Programmable Fractional Sample Delay Filters with Flatness Compromise Between Magnitude Response and Group Delay

Soo-Chang Pei, Bi-Ruei Chiou, and Peng-Hua Wang

***Abstract*—In this paper, a new design of fractional sample delay filter is presented. This method is based on the Stancu polynomial, which possesses an extra parameter α. Under special choice of this parameter, the results obtained are identical to previous ones based on Lagrange interpolation formula. Much more flatness of group-delay response can be achieved at the sacrifice of the filter magnitude response. Thus, this method provides designers more flexibility for trading between these two performance criterion.**

***Index Terms*—Flatness of group delay, fractional sample delay filter, Stancu polynomial.**

I. Introduction

In digital signal processing system, sometimes it is desired to have the linear phase shifter or, equivalently, constant delay of a signal. A signal delayed for a duration of integer multiples sampling period can be easily obtained by passing it through cascading unit-delay elements. However, it is desired to delay a signal with fractional multiples of sampling period [3]–[6]. For example, transferring discrete time data samples between two digital systems that have the same clock rate but use separate clock generators might require an fractional sample delayer to compensate for the delay time between two clocks.

The ideal frequency response of a phase shifter is

$$H(\omega) = e^{-j\omega\tau} \tag{1}$$

where τ is the amount of constant delay that the filter would like to achieve. Since $|H(\omega)| = |e^{-j\omega\tau}| = 1$, the ideal magnitude response is 1 for all frequency components. Also, the desired group delay of a filter is constant τ frequency components. Thus, both are used as criterion for grading fraction delay filters.

In [3] and [4], fractional sample delay filters are designed based on the interpolation method. To estimate the signal value at fractional sampling period, the Lagrange interpolation formula is utilized. In this paper, we present another design of fractional sample delay filters by using the Stancu polynomial, which is first introduced by Stancu in [2]. Under certain conditions, the designed filters are identical to those in [3]. But they enjoy a flexibility of trading between the magnitude response and group delay of filter responses.

This paper is organized as followed. In Section II, the definition and some properties of the Stancu polynomial are introduced. We briefly review previous method in [3] and then demonstrate our design in Section III. Illustrative examples are also given. Section IV is devoted to conclusions.

Manuscript received December 1998; revised April 2000. This paper was recommended by Associate Editor V. Madisetti.

The authors are with the Department of Electrical Engineering, National Taiwan University, Taipei, Taiwan, R.O.C.

Publisher Item Identifier S 1057-7130(00)06570-8.

II. Definition and Properties of Stancu Polynomial

A. Definition of the Stancu Polynomial

The Stancu polynomial operator $S^N(f, x; \alpha) = S^N(f(t), x; \alpha)$, corresponding to a function $f = f(x)$ defined on the interval (0, 1), and to a parameter α, is defined as [2]

$$S^N(f, x; \alpha) = \sum_{i=0}^{n} S_i{}^N(x;\alpha) f\left(\frac{i}{N}\right) \tag{2}$$

where we have (3), shown at the bottom of the next page.

There is another representation of the Stancu polynomial. This is dividing (3) the numerator and denominator with α^N, yielding (4), shown at the bottom of the next page.

By using the recurrence property of gamma function, which is defined as $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$, $\Gamma(x+1) = x\Gamma(x)$, we may write down the following equations:

$$\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\cdots\left(\frac{x}{\alpha}+i-1\right) = \frac{\Gamma\left(\frac{x}{\alpha}+i\right)}{\Gamma\left(\frac{x}{\alpha}\right)} \tag{5}$$

$$\frac{1-x}{\alpha}\left(\frac{1-x}{\alpha}+1\right)\cdots\left(\frac{1-x}{\alpha}+N-i-1\right) = \frac{\Gamma\left(\frac{1-x}{\alpha}+N-i\right)}{\Gamma\left(\frac{1-x}{\alpha}\right)} \tag{6}$$

$$\frac{1}{\alpha}\left(\frac{1}{\alpha}+1\right)\cdots\left(\frac{1}{\alpha}+N-1\right) = \frac{\Gamma\left(\frac{1}{\alpha}+N\right)}{\Gamma\left(\frac{1}{\alpha}\right)}. \tag{7}$$

