Design of Arbitrary Cutoff 2-D Diamond-Shaped FIR Filters Using the Bernstein Polynomial

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***Abstract*—In this paper, we propose a design of the two-dimensional (2-D) linear-phase, diamond-shaped (DS) finite impulse response (FIR) filters by using the Bernstein polynomials. Although the one-dimensional (1-D) FIR filter designed by the Bernstein polynomials has been well investigated, this approach is not broadly applied to 2-D filter design yet. We present a novel method of designing the 2-D FIR DS filter. In order to be approximated by a 2-D Bernstein polynomial, the 2-D symmetrical frequency response is transformed into a new domain. The key observation is that the region of support of the transformed frequency response is not diamond-shaped. The boundary of the new region of support represents an ellipse, a circle, or a line, and is analytically derived. The resultant magnitude responses are flat in the passband and stopband.**

***Index Terms*—Bernstein polynomials, cutoff frequency, two-dimensional (2-D) diamond-shaped FIR filter.**

I. Introduction

GIVEN a function $f_d(x)$ defined on [0,1], it can be approximated by the *Bernstein polynomial*

$$f(x) = \sum_{i=0}^{m} f_d\left(\frac{i}{m}\right)\binom{m}{i}x^i(1-x)^{m-i} \tag{1}$$

where m is called the order of $f(x)$ [1]. In other words, $f(x)$ is synthesized by the kernels $x^i(1-x)^{m-i}$ with weighting coefficients obtained by $f_d(x)$ evaluated at m points evenly distributed on [0, 1]. As m approaches infinity, $f(x)$ approaches $f_d(x)$ uniformly if $f_d(x)$ is bounded and continuous on [0,1] [1]. In [4], the authors showed that the linear phase, maximally-flat (MF) lowpass FIR filters could be designed using the Bernstein polynomial where the coefficients of the linear phase MF lowpass FIR filters were analytically obtained by a method based on the Hermite interpolation in [2]. To design a linear phase low pass FIR filter using the Bernstein polynomial, the desired frequency response $H_d(\omega)$ on the ω domain is transformed into a function $f_d(x)$ on the x domain by the relation

$$x = \frac{1-\cos\omega}{2}. \tag{2}$$

Since $H_d(\omega)$ is a low pass function, which is symmetrical about $\omega = 0$ and defined on $[-\pi, \pi]$, $f_d(x)$ is also a low pass function defined on [0, 1]. The cutoff frequency ω_c of $H_d(\omega)$ corresponds with the cutoff point $x_c = (1-\cos\omega_c)/2$ of $f_d(x)$ on the x domain. Based on (1), the frequency response of the designed filter can be expressed by

$$f(x) = \sum_{i=0}^{K}\binom{m}{i}x^i(1-x)^{m-i}$$

on the x domain, or represented by

$$H(\omega) = \sum_{i=0}^{K}\binom{m}{i}\left(\frac{1-\cos\omega}{2}\right)^i\left(\frac{1+\cos\omega}{2}\right)^{m-i} \tag{3}$$

on the ω domain. K is the order of flatness at $\omega = \pi$ and determined by the following formula proposed by Herrmann [2]

$$K = m - [mx_c + 0.5]$$

where $[p]$ denotes the integer part of p. The causal filter transfer function with the frequency response expressed in (3) is

$$H(z) = \sum_{i=0}^{K}\binom{m}{i}(1-z^{-1})^{2i}(1+z^{-1})^{2m-2i}(-1)^i \tag{4}$$

which has been proposed by Miller [3].

The design method based on the Bernstein polynomial is not only applied to the design of the low pass filters but is also used for the other type of frequency responses. Design of the MF FIR Hilbert transformers was proposed in [5] with a modified Bernstein approximant, and design of the FIR notch filters was proposed in [7], where the notch frequency was regarded as the zero crossing of the frequency response. In [6], the authors proposed a closed-form design of the two-dimensional (2-D) MF half-band (HB) DS FIR filters by a similar method based on the one-dimensional (1-D) Bernstein polynomial. Since the desired frequency response of the DS filter is a 2-D function, the design problem is to approximate a desired function $f_d(x, y)$ using the $m \times n$ order Bernstein polynomial

$$f(x, y) = \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j} f_d\cdot\left(\frac{i}{m}, \frac{j}{n}\right)x^i(1-x)^{m-i}y^j(1-y)^{n-j}. \tag{5}$$

