A Two-Channel Nonuniform Perfect Reconstruction Filter Bank With Irrational Down-Sampling Factors

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***Abstract*—Most of the existing methods designed to implement fractional interpolation and decimation are limited by rational scaling factors such as L/M, where L and M are positive integers. The general procedure is usually done with up-sampling by L first, and then followed by down-sampling by M. This scheme, however, requires two steps and higher sampling rates to fulfill and is not capable in dealing with irrational scaling factors, which cannot be represented by L/M. In this paper, we propose a new method to solve the above two difficulties by a single step. Furthermore, a new structure of a two-channel nonuniform perfect reconstruction filter bank is derived using this new method, and the experimental results are presented.**

***Index Terms*—Decimation, interpolation, irrational scaling, nonuniform filter bank, perfect reconstruction.**

I. Introduction

In a conventional filter bank (FB) system, the scaling operation is very important in that it controls the sampling rate to meet the system requirements. However, as long as a discrete time system is considered, most previous works fail to realize a scaling operation with irrational factors. Although rational scaling can approximate any irrational scaling to reasonable accuracy, i.e., $r = M/N$ [1], M and N might have to be very large with higher sampling rates.

For example, given a discrete time signal $x[n]$, how can we scale it by $\sqrt{2}$? Although it seems quite difficult to be realized in time domain, Rao [2]–[4] developed a practical method to accomplish it using the concept of frequency domain warping. This method, however, requires the computation of inverse discrete-time Fourier transform (IDTFT), which lacks a closed-form formula to avoid the computational approximation. Furthermore, it is not well suited in a perfect reconstruction (PR) FB in practice.

Partly motivated by the frequency warping concept, we propose a new method devoted to maintaining the linearity while doing the frequency warping. Fortunately, this leads to a concise closed form formula in time domain, which is invulnerable to any computational approximation. It also works very well in the proposed PR FB structure.

Manuscript received April 17, 2004; revised June 22, 2004. This work was supported by the National Science Council, R. O. C., under Contracts NSC93-2219-E-002-004 and NSC93-2752-E-002-006-PAE. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. See-May Phoong.

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Digital Object Identifier 10.1109/LSP.2004.840865

The paper is organized as follows. Section II provides the derivation of the new scaling method, and the structure of the irrational scaling PR FB is shown in Section III. Section IV presents the simulation results of the above scheme. Finally, the conclusion is made in Section V.

II. Proposed Irrational Scaling Algorithm

Given a discrete time signal $x[n]$, suppose its scaled version is $x_a[n]$, where a is the scaling factor, and $a \in \Re$, $a > 0$. In this way, $a > 1$ is equivalent to interpolation, and $0 < a < 1$ is for decimation. Referring to the scaling property of continuous time Fourier transform (CTFT) as below

$$x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(j\omega)$$
$$x\left(\frac{t}{a}\right) \overset{\text{CTFT}}{\longleftrightarrow} |a|X(j\omega a) \tag{1}$$

we could reasonably infer (1) to its DTFT counterpart, i.e.,

$$x[n] \overset{\text{DTFT}}{\longleftrightarrow} X(e^{j\omega})$$
$$x_a[n] \overset{\text{DTFT}}{\longleftrightarrow} |a|X(e^{j\omega a}). \tag{2}$$

Because $X(e^{j\omega})$ is a continuous time periodic function with period 2π, $X(e^{j\omega a})$, by definition, exists whenever $a \in \Re$.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

By (2)

$$x_a[n] = \text{IDTFT}\{aX(e^{j\omega a})\} = \frac{a}{2\pi}\sum_{k=-\infty}^{\infty} x[k]\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} e^{j(n-ak)\omega}d\omega. \tag{3}$$

We now divide (3) into two disjoint cases: one being $k = n/a$ and the other being $k \neq n/a$. For $k = n/a$

$$\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} e^{j(n-ak)\omega}d\omega = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} d\omega = \frac{2\pi}{a}. \tag{4}$$

For $k \neq n/a$

$$\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} e^{j(n-ak)\omega}d\omega = \frac{2\pi}{a}\text{sinc}\left(\frac{n}{a} - k\right). \tag{5}$$

TABLE I
IRRATIONAL SCALING FORMULAS

Interpolation	$a>1$	$\sum_{k=-\infty}^{\infty} x[k]\mathrm{sinc}\left(\frac{n}{a}-k\right)$
Decimation	$a<1$	$a\sum_{k=-\infty}^{\infty} x[k]\mathrm{sinc}(n-ak)$