Substituting them into (4), we obtain another form of the Stancu polynomial

$$S^N(f, x; \alpha) = \sum_{i=0}^{N} f\left(\frac{i}{N}\right)\binom{N}{i}\frac{\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}+N\right)} \cdot \frac{\Gamma\left(\frac{x}{\alpha}+i\right)\Gamma\left(\frac{1-x}{\alpha}+N-i\right)}{\Gamma\left(\frac{x}{\alpha}\right)\Gamma\left(\frac{1-x}{\alpha}\right)}. \tag{8}$$

Moreover, using the relation between Gamma function and Beta function, which is defined as $B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt$, that

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{9}$$

$S_i{}^N(x;\alpha)$ can further be written as

$$S_i{}^N(x) = \binom{N}{i}\frac{B\left(\frac{x}{\alpha}+i, \frac{1-x}{\alpha}+N-i\right)}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)}. \tag{10}$$

Thus, we have a more compact form of the Stancu polynomial

$$S^N(f, x; \alpha) = \sum_{i=0}^{N} f\left(\frac{i}{N}\right)\binom{N}{i}\frac{B\left(\frac{x}{\alpha}+i, \frac{1-x}{\alpha}+N-i\right)}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)}. \tag{11}$$

B. Properties of the Stancu Polynomial

Some properties of the Stancu polynomial listed in [2], [1] are revisited.

1) It is obvious that the Stancu approximation of $f(t)$ always passes through $f(0)$ and $f(1)$, i.e.,

$$S^N(f,0;\alpha)=f(0),\quad S^N(f,1;\alpha)=f(1). \tag{12}$$

2) We note that when $\alpha=0$, the Stancu polynomial reduces to well-known Bernstein polynomial

$$B^N(f,x)=\sum_{i=0}^{N} f\left(\frac{i}{N}\right)\binom{N}{i}x^i(1-x)^{N-i}. \tag{13}$$

And, when $\alpha=-(1/N)$, the Stancu polynomial becomes as in (14), shown at the bottom of the page, which is the equally spaced Lagrange interpolation of $f(t)$ on the interval $t\in(0,1)$.

3) There is an identity of the Stancu polynomial; this is

$$S^N(1,x;\alpha)=\sum_{i=0}^{N} {S_i}^N(x;\alpha)=1. \tag{15}$$

III. Design Method

A. Previous Design

We first describe the design method in [3].

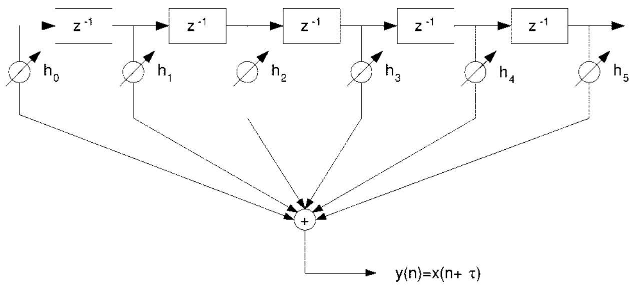

Fig. 1. Transversal filter structure for implementing fractional sample delay filters.

A $2N$-order Lagrange interpolation formula for an equally spaced data sequence $\{x(n+N),x(n+N-1),\cdots,x(n+1),x(n),x(n-1),\cdots,x(n-N)\}$ is constructed

$$x(t)=\sum_{i=-N}^{N} L_i(t)x(n+i) \tag{16}$$

where $L_i(t)$ is defined as (17), shown at the bottom of the next page.