The resultant frequency response is obtained by

$$H(\omega_1, \omega_2) = f(x, y)\big|_{x=(1-\cos\omega_1)/2,\, y=(1-\cos\omega_2)/2}. \tag{6}$$

In [6], $f_d(i/m, j/n)$ is defined by

$$f_d\left(\frac{i}{m}, \frac{j}{n}\right) = \begin{cases} 1, & \text{if } i+j < m-1 \\ 0.5, & \text{if } i+j = m \\ 0, & \text{otherwise}\end{cases}$$

Manuscript received February 16, 2000. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. R. L. de Queiroz.

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Publisher Item Identifier S 1070-9908(00)09408-6.

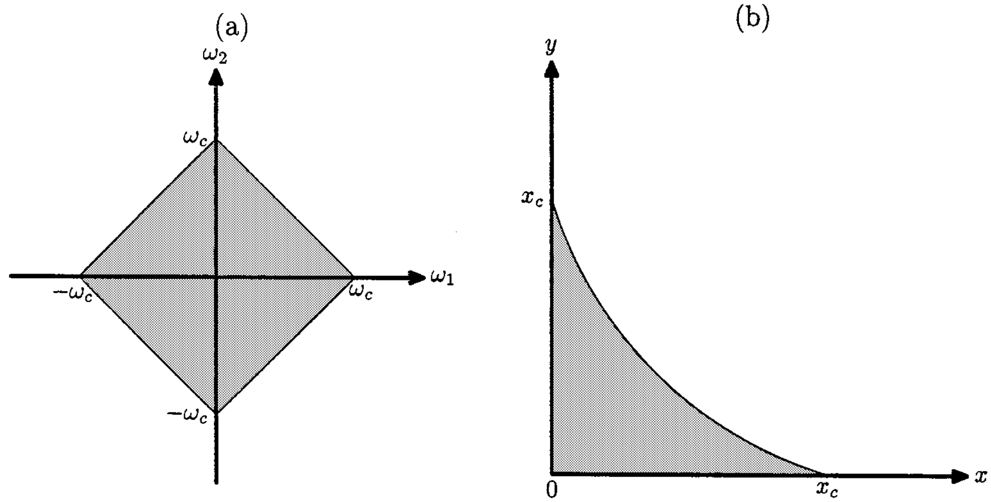

Fig. 1. (a) Support region of the ideal diamond-shaped magnitude response on the $\omega_1 - \omega_2$ plane with cutoff frequency ω_c. (b) Support region of the corresponding desired magnitude response on the $x - y$ plane. The cutoff frequency is $x_c = (1 - \cos\omega_c)/2$ on the $x - y$ plane.

and $m = n$. This specification can be regarded as a sampled HB DS function on the first quarter of the $x - y$ plane with a careful consideration at the boundary of the region of support. However, if the desired DS frequency response $H_d(\omega_1, \omega_2)$ is not HB on the $\omega_1 - \omega_2$ plane, the desired function $f_d(x, y)$ on the $x - y$ plane is neither DS nor HB. Fig. 1 shows the region of support of $H_d(\omega_1, \omega_2)$ and a suitable region of support of $f_d(x, y)$ on the $x - y$ plane. In this paper, we will investigate the design of general DS FIR filters based on the Bernstein polynomial by finding the analytical expression of the region of support of $f_d(x, y)$.