From (3)–(5)

$$x_a[n] = \frac{a}{2\pi}\left[\sum_{k=-\infty,k=\frac{n}{a}}^{\infty} x[k]\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} e^{j(n-ak)\omega}d\omega + \sum_{k=-\infty,k\neq\frac{n}{a}}^{\infty} x[k]\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} e^{j(n-ak)\omega}d\omega\right]$$
$$= \sum_{k=-\infty,k=\frac{n}{a}}^{\infty} x[k] + \sum_{k=-\infty,k\neq\frac{n}{a}}^{\infty} x[k]\mathrm{sinc}\left(\frac{n}{a}-k\right)$$
$$= \sum_{k=-\infty}^{\infty} x[k]\mathrm{sinc}\left(\frac{n}{a}-k\right). \quad (6)$$

Note that we have changed the upper bound and lower bound of the definite integral of (3) from $[-\pi \sim +\pi]$ to $[-\pi/a \sim +\pi/a]$ for the purpose of removing the high-frequency spectrum. The unwanted spectrum is introduced when scaling a periodic signal to a different new period, which is, in this case, from 2π to $2\pi/a$. In other words, changing the integral upper/lower bound is equivalent to applying an ideal lowpass filter after up-sampling. We also ignore the absolute value of $|a|$ because $a > 0$ throughout this paper. Equation (6) is only suitable in the case where $a > 1$, i.e., interpolation. Since the new period of $X(e^{j\omega a})$ due to interpolation equals to $2\pi/a$, which is smaller than 2π in this case, we need only preserve those frequency components inside $[-\pi/a \sim +\pi/a]$ for latter processing.

For the case where $0 < a < 1$, i.e., decimation, the upper and lower bound should be changed back to $[-\pi \sim +\pi]$ again to avoid the aliasing due to down-sampling. This is equivalent to applying an ideal lowpass filter before down-sampling. Here, the new period of $X(e^{j\omega a})$ due to decimation is larger than 2π, which forces us to abandon those frequency components outside 2π or, otherwise, they will inevitably have aliasing images reflected into the basic 2π period and damage the algorithms.

$$x_a[n] = \mathrm{IDTFT}\left\{aX(e^{j\omega a})\right\}$$
$$= \frac{a}{2\pi}\sum_{k=-\infty}^{\infty} x[k]\int_{-\pi}^{\pi} e^{j(n-ak)\omega}d\omega$$
$$= a\sum_{k=-\infty}^{\infty} x[k]\mathrm{sinc}(n-ak) \quad (7)$$

For convenience, the above two formulas are summarized in Table I.

This is not the only way to derive irrational scaling formulas. Adams [5] has given a different approach to doing irrational interpolations from the view point of sinc-interpolated discrete time signals. However, using this method presents difficulties in doing irrational decimations without aliasing. It is a surprise that after a modification of the down-sampling counterpart, identical formulas can be achieved, as those shown in Table I. The details are described in the Appendix.

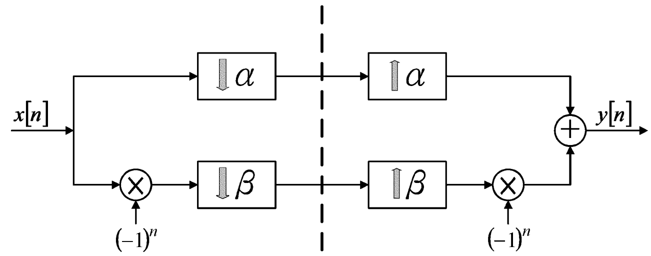

Fig. 1. Irrational scaling two-channel perfect reconstruction filter bank structure.

TABLE II
IRRATIONAL UP AND DOWN SAMPLING KERNELS

⇑	a	Up Sample by "a"	$a>1$	$\sum_{k=-\infty}^{\infty} x[k]\mathrm{sinc}\left(\frac{n}{a}-k\right)$
⇓	a	Down Sample by "a"		$\frac{1}{a}\sum_{k=-\infty}^{\infty} x[k]\mathrm{sinc}\left(n-\frac{k}{a}\right)$

III. IRRATIONAL SCALING PERFECT RECONSTRUCTION FILTER BANK STRUCTURE

The fundamental structure of the two-channel nonuniform PR FB using irrational scaling factors is shown in Fig. 1, where α, $\beta \in R$, α, $\beta > 1$, and $(1/\alpha) + (1/\beta) = 1$. Note that there is no need to design the prefilters and post-filters in the PR FB structure since those filters have already been merged into the down-sampling and up-sampling kernels, respectively, which could be revealed from the derivations in the previous section. The equivalent prefilter addressed is the ideal lowpass filter whose cutoff frequencies are located at $\omega_c = \pm\pi$, and the equivalent post-filter is the ideal lowpass filter whose cut-off frequencies are located at $\omega_c = \pm(\pi/a)$. The basic goal of the proposed structure is to perfectly reconstruct the output such that $y[n] = x[n]$, under any circumstances, even if α, β are irrational numbers.