Then the value of the waveform at the fractional sample point $(n+\tau)$ is evaluated using this Lagrange interpolation formula

$$x(n+\tau)=\sum_{i=-N}^{N} L_i(n+\tau)x(n+i). \tag{18}$$

It is easily seen that $x(n+\tau)$ is linear combination of the input data sequence $x(n+i)$, $-N\le i\le N$. Let $x(n+\tau)$ be the output of a fractional delay filter with delay time τ, i.e., $y(n)$, then (18) becomes the

$${S_i}^N(x;\alpha)=\binom{N}{i}\frac{x(x+\alpha)\cdots(x+(i-1)\alpha)(1-x)(1-x+\alpha)\cdots(1-x+(N-i-1)\alpha)}{(1+\alpha)(1+2\alpha)\cdots(1+(N-1)\alpha)} \tag{3}$$

$$S^N(f,x;\alpha)=\sum_{i=0}^{N} f\left(\frac{i}{N}\right)\binom{N}{i}\frac{\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\cdots\left(\frac{x}{\alpha}+(i-1)\right)\frac{1-x}{\alpha}\left(\frac{1-x}{\alpha}+1\right)\cdots\left(\frac{1-x}{\alpha}+N-i-1\right)}{\frac{1}{\alpha}\left(\frac{1}{\alpha}+1\right)\cdots\left(\frac{1}{\alpha}+N-1\right)}. \tag{4}$$

$$\begin{aligned}S^N\left(f,x;-\frac{1}{N}\right)&=\sum_{i=0}^{N} f\left(\frac{i}{N}\right)\frac{N!}{i!(N-i)!}\\&\quad\cdot\frac{x\left(x-\frac{1}{N}\right)\cdots\left(x-\frac{i-1}{N}\right)(1-x)\left(1-x-\frac{1}{N}\right)\cdots\left(1-x-\frac{N-i-1}{N}\right)}{\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{N-1}{N}\right)}\\&=\sum_{i=0}^{N} f\left(\frac{i}{N}\right)\frac{N(N-1)\cdots1}{(N-i)\cdots1\cdot i(i-1)\cdots1}\\&\quad\cdot\frac{(x-0)\left(x-\frac{1}{N}\right)\cdots\left(x-\frac{i-1}{N}\right)(1-x)\left(1-x-\frac{1}{N}\right)\cdots\left(1-x-\frac{N-i-1}{N}\right)}{\frac{N}{N}\,\frac{N-1}{N}\cdots\frac{1}{N}}\\&=\sum_{i=0}^{N} f\left(\frac{i}{N}\right)\frac{(x-0)\left(x-\frac{1}{N}\right)\cdots\left(x-\frac{i-1}{N}\right)\left(x-\frac{i+1}{N}\right)\left(x-\frac{i+2}{N}\right)\cdots(x-1)}{\left(\frac{i}{N}-0\right)\left(\frac{i}{N}-\frac{1}{N}\right)\cdots\left(\frac{i}{N}-\frac{i-1}{N}\right)\left(\frac{i}{N}-\frac{i+1}{N}\right)\cdots\left(\frac{i}{N}-\frac{N}{N}\right)}\end{aligned} \tag{14}$$

difference equation which relates the input and output of filter. Thus, the coefficients of filter is $L_i(n+\tau)$. This filter can be implemented by using transversal filter structure shown in Fig. 1. We also note that although the designed filter is noncausal at this moment, we may make it causal by simply shifting the coefficients with integer multiples of delay. Unavoidable tape delay is introduced by this procedure.

B. Proposed Design

Now we describe how to design fractional delay filters using the Stancu polynomial.