II. Design of Arbitrary Cutoff 2-D Linear Phase DS FIR Filters

The ideal frequency response of the DS filter is defined by

$$H_d(\omega_1, \omega_2) = \begin{cases} 1, & |\omega_1/\omega_{c1}| + |\omega_2/\omega_{c2}| \le 1 \\ 0, & |\omega_1/\omega_{c1}| + |\omega_2/\omega_{c2}| \ge 1 \end{cases} \quad (7)$$

In this paper, we consider the case of $\omega_{c1} = \omega_{c2} = \omega_c$ only. Therefore, the boundary of the support region can be expressed by

$$|\omega_1| + |\omega_2| = \omega_c. \quad (8)$$

Since we will use the relations of

$$x = \frac{1 - \cos\omega_1}{2}, \quad y = \frac{1 - \cos\omega_2}{2} \quad (9)$$

or, equivalently

$$\omega_1 = \arccos(1 - 2x), \quad \omega_2 = \arccos(1 - 2y) \quad (10)$$

for transforming the ideal frequency response $H_d(\omega_1, \omega_2)$ into the desired function $f_d(x, y)$ on the $x - y$ domain, the boundary of the support region of $f_d(x, y)$ has to be transformed accordingly. Note that $0 \le x, y \le 1$. Because $H_d(\omega_1, \omega_2)$ is quarter-symmetrical about the origin, it is sufficient to consider the boundary on the first quarter, i.e.,

$$\omega_1 + \omega_2 = \omega_c. \quad (11)$$

Substituting (10) for ω_1 and ω_2 in the above equation, the boundary function of the support region is expressed by

$$\arccos(1 - 2x) + \arccos(1 - 2y) = \omega_c. \quad (12)$$

on the $x - y$ plane. That is, the support region showed on Fig. 1(b) is defined by

$$\arccos(1 - 2x) + \arccos(1 - 2y) \le \omega_c. \quad (13)$$

The boundary function in (12) could be simplified further. We first express (12) as

$$1 - 2y = \cos(\omega_c - \arccos(1 - 2x))$$

and then use the addition formula of cosine function to obtain

$$1 - 2y = (1 - 2x)\cos\omega_c + \sin\omega_c \sin\arccos(1 - 2x).$$

Since $\sin\arccos(1 - 2x) = \sqrt{1 - (1 - 2x)^2}$, the above equation can be expressed by

$$x^2 - 2xy\cos\omega_c + y^2 - 2(x + y)x_c + x_c^2 = 0 \quad (14)$$

for $0 \le x \le x_c$ where $x_c = (1 - \cos\omega_c)/2$. Since the discriminant

$$\Delta = (2\cos\omega_c)^2 - 4 = 4(\cos^2\omega_c - 1) \le 0$$

Equation (14) represents an ellipse, a circle, or a line [8]. Fig. 2 shows the boundary functions on the first quarter. Fig. 2(a) shows the boundary functions in (11) with various cutoff frequencies on the $\omega_1 - \omega_2$ plane, and Fig. 2(b) shows the corresponding boundary functions in (14) on the $x - y$ plane.

If the cutoff frequency $\omega_c = \pi$, the discriminant $\Delta = 0$, and (14) represents a section of line on the first quarter of the $x - y$ plane. In fact, (14) is reduced to a very simple linear equation

$$x + y = 1$$

for $0 \le x \le 1$. This case is the very special one that both $H_d(\omega_1, \omega_2)$ and $f_d(x, y)$ are of DS support regions.

