Multiport Scattering Matrix Measurement Using a Reduced-Port Network Analyzer

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***Abstract*—A novel method for acquiring the scattering matrix of an n-port network from measurements using a reduced-port network analyzer is developed. This method can obtain the scattering matrix of a nonreciprocal or reciprocal n-port network with the use of a three- or two-port network analyzer. The formulation of this method considers the imperfection of terminators used in the measurement, and only two of the terminators are required to be known. Experimental results from a four-port microstrip circuit show good accuracy using the developed method.**

***Index Terms*—Multiport network, scattering matrix measurement.**

I. Introduction

THE multiport scattering matrix measurement of an n-port network may need to design a special multiport network analyzer [1], [2] or to use a two-port vector network analyzer with all other $(n-2)$-ports of the test network connected with perfect terminators based on the definition of the scattering matrix. The multiport network analyzer requires specific calibration methods, e.g., [2]–[5]. On the other hand, the instruments and calibration methods needed to measure a two-port scattering matrix are well developed. Agilent 8510C and Anritsu ME7808A are two typical two-port vector network analyzers and the 16-term error model is the most general approach for calibration [6]–[8]. In practice, the imperfect terminators must be taken into consideration when using a two-port network analyzer to measure an n-port network accurately. The rigorous methods for solving the scattering matrix of a multiport network using a two-port vector network analyzer with known terminators were described in [9]–[12]. A multiport network analyzer using a two-port network analyzer with a calibration method was given in [13]. It uses the method proposed in [9] to reconstruct the n-port scattering matrix from two-port scattering matrices. Currently, multiport vector analyzers are available, e.g., Agilent N4446A consists of an 8720ES two-port vector network analyzer and an N4418A four-port test set for port extension.

In [9], the n-port scattering matrix is calculated directly from C_2^n sets of two-port scattering matrices. In [14], Lin and Ruan proposed an approach from the port reduction point-of-view. As a terminator is connected to an n-port network, the order of measured ports is reduced by one. With their method, the n-port scattering matrix can be reconstructed from n sets of the reduced $(n-1)$-port scattering matrix by connecting n known terminators to each port one at a time. This port reduction process can be continued to reduce the port order, and the resulting minimal reduced port order is three. With the idea of reconstructing an n-port scattering matrix from $(n-1)$-port scattering matrices, two-port reduction methods (PRMs) are proposed in [15]. They are called type-I and type-II PRMs. Only three terminators are required in each step of port reduction in both methods, and the order of measured ports can be reduced to two for both reciprocal and nonreciprocal networks.

In this paper, we present a new formulation of the PRM, called a type-III PRM, for solving the n-port scattering matrix from n sets of the reduced $(n-1)$-port scattering matrix by connecting n terminators to each port one at a time. The minimal order of measured ports can be reduced to two for a reciprocal n-port network. In addition, only two of the n terminators are required to be known instead of n in [9] or [14].

In the following sections, the basic formulation of the developed type-III PRM is described in Section II. Experimental results of four-port reciprocal and nonreciprocal circuits are given in Section III. The measured results are verified and compared with those measured with the assumption of using perfect termination. The accuracy of experimental results is also analyzed. Finally, a conclusion is given in Section IV.

II. Formulation

For an n-port network, when a terminator with a reflection coefficient of Γ_k is connected at the kth port, the relationship between $S_{ij}^{(k)}$ of this reduced $(n-1)$-port network and S_{ij} of the n-port network is given as

$$S_{ij}^{(k)} = S_{ij} + \frac{S_{ik}S_{kj}\Gamma_k}{1 - S_{kk}\Gamma_k}. \tag{1}$$

In (1), the port numbering is the same for the n-port network and the $(n-1)$-port network. This equation means that there are a total of $n(n-1)^2$ values of $S_{ij}^{(k)}$ to be measured.

In the following, we will first present the formulation for diagonal elements in the scattering matrix of an n-port network, and then derive the formulation for off-diagonal elements.

A. Diagonal Elements

From (1), the ith diagonal element can be written as

$$S_{ii}^{(k)} = S_{ii} + \frac{S_{ik}S_{ki}\Gamma_k}{1 - S_{kk}\Gamma_k}, \qquad i \neq k. \tag{2}$$

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As given in [14], the matrix equation to relate the diagonal elements of an n-port scattering matrix and the elements of reduced $(n-1)$-port scattering matrices is

$$[R_{n\times n}]\,[S_d] = [S_r] \tag{3}$$

with (4)–(6), shown at the bottom of this page, for $n \geq 3$. In (3), $[R_{n\times n}]$ and $[S_r]$ are matrices related to the reduced $(n-1)$-port scattering parameters and the reflection coefficients of terminators used, whereas $[S_d]$ contains the diagonal elements to be solved.