Since the Stancu polynomial is defined on the interval (0, 1), the mapping $y = 2Nx - N$ maps the interval (0, 1) into $(-N, N)$. We then start with a $2N$-order Stancu polynomial (19), shown at the bottom of the page. Using the change of variable $x = (y/2N) + \frac{1}{2}$, we obtain a Stancu polynomial defined on $y \in (-N, N)$ (20), shown at the bottom of the page. Applying a similar technique, the output of a fractional delay filter with delay τ is taken by evaluating $S^{2N}(f, y; \alpha)$ at $y = \tau$. The coefficients of filter are found in (21), shown at the bottom of the page. A causal delay filter is obtained by re-indexing the coefficients defined above

$$h(i) = c_{i-N}, \qquad i = 0, \cdots, 2N. \tag{22}$$

Here, we claim that this result is identical to that in [3] when $\alpha = -(1/2N)$. We would not prove it explicitly but state out argument intuitively. First, since $S^{2N}(f, x; -(1/2N))$ is the Lagrange interpolation formula for equally spaced data points at $x = (i/2N)$, then $S^{2N}(f, x; -(1/2N))$ passes through these data points. By using the change of variable $y = 2Nx - N$, the points $x = (i/2N), i = 0, \cdots, 2N$ are mapped to $y = -N, \cdots, N$. So the mapped polynomial $S^{2N}(f, y; \alpha)$ coincides with function f at $y = -N, \cdots, N$. By the uniqueness of Lagrange interpolation formula, it is clear that $S^{2N}(f, y; \alpha)$ is the $2N$-order Lagrange interpolation for equal spaced data at $y = -N, \cdots, N$, which is used in [3] for designing filter.

Using (8) and (11), we may rewrite the weighting coefficients in more compact forms. Applying the change of variable $x = (y/2N) + (1/2)$ to the $2N$-order alternative Stancu epresentations and evaluating them at $y = \tau$ yields

$$h(i) = c_{i-N} = \binom{2N}{i} \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha} + 2N\right)} \cdot \frac{\Gamma\left(\frac{\frac{\tau}{2N} + \frac{1}{2}}{\alpha} + i\right) \Gamma\left(\frac{\frac{1}{2} - \frac{\tau}{2N}}{\alpha} + 2N - i\right)}{\Gamma\left(\frac{\frac{\tau}{2N} + \frac{1}{2}}{\alpha}\right) \Gamma\left(\frac{\frac{1}{2} - \frac{\tau}{2N}}{\alpha}\right)}, \quad i = 0, \cdots, 2N \tag{23}$$

and

$$h(i) = c_{i-N} = \binom{2N}{i} \frac{B\left(\frac{\frac{\tau}{2N} + \frac{1}{2}}{\alpha} + i, \frac{\frac{1}{2} - \frac{\tau}{2N}}{\alpha} + 2N - i\right)}{B\left(\frac{\frac{\tau}{2N} + \frac{1}{2}}{\alpha}, \frac{\frac{1}{2} - \frac{\tau}{2N}}{\alpha}\right)}, \quad i = 0, \cdots, 2N \tag{24}$$

C. Design Example

Example 1: We illustrate an example of fractional delay filters which are of length 5 , i.e., $2N = 4$, and with various delay values. First we select $\alpha = -\frac{1}{4}$, and the noninteger part of delay value varies from -0.45 to 0.45. For this α, designed filters are equivalent to those designed using Lagrange interpolation formula in [3]. Magnitude response and group

$$L_i(t) = \frac{(t-n-N)(t-n-N+1)\cdots(t-n-i+1)(t-n-i-1)\cdots(t-n+N)}{(N-i)(N-i-1)\cdots(1-i)(i+1)\cdots(i+N)} \tag{17}$$

$$S^{2N}(f, x; \alpha) = \sum_{i=0}^{2N} f\left(\frac{i}{N}\right) \binom{2N}{i} \frac{x(x+\alpha)\cdots(x+(i-1)\alpha)(1-x)\cdots(1-x+(2N-i-1)\alpha)}{(1+\alpha)(1+2\alpha)\cdots(1+(2N-1)\alpha)} \tag{19}$$