In Fig. 1, the up-sampling and down-sampling kernel is almost identical to the results in Section II, except for small modifications by replacing a by $1/a$ in the decimation case, which we rearrange in Table II.

IV. EXPERIMENTAL RESULTS

To test the proposed PR FB structure in Fig. 1, we take a 256-point normalized Gaussian random signal with zero mean and unit variance as input sequence $x[n]$, choosing $\alpha_1 = \sqrt{2}$ and $\alpha_2 = 2 + \sqrt{2}$ to construct a two-channel nonuniform PR FB. Since the index k in Table II has a range from minus infinity to infinity, which is not feasible in practical implementations, here, we choose $k = [-256 \sim +256]$ in this experiment and observe the reconstruction errors due to truncation of k. After reconstruction, the mean absolute value of the reconstruction error is only 0.0026, which is showed in Fig. 2(e), together with

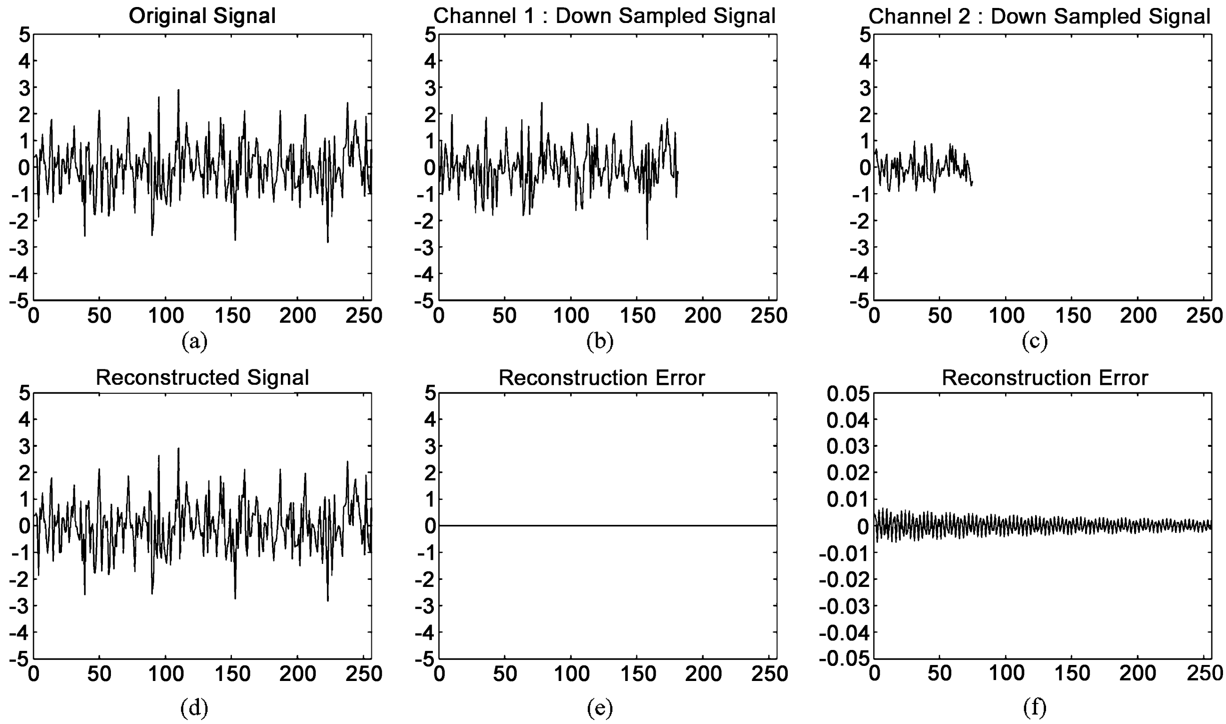

Fig. 2. (a) Original input signal. (b), (c) Down-sampled signals in channels 1 and 2, respectively. (d) Perfectly reconstructed output signal. (e) Reconstruction error. (f) Scaled version of (e) by 100 times.