In Appendix A, the determinant of $[R_{n\times n}]$ is proven to be zero. This means the elements of S_{ii} in (4) cannot be solved explicitly, but can be expressed in a polynomial form in terms of one element, such as S_{11}. The equations to solve S_{11} are given in the following derivation of the formulation for solving off-diagonal elements of S_{ij}. In addition, $\det[R_{n\times n}] = 0$ provides a relationship between the S_{ii}'s of $(n-1)$-port networks and the reflection coefficients of all terminators. One can then utilize this relation in the following two ways. One is to verify the measurement consistency by calculating $\det[R_{n\times n}]$ with known values of S_{ii}'s and Γ_i's. The determinate of (6) should be close to zero if all the values are correct. Alternately, one can use this equation to reduce the number of terminators that must be known in the measurement. It is proven in Appendix B that only two of the terminators used to reduce the measured ports are required to be known instead of n, as in [9] or [14]. This then relaxes the measurement requirements.

B. Off-Diagonal Elements

As a terminator is connected at the jth port, (2) can be written as

$$\left(S_{ii}^{(j)} - S_{ii}\right)\left(\frac{1}{\Gamma_j} - S_{jj}\right) = S_{ij}S_{ji}. \tag{7}$$

By substituting (7) into (10) in [14] given by

$$\begin{aligned} S_{ji}^{(k)}S_{ij} + S_{ij}^{(k)}S_{ji} = {} & S_{ji}^{(k)}S_{ij}^{(k)} + S_{ij}S_{ji} \\ & - \left(S_{ii}^{(k)} - S_{ii}\right)\left(S_{jj}^{(k)} - S_{jj}\right), \\ & i,\, j = 1,\, 2,\, \ldots\, n; \quad i \neq j; \quad i,\, j \neq k \end{aligned} \tag{8}$$

one can obtain

$$\begin{aligned} S_{ji}^{(k)}S_{ij} + S_{ij}^{(k)}S_{ji} = {} & S_{ji}^{(k)}S_{ij}^{(k)} + \frac{S_{ii}^{(j)}}{\Gamma_j} - S_{ii}^{(k)}S_{jj}^{(k)} \\ & + \left(S_{jj}^{(k)} - \frac{1}{\Gamma_j}\right)S_{ii} + \left(S_{ii}^{(k)} - S_{ii}^{(j)}\right)S_{jj}. \end{aligned} \tag{9}$$

As described in Section II-A, all the diagonal elements can be expressed in terms of S_{11} (or designated as t in the following for simplicity). The right-hand side of (9) is then a polynomial of t with the order of one.

For $n \geq 4$, a different port h that is different from i, j, and k can be taken and (9) becomes

$$\begin{aligned} S_{ji}^{(h)}S_{ij} + S_{ij}^{(h)}S_{ji} = {} & S_{ji}^{(h)}S_{ij}^{(h)} + \frac{S_{ii}^{(j)}}{\Gamma_j} - S_{ii}^{(h)}S_{jj}^{(h)} \\ & + \left(S_{jj}^{(h)} - \frac{1}{\Gamma_j}\right)S_{ii} + \left(S_{ii}^{(h)} - S_{ii}^{(j)}\right)S_{jj}. \end{aligned} \tag{10}$$

From (9) and (10), one can then express the off-diagonal elements S_{ij} and S_{ji} in terms of t. In other words, S_{ij} and S_{ji} are polynomials of t with the order of one. Let

$$S_{ii} = f_1(t) \quad S_{jj} = f_2(t) \quad S_{ij} = f_3(t) \quad S_{ji} = f_4(t) \tag{11}$$

then (7) becomes

$$\left(S_{ii}^{(j)} - f_1(t)\right)\left(\frac{1}{\Gamma_j} - f_2(t)\right) = f_3(t)f_4(t) \tag{12}$$

which is a second-order polynomial equation of t. Two solutions of t (or S_{11}) can be found. As t is solved, all the diagonal and off-diagonal elements can be calculated from (11).

Note there are two possible solutions of S_{ij} for the two different values of t. To determine the correct value of t, one can substitute the resulting values into

$$\Delta = S_{ik}^{(j)} - S_{ik} - \frac{S_{ij}S_{jk}\Gamma_j}{1 - S_{jj}\Gamma_j}. \tag{13}$$

The correct value of t then gives a very small value of Δ based on (1).