$$S^{2N}(f, y; \alpha) = \sum_{i=-N}^{N} f\left(\frac{i+N}{2N}\right) \binom{2N}{i+N} \frac{\left(\frac{y}{2N} + \frac{1}{2}\right)\cdots\left(\frac{y}{2N} + \frac{1}{2} + (i-1)\alpha\right)\left(\frac{1}{2} - \frac{y}{2N}\right)\cdots\left(\frac{1}{2} - \frac{y}{2N} + (2N-i-1)\alpha\right)}{(1+\alpha)(1+2\alpha)\cdots(1+(2N-1)\alpha)} \tag{20}$$

$$c_i = \binom{2N}{i+N} \frac{\left(\frac{\tau}{2N} + \frac{1}{2}\right)\cdots\left(\frac{\tau}{2N} + \frac{1}{2} + (i-1)\alpha\right)\left(\frac{1}{2} - \frac{\tau}{2N}\right)\cdots\left(\frac{1}{2} - \frac{\tau}{2N} + (2N-i-1)\alpha\right)}{(1+\alpha)(1+2\alpha)\cdots(1+(2N-1)\alpha)}, \quad i = -N, \cdots, N. \tag{21}$$

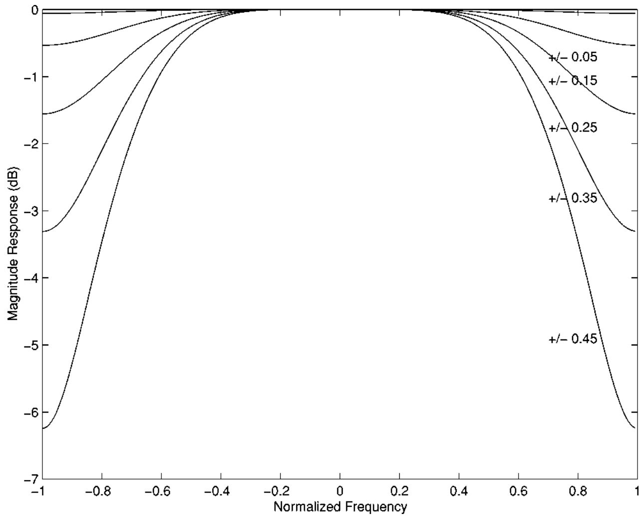

Fig. 2. Fractional delay filters with $\tau = -0.45, \cdots, 0.45$, and $\alpha = -0.25$.

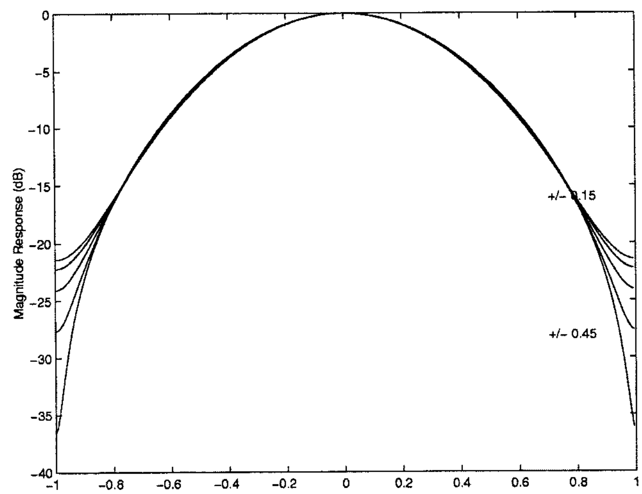

Fig. 4. Fractional delay filters with $\tau = -0.45, \cdots, 0.45$, and $\alpha = -0.125$.

