltimore, MD: The Johns Hopkins Univ. Press, 1989.

[6] B. Noble and J. W. Daniel, *Applied Linear Algebra*. Englewood Cliffs, NJ: Prentice-Hall, 1988.