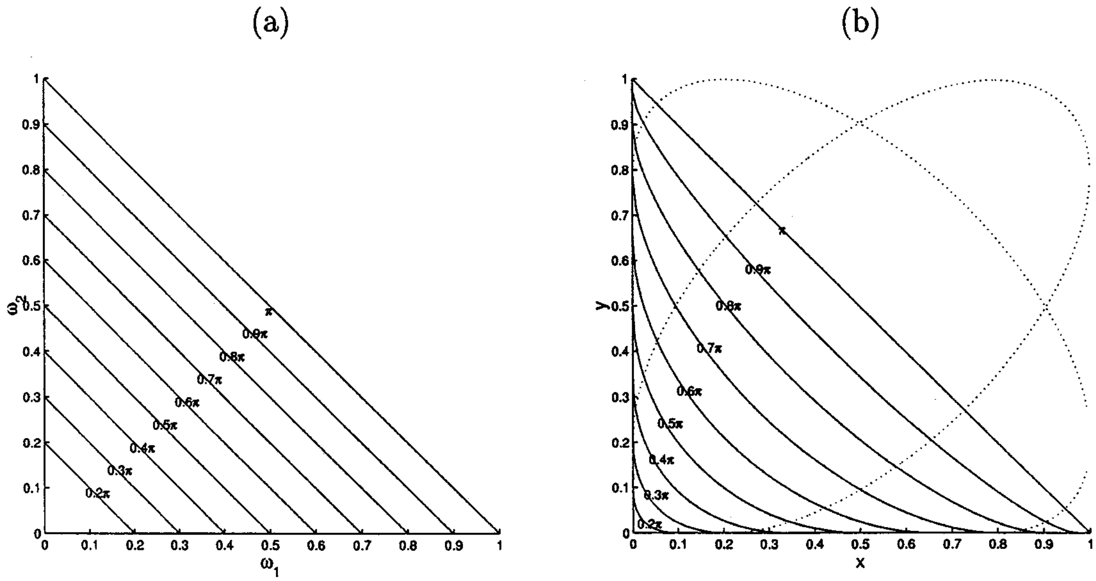

Fig. 2. (a) Boundaries of the support regions of the ideal diamond-shaped magnitude responses on the $\omega_1 - \omega_2$ plane with various cutoff frequencies. (b) Boundaries of the support regions of the corresponding desired magnitude responses on the $x - y$ plane. The two dotted-line ellipses corresponding to $\omega_c = 0.3\pi$ and $\omega_c = 0.7\pi$ indicate that these boundaries are segments of ellipses.

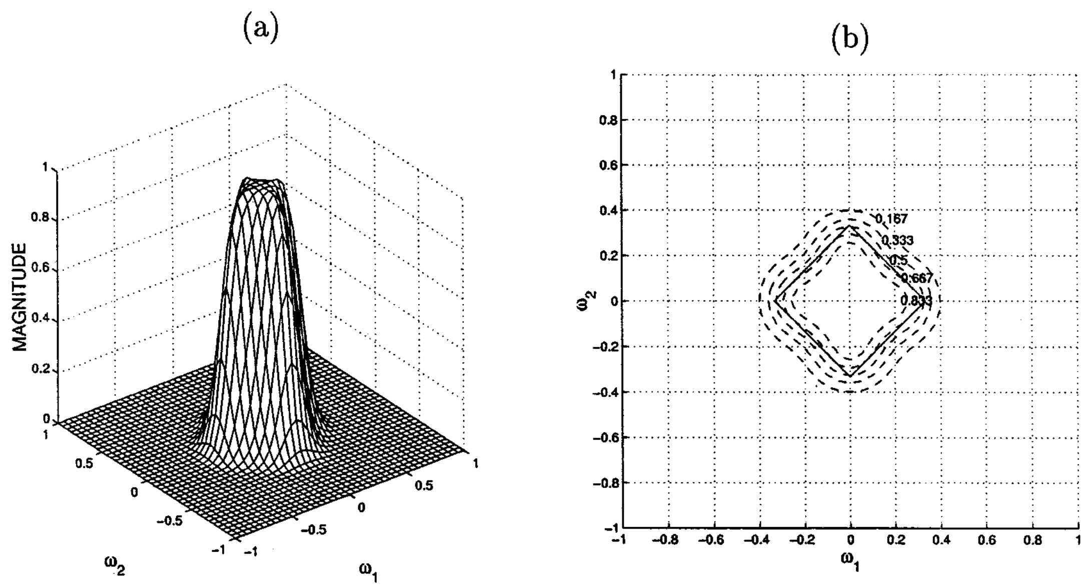

Fig. 3. Design of the 41 × 41 DS linear-phase FIR filter with $\omega_c = \pi/3$. (a) Perspective plot of the magnitude response and (b) a contour plot of the magnitude response where the square shown in solid line represents the boundary of the support region of the desired magnitude response.

The steps in the design of linear-phase DS FIR filters are summarized as follows.

Step 1) Specify the filter order N and the cutoff frequency ω_c. N must be an odd integer. $m = (N-1)/2$.

Step 2) Evaluate the sampled version of the desired function $f_d(i/m, j/m)$ by

$$f_d(i/m, j/m) = \begin{cases} 1, & \arccos(1-2i/m) + \arccos(1-2j/m) < \omega_c \\ 0, & \arccos(1-2i/m) + \arccos(1-2j/m) \geq \omega_c \end{cases} \tag{15}$$

for $0 \leq i, j \leq m$.