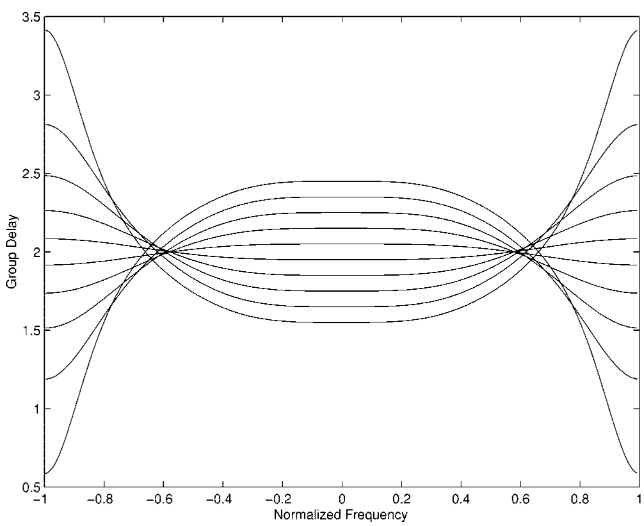

Fig. 3. Fractional delay filters with $\tau = -0.45, \cdots, 0.45$, and $\alpha = -0.25$.

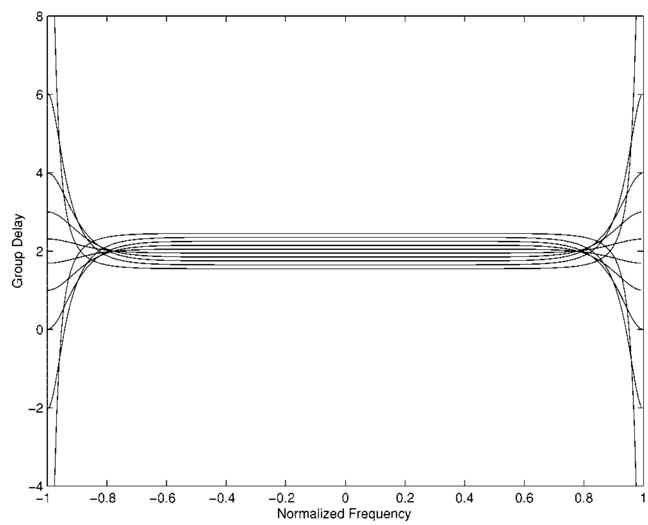

Fig. 5. Fractional delay filters with $\tau = -0.45, \cdots, 0.45$, and $\alpha = -0.125$.

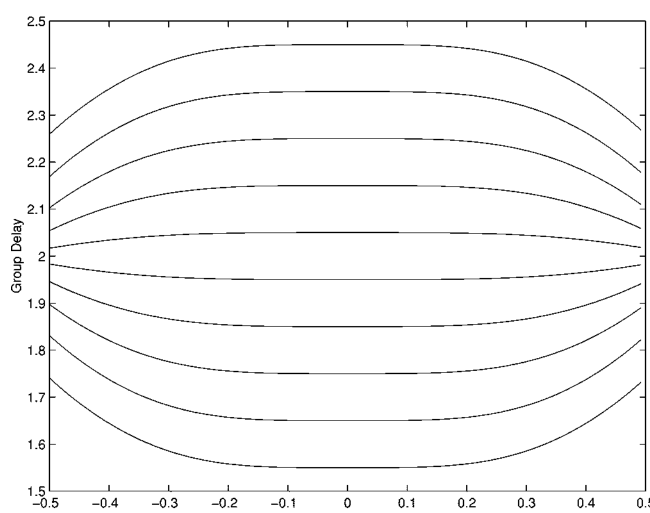

Fig. 6. Zoom-in graph of Fig. 3 for low-frequency component.

delay of these filters are shown in Figs. 2 and 3. Then we illustrate another example that α is chosen as $-\frac{1}{8}$. Their magnitude responses and group delays are shown in Figs. 4 and 5. It can be seen that although their magnitude responses deteriorate, their group delays are much more flat than the first example. A zoom-in graph of group delays in low frequency for both design are shown in Figs. 6 and 7.