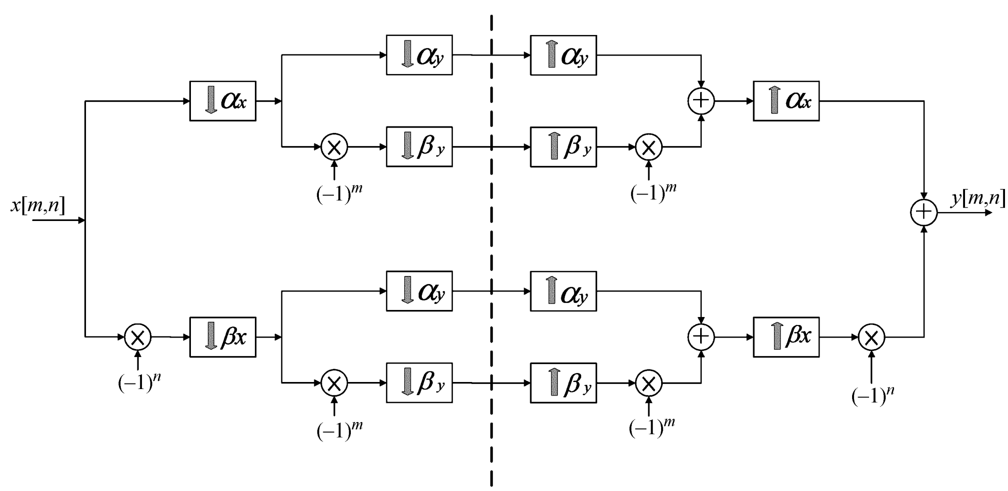

Fig. 3. Two-dimensional irrational scaling perfect reconstruction filter bank structure.

the corresponding magnified version of 100 times in Fig. 2(f). Note that these errors are introduced by truncation of k, which makes the equivalent lowpass filters no longer ideal. However, the errors keep decreasing as k grows larger, and the FB structure virtually becomes a perfect reconstruction as $k \to \infty$.

Although the above formulas are designed for one-dimensional signals, they could be well extended to work with two-dimensional images via two separable scalings in horizontal and vertical directions. The modified structure is shown in Fig. 3 for the four-channel nonuniform PR FB case.

As a result, we take a gray level $256 * 256$ "Lena" image as an input, choosing $\alpha_x = \sqrt{3}$, $\beta_x = (3+\sqrt{3})/2$ and $\alpha_y = \sqrt{2}$, $\beta_y = 2+\sqrt{2}$ to produce a rectangular down-sampled image and using the system in Fig. 3 to reconstruct the original image perfectly. The results are shown in Fig. 4. As we can see, the down-sampled image in the upper channel is of $182 * 148$ $(\lceil 256/\sqrt{2} \rceil * \lceil 256/\sqrt{3} \rceil)$ resolutions. The reconstructed image is of $256 * 256$ resolutions and is almost identical to the original image. The mean absolute value of the reconstruction error is about 0.05.

V. Conclusions

In this paper, a nonuniform perfect reconstruction filter bank using irrational scaling factor is proposed. The advantage of the new scaling algorithm is that it is a concise closed form formula in time domain, and it completes an irrational scaling in a single step with removing unwanted images and avoiding aliasing simultaneously. The experimental results have shown the perfect reconstruction after several irrational scalings.

(a) (b) (c) (d)

Fig. 4. (a) Original input image. (b) Down sampled image in the upper channel. (c) Perfectly reconstructed output image. (d) Reconstruction error.

APPENDIX

At first, we consider $x[n]$ as a nonaliasing sampled version of a continuous time signal $x(t)$, i.e.,

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin\left[\frac{\pi(t-nT)}{T}\right]}{\frac{\pi(t-nT)}{T}} = \sum_{n=-\infty}^{\infty} x[n]\mathrm{sinc}(t-n)$$

where we have assumed the sampling period $T = 1$. Choosing the new sampling period $T' = 1/a$, we get

$$x_a[n] = x\left(\frac{n}{a}\right) = \sum_{k=-\infty}^{\infty} x[k]\mathrm{sinc}\left(\frac{n}{a} - k\right). \quad (8)$$

For $a \geq 1$, (8) is the same as (6). On the other hand, as for $a < 1$, we prefilter $x(t)$ with an ideal lowpass filter $H(j\omega)$ to avoid the aliasing effect caused by decimation, where

$$H(j\omega) = \begin{cases} 1, & \text{when } |\omega| \leq a\pi \\ 0, & \text{else} \end{cases}$$

and

$$h(t) = \text{ICTFT}\{H(j\omega)\} = a\,\mathrm{sinc}\,(at)$$

is the impulse response of $H(j\omega)$.

$$\begin{aligned} x_{LP}(t) &= x(t) \otimes h(t) \\ &= \left\{ \sum_{n=-\infty}^{\infty} x[n]\mathrm{sinc}(t-n) \right\} \otimes h(t) \\ &= a \sum_{n=-\infty}^{\infty} x[n]\mathrm{sinc}(at - an) \end{aligned}$$

where $x_{LP}(t)$ denotes the lowpass filtered signal of $x(t)$. The last step is to resample $x_{LP}(t)$ with new period $T' = 1/a$, i.e.,

$$x_a[n] = x_{LP}\left(\frac{n}{a}\right) = a \sum_{k=-\infty}^{\infty} x[k]\mathrm{sinc}(n - ak). \quad (9)$$

Note that (9) is the same as (7), which completes the derivation of the alternative approach.

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