Step 3) Evaluate the designed frequency response by (5) and (6).

Step 4) Calculate the impulse response by substituting $(z_1 + z_1^{-1})/2$ and $(z_2 + z_2^{-1})/2$ for $\cos\omega_1$ and $\cos\omega_2$ in $H(\omega_1, \omega_2)$ obtained in Step 3.

III. Design Examples

In this section, we will illustrate the proposed method by two design examples. The filter order is 41 × 41 in both two examples, i.e., $m = 20$.

Example 1: The cutoff frequency is $\omega_c = \pi/3$ in this example. Fig. 3(a) shows the resultant magnitude response and Fig. 3(b) shows the contour plot of the magnitude response. The

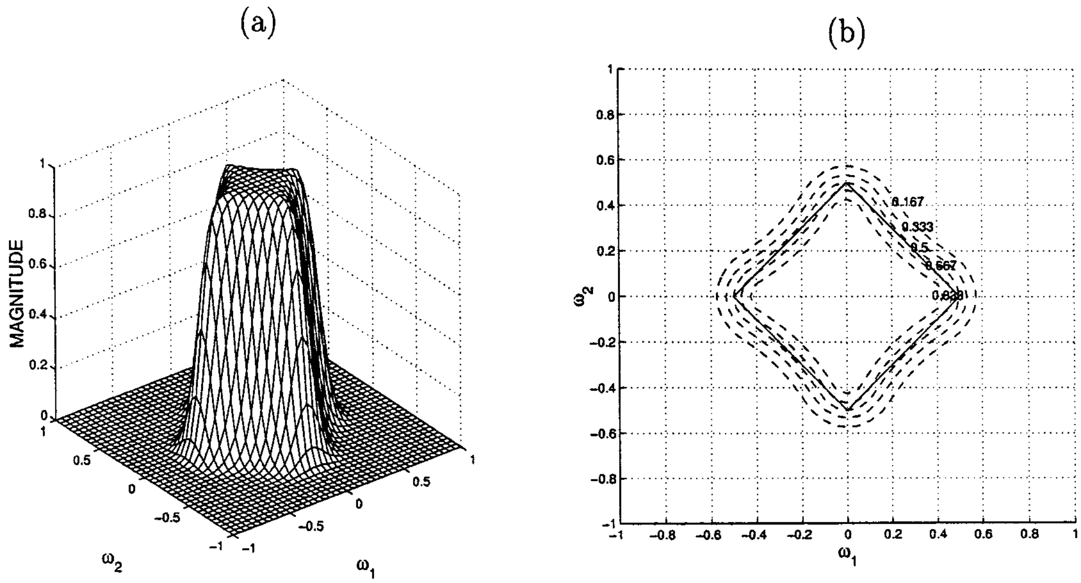

Fig. 4. Design of the 41 × 41 DS linear-phase FIR filter with $\omega_c = \pi/2$. (a) Perspective plot of the magnitude response and (b) a contour plot of the magnitude response where the square shown in solid line represents the boundary of the support region of the desired magnitude response.

magnitude response is flat. However, the transition band is wide since the Bernstein polynomial converges slowly.

Example 2: In this example, the desired cutoff frequency is $\omega_c = \pi/2$. Fig. 4(a) and (b) shows the magnitude response and the contour plot, respectively. The magnitude response is still flat. It is obvious that the boundary in the example is more linear than the one in Example 1, since the nonzero elements in $f(i/m, j/n)$ in this example are more than the ones in Example 1.

IV. Conclusion

In this paper, we proposed a design of the linear-phase DS FIR filters with arbitrary cutoff frequency using the Bernstein polynomial. The ideal frequency response is transformed into another function whose sampled values are used for the weighting coefficients. This function is approximated by a 2-D Bernstein polynomial with these weighting coefficients. We have analytically derived that the boundary of the transformed function region of support is an ellipse, a circle, or a line. Transforming the function back to a function on the frequency domain, the resultant filter is obtained. Design examples suggest that the resultant frequency responses have flat passband and stopband regions.

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