Example 2: Next we illustrate another example of fractional delay filters which are also of length 5 and the fractional value is fixed at 0.2, but now we varies with different α values, $\alpha = -\frac{1}{16}$, $-\frac{1}{8}$ and $-\frac{1}{4}$. Both their magnitude responses and group delays are shown in Figs. 8 and 9. It is obvious that when α moves from $-\frac{1}{4}$ toward 0, the magnitude response deteriorates continuously but the group delay first becomes more flat and, after $\alpha \geq -(1/4)$, then bending again. Thus we suggest that designers had better select a α value between $-(1/2N)$ and $-(1/4N)$ for delay filters of length $2N + 1$.

IV. Conclusion

In this paper, we have presented a new design method, which is based on the Stancu polynomial, for fractional sample delay filters. By controlling a parameter α, more flexibility can be achieved. We may find compromise for the flatness between their magnitude response and group delay. A previous design based on Lagrange interpolation is a special case of our results. Compact forms of the weighting coeffici-

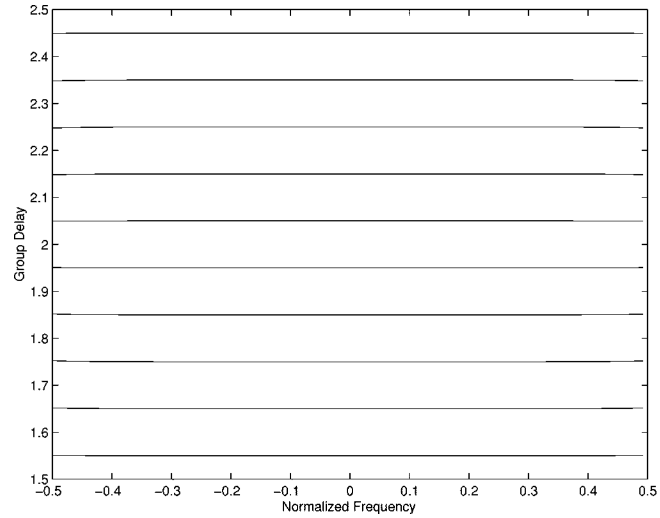

Fig. 7. Zoom-in graph of Fig. 5 for low-frequency component.

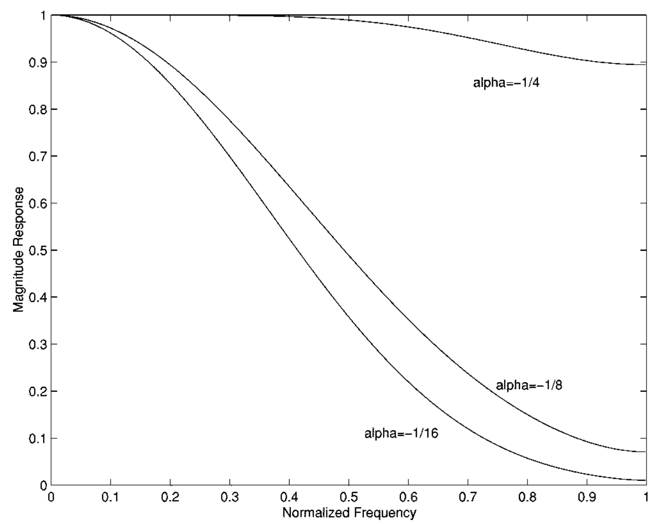

Fig. 8. Fractional delay filters with $\tau = 0.2$, and $\alpha = -0.25, -0.125, -0.0625$.