The derivation of the PRM formulation given above by reducing the order of n ports to be $n-1$ is valid for a nonreciprocal network in general. One can repeat this port reduction process to reduce the measured ports to a minimal order of three,

$$[S_d] = [\,\Gamma_1 S_{11} \quad \Gamma_2 S_{22} \quad \cdots \quad \Gamma_n S_{nn}\,] \tag{4}$$

$$[S_r] = \left[\,\Gamma_1 S_{11}^{(2)} - \Gamma_2 S_{22}^{(1)} \quad \Gamma_2 S_{22}^{(3)} - \Gamma_3 S_{33}^{(2)} \quad \cdots \quad \Gamma_1 S_{11}^{(n)} - \Gamma_n S_{nn}^{(1)}\,\right]^t \tag{5}$$

and

$$[R_{n\times n}] = \begin{bmatrix} 1-\Gamma_2 S_{22}^{(1)} & -\left(1-\Gamma_1 S_{11}^{(2)}\right) & 0 & \cdots & 0 & 0 \\ 0 & 1-\Gamma_3 S_{33}^{(2)} & -\left(1-\Gamma_2 S_{22}^{(3)}\right) & \cdots & 0 & 0 \\ 0 & 0 & 1-\Gamma_4 S_{44}^{(3)} & \cdots & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-\Gamma_n S_{nn}^{(n-1)} & -\left(1-\Gamma_{n-1} S_{n-1\,n-1}^{(n)}\right) \\ 1-\Gamma_n S_{nn}^{(1)} & 0 & 0 & \cdots & 0 & -\left(1-\Gamma_1 S_{11}^{(n)}\right) \end{bmatrix} \tag{6}$$

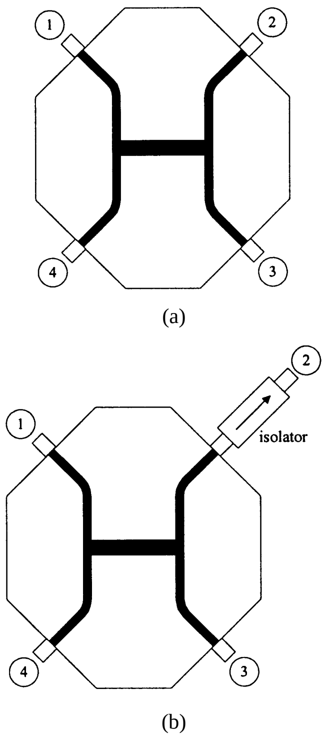


Fig. 1. Circuit layouts of: (a) reciprocal four-port network (or "R" network) and (b) nonreciprocal four-port network (or "NR" network).

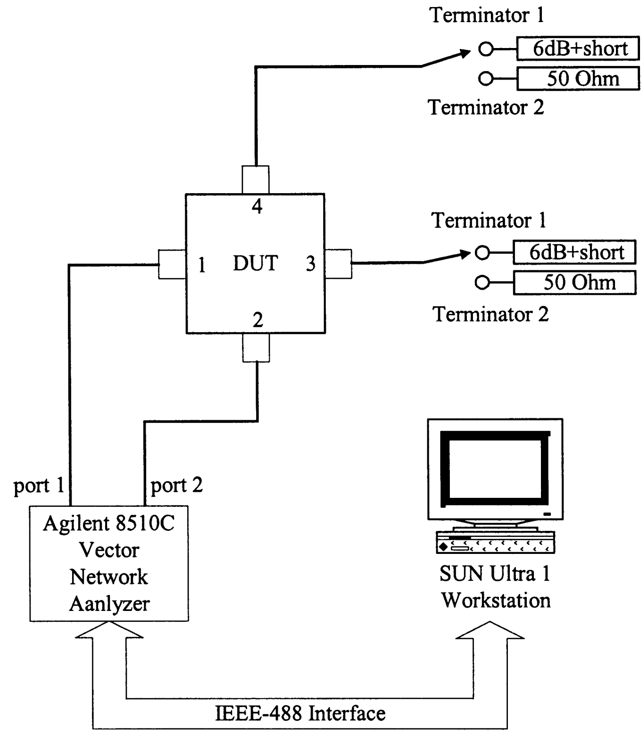


Fig. 2. Measurement arrangement of a four-port network (DUT) with its ports 1 and 2 connected to an Agilent 8510C.

as given in [14]. In other words, the n-port scattering matrix can be reconstructed from the measurement of terminated three-port networks. In addition, the terminators used to reduce the measured ports can be partially known. In the following, we will show that, for a reciprocal network, the measured ports can be further reduced to be two.