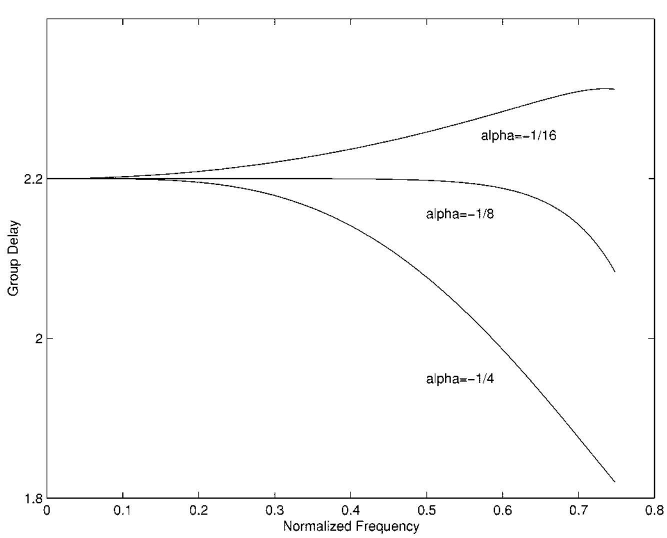

Fig. 9. Fractional delay filters with $\tau = 0.2$ and $\alpha = -0.25, -0.125, -0.0625$.

ents of filters are obtained. Design examples are also given for illustrative purpose.

REFERENCES

[1] S. Samadi, A. Nishihara, and N. Fujii, "Stancu operator and FIR digital filters," in *Proc. 8th DSP Symp.*, Sen-Dai, Japan, 1993, pp. 275–281.
[2] D. D. Stancu, "On a new positive linear polynomial operator," in *Proc. Japan Acad.*, vol. 44, no. 4, 1968, pp. 221–224.
[3] G. S. Liu and C. H. Wei, "Programmable fractional sample delay filter with Lagrange interpolation," *Electron. Lett.*, vol. 26, no. 19, pp. 1608–1610, 1990.
[4] ——, "A new variable fractional sample delay filter with nonlinear interpolation," *IEEE Trans. Circuits Syst. II*, vol. 39, pp. 123–126, Feb. 1992.
[5] C. W. Farrow, "A continuously variable digital delay element," in *Proc. IEEE Int. Symp. Circuits and Systems*, Espoo, Finland, 1988, pp. 2461–2465.
[6] R. E. Crochiene, L. R. Rabiner, and R. R. Shively, "A novel implementation of digital phase shifters," *Bell Syst. Tech. J.*, vol. 54, pp. 1497–1502, 1975.

Eigenfilter Design of Real and Complex Coefficient QMF Prototypes

Fabrizio Argenti and Enrico Del Re

***Abstract*—In this brief, we propose a new method to design pseudo-QMF prototypes to implement near perfect reconstruction (NPR) modulated filter banks. The proposed method is based on the eigenfilter approach, simple to implement, but nevertheless, very efficient in designing high attenuation filters. The method also allows to design complex coefficient prototypes that may be used to build nonuniform filter banks. Several examples of both uniform and nonuniform filter bank design are presented.**

***Index Terms*—Eigenfilter design, modulated filter banks, uniform and nonuniform subband decomposition.**

I. INTRODUCTION

The eigenfilter approach is an efficient method to design a large variety of digital filters having both finite-impulse response (FIR) [1]–[3] and infinite-impulse response (IIR) [4]–[6]. The method is flexible and easy to implement, since the problem is reduced to finding the eigenvector corresponding to the minimum eigenvalue of a positive-definite matrix. The design of complex coefficient FIR eigenfilters has been described in [2], where the problem is converted into a real coefficient design problem, and in [3], where the design involves the search of the eigenvector of a complex Hermitian symmetric positive-definite matrix. The eigenfilter approach has been applied to multirate signal processing and two-channel QMF bank design in [7] and [8], respectively.

The purpose of this brief is extending the application of the eigenfilter approach to the design of linear phase prototypes with real and complex coefficients to implement uniform and nonuniform NPR modulated filter banks. An M-channel *nonuniform* filter bank with integer decimation factors is shown in Fig. 1. If all the downsampling/upsampling factors are equal to M, we have a *uniform* filter bank.

Manuscript received July 1998; revised April 2000. This paper was recommended by Associate Editor J. Chicharo.
The authors are with the Dipartimento di Elettronica e Telecomunicazioni, Universitá di Firenze, 50139 Firenze, Italy.
Publisher Item Identifier S 1057-7130(00)06573-3.