C. Reciprocal Network

For a reciprocal network

$$S_{ij} = S_{ji} \text{ and } S_{ij}^{(k)} = S_{ji}^{(k)}. \tag{14}$$

By introducing (14) into (9), it becomes

$$2S_{ij}^{(k)} S_{ij} = S_{ij}^{(k)^2} + \frac{S_{ii}^{(j)}}{\Gamma_j} - S_{ii}^{(k)} S_{jj}^{(k)} + \left(S_{jj}^{(k)} - \frac{1}{\Gamma_j} \right) S_{ii} + \left(S_{ii}^{(k)} - S_{ii}^{(j)} \right) S_{jj} \tag{15}$$

which means S_{ij} can be expressed as a polynomial of t with the order of one. Therefore, by letting $S_{ij} = f_5(t)$ and substituting it into (7), it becomes a second-order polynomial equation of t. Similarly, one can solve the correct value of t and reconstruct the n-port scattering matrix using the same procedure as in the nonreciprocal case.

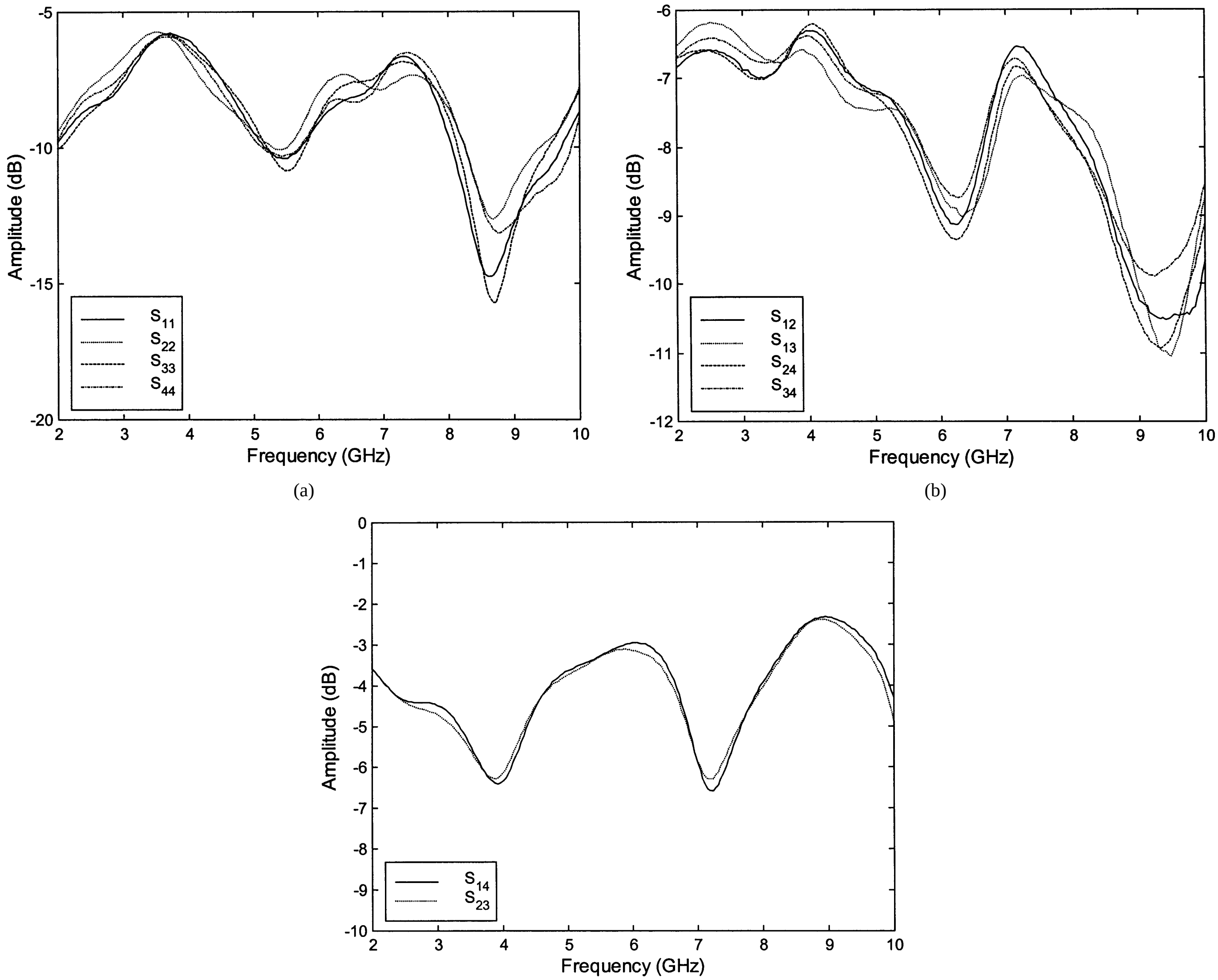
For a reciprocal network, one can find that all formulations given above are valid until n is three. This means that the minimal order of measured ports can be reduced to be two. In other words, one can use the derived formulation of the type-III PRM to acquire the scattering matrix of a reciprocal n-port network using a two-port network analyzer.

TABLE I
PORT DESCRIPTION OF THE SCATTERING MATRIX IN THE PRM PROCESS

Ports of resulting four-port S-matrix	Ports of intermediate three-port S-matrix	Ports of measured two-port S-matrix	Ports of actually measured two-port S-matrix
1234	123_1	12_11	12_11
		13_11	13_11
		23_11	23_11
	124_1	12_11	---
		14_11	14_11
		24_11	24_11
	134_1	13_11	---
		14_11	---
		34_11	34_11
	234_1	23_11	---
		24_11	---
		34_11	---

III. EXPERIMENTS AND VERIFICATION

The experiments using the developed type-III PRM contain two parts: experiment 1 for a reciprocal network and experiment 2 for a nonreciprocal network. In experiment 1, the scattering matrix of a reciprocal four-port network is reconstructed from the two-port measurements using an Agilent 8510C. In experiment 2, this reciprocal network is connected with an isolator at one port to become a nonreciprocal four-port network. As shown in Section II-B, the scattering matrix of this four-port network can be reconstructed from its terminated three-port networks. Since the multiport vector network analyzer is not available in our laboratory, the three-port scattering matrices for the PRM process of this nonreciprocal network are calculated using the reconstructed four-port scattering matrix of a reciprocal network from experiment 1 and the measured scattering matrices of the isolator and terminators used. The



(c)

Fig. 3. Reconstructed results of: (a) input, (b) coupled path, and (c) direct path characteristics of the "R" network.

experiment 2 results are finally verified with the measured two-port scattering matrices of the terminated four-port nonreciprocal network.

A. Measurement Arrangement

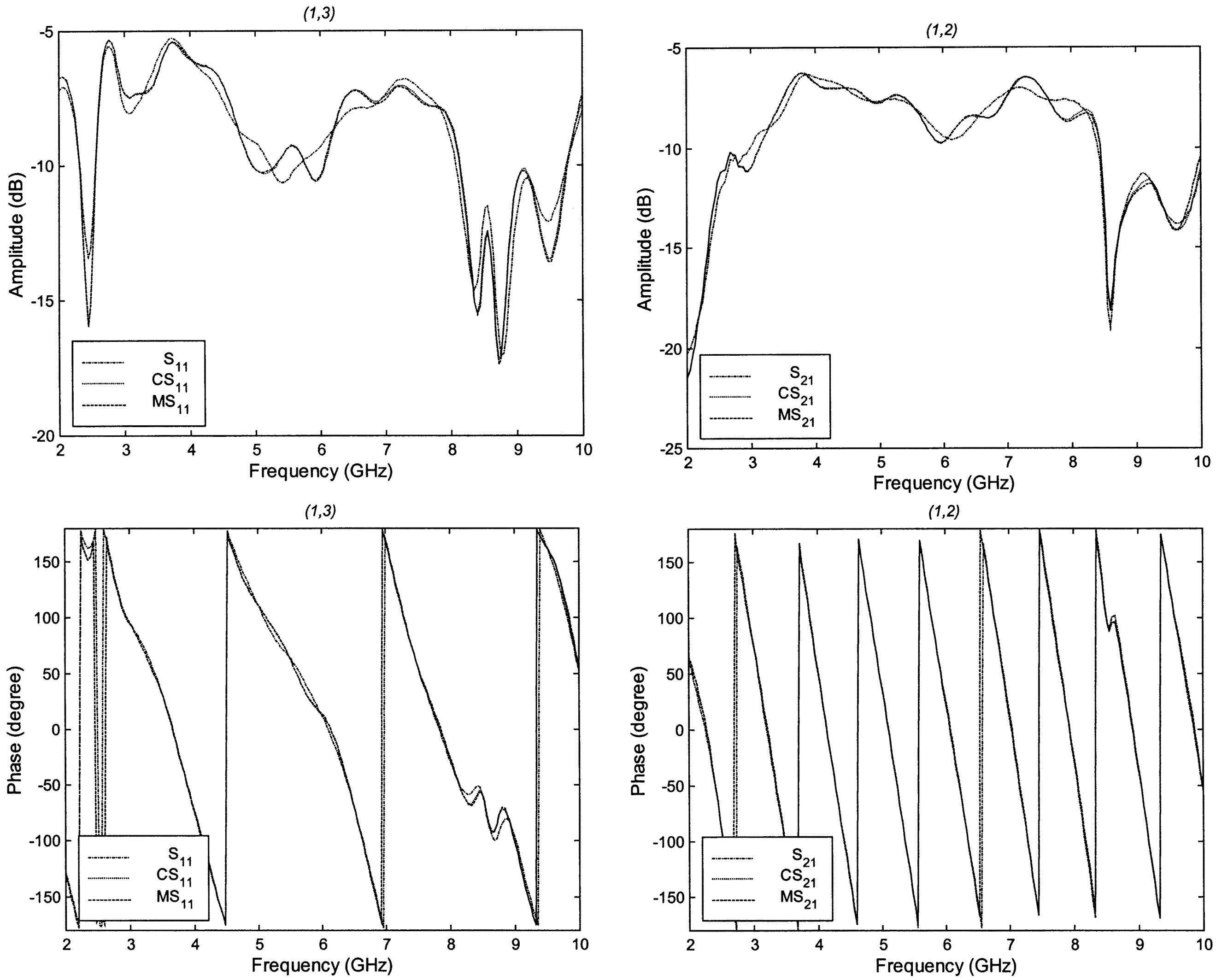
Fig. 1(a) and (b) shows the reciprocal four-port network (or "R" network) and the nonreciprocal four-port network (or "NR" network) used in the experiments. The "R" network is a four-port microstrip circuit with a 50-mil-thick RT/Duroid 6006 substrate. By connecting an isolator (Narda IOS-4080) at port 2, as shown in Fig. 1(b), it becomes a four-port nonreciprocal network.

A typical two-port scattering matrix measurement arrangement by terminating two selected ports of the device-under-test (DUT) is illustrated in Fig. 2. In the measurement, an Agilent 8510C is calibrated with the use of full two-port calibration and an adapter removal technique. A SUN Ultra 1 workstation is linked through an IEEE-488 interface for data recording and PRM calculation. In Fig. 2, an Agilent 8510C is shown connected to ports 1 and 2 of the DUT, whereas ports 3 and 4 are connected with two terminators. Terminator 1 is a 6-dB attenuator with a short load for the PRM measurement. Terminator 2 is a 50-Ω load for the PRM verification. Note that the pair of ports 1 and 2 connected to an Agilent 8510C is only one typical measurement arrangement. The actual measured ports of the DUT are discussed in the following.

B. DUT Measurement Ports

Table I illustrates the port arrangement of the scattering matrix at different orders in the PRM process. The first column is the ports of the DUT, i.e., 1234 represents ports 1–4 of the resulting DUT four-port scattering matrix. The ports of intermediate three-port scattering matrices required to reconstruct this four-port scattering matrix are shown in the second column. The type of terminator connected is also given. For example, the second element 123_1 represents a three-port scattering matrix of ports 1–3 with terminator 1 connected at port 4.

The third column describes the ports and terminators for the two-port scattering matrix measurement. Similarly, the first two digits represent the measured ports connected to an



(a) (b)

Fig. 4. Comparison of: (a) S_{11} and (b) S_{21} of the "NR" network with S_{ij} from the four-port scattering matrix reconstructed by the PRM, MS_{ij} from the measured two-port scattering matrix, and CS_{ij} from the calculated two-port scattering matrix.

Agilent 8510C and the following two digits represent the type of terminators connected. The actual measured ports are then given in the last column. As listed in Table I, there is a total of six sets of the two-port scattering matrix to be measured for the PRM calculation in order to reconstruct the four-port scattering matrix of the "R" network given in Fig. 1(a).

C. Experiment 1

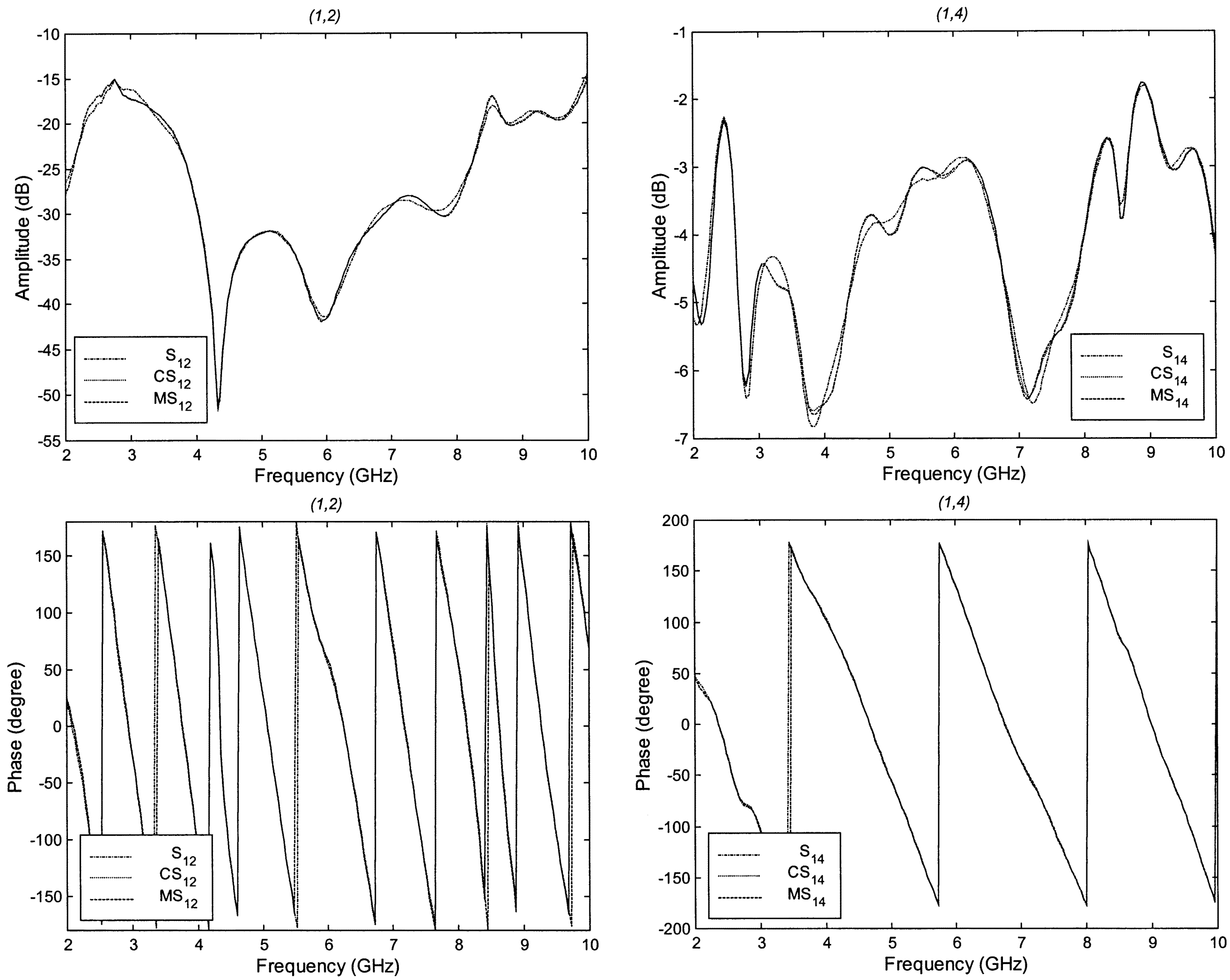
As described in Section II, the first process in the PRM calculation of the "R" network is to solve the intermediate three-port scattering matrices from the six measured sets of two-port scattering matrices. The four-port scattering matrix is then reconstructed from the four three-port scattering matrices, as given in Table I. The measurement frequency range is 2 GHz$\sim$10 GHz. Since the "R" network is reciprocal, there are only ten elements of the four-port scattering matrix to be solved. The results are shown in Fig. 3.

Fig. 3(a) shows the resulting four reflection coefficients of S_{11}, S_{22}, S_{33}, and S_{44}, which are very close to each other. Their differences are due to the soldering and that four SMA connectors are not identical. The scattering parameters for the coupled paths S_{12}, S_{13}, S_{24}, and S_{34} are shown in Fig. 3(b). Fig. 3(c) gives the direct path characteristics of S_{14} and S_{23}. Similarly, these characteristics are very close, respectively, because the "R" network is symmetrical.

In this experiment, the reflection coefficients of four terminators used for each port are all given without using $\det\,[R_{n\times n}] = 0$ to solve them, as described in Appendix B. Instead, $\det\,[R_{n\times n}] = 0$ is used as a criterion to verify the measurement consistency. For each two- to three-port scattering matrix reconstruction, as in Table I, the value of $\det\,[R^{(3)}]$ is very low and within the range of -60 dB.

D. Experiment 2

As described in Section II, the minimal order of the reduced ports for the "NR" network is three. In this experiment, the three-port scattering matrices required for the reconstruction of the "NR" network are calculated using the results of the "R"



(c) (d)

Fig. 4. *(Continued.)* Comparison of: (c) S_{12} and (d) S_{14} of the "NR" network with S_{ij} from the four-port scattering matrix reconstructed by the PRM, MS_{ij} from the measured two-port scattering matrix, and CS_{ij} from the calculated two-port scattering matrix.

network in experiment 1 and the measured scattering matrices of the isolator and terminators used. Based on the PRM formulation for the nonreciprocal network developed in Section II, the resulting four-port scattering matrix of this "NR" network is shown in Fig. 4.

The reconstructed four-port scattering matrix is verified with the measured two-port scattering matrices by connecting two terminators 2 (50-Ω loads), as illustrated in Fig. 2. In addition, these two-port scattering matrices are calculated using the four-port scattering matrix of the "NR" network with the measured reflection coefficients of 50-Ω loads. The reflection coefficients of all four 50-Ω loads are approximately -20 dB. The measured and calculated two-port scattering parameters (denoted as MS_{ij} and CS_{ij}) are also given in Fig. 4, and they are shown to be identical. This shows that the reconstructed S_{ij} of the "NR" network have good accuracy. In addition, a quantitative discussion on the accuracy of reconstructed scattering parameters is given in Appendix C.

The measured port pair for MS_{ij} is given on the top of each figure. Note that not all MS_{ij}'s are shown. For example, there are three MS_{11}'s with port pairs $(1, 2)$, $(1, 3)$, or $(1, 4)$, as measured with an Agilent 8510C. However, the difference between them is not noticeable, therefore, only MS_{11} from the port pair $(1, 3)$ is given in Fig. 4(a). The results in Fig. 4 show that the measured results MS_{ij} are identical to calculated results CS_{ij}. However, there are some discrepancies between MS_{ij} and S_{ij}. This is because the reflections from two imperfect 50-Ω terminators contaminate the DUT S_{ij} in the measurement when using a two-port vector network analyzer. In addition, the discrepancy between MS_{14} and S_{14} is in the range of a few tenths of decibels, which is considerable smaller than those in the other scattering parameters shown in Fig. 4. This can be explained by the reflection from the terminator at port 2 being blocked by the isolator, hence, only the reflection from the terminator at port 3 affects S_{14} through the coupled path. Since the coupling in the direct path is much larger than that of the coupled path, the discrepancy between MS_{11} and S_{11} in Fig. 4(a) is more severe than that for MS_{14} and S_{14} in Fig. 4(d). The discussion on the accuracy comparison of this type-III PRM with a type-I and type-II PRM in [15] is given in Appendix D.

IV. Conclusion

In this paper, a novel PRM (type-III PRM) has been developed to reconstruct the scattering matrix of a multiport network from the measured reduced-port scattering matrices. The effects of nonideal terminators are completely taken into consideration in the derived formulation. The equation $\det\left[R_{n\times n}\right]=0$ to relate the reflection coefficients of terminators and measured scattering parameters is used as a criterion for verifying measurement consistency. It can alternatively be used to reduce the required number of known terminators to two. With the developed PRM, one can measure the scattering matrix of a reciprocal n-port network with a conventional two-port network analyzer. For a nonreciprocal n-port network, the measured ports can be reduced to the order of three.

Appendix

A. Proof of $\det\left[R_{n\times n}\right]=0$

The following proof uses the induction method starting from a 3×3 matrix $\left[R^{(3)}\right]$.

Step 1: Prove $\det\left[R^{(3)}\right]=0$.

Let

$$\left[R^{(3)}\right]=\begin{bmatrix} 1-\Gamma_2 S_{22}^{(1)} & -\left(1-\Gamma_1 S_{11}^{(2)}\right) & 0 \\ 0 & 1-\Gamma_3 S_{33}^{(2)} & -\left(1-\Gamma_2 S_{22}^{(3)}\right) \\ 1-\Gamma_3 S_{33}^{(1)} & 0 & -\left(1-\Gamma_1 S_{11}^{(3)}\right) \end{bmatrix}. \tag{A.1}$$

By substituting $S_{kk}^{(i)}$ given in (2) into (A.1) and after proper manipulation, it can be shown that $\det\left[R^{(3)}\right]=0$. In addition, by changing the subscripts and superscripts in $\left[R^{(3)}\right]$ to be $i,\ j$ and k as

$$\left[R^{(3)'}\right]=\begin{bmatrix} 1-\Gamma_j S_{jj}^{(i)} & -\left(1-\Gamma_i S_{ii}^{(j)}\right) & 0 \\ 0 & 1-\Gamma_k S_{kk}^{(j)} & -\left(1-\Gamma_j S_{jj}^{(k)}\right) \\ 1-\Gamma_k S_{kk}^{(i)} & 0 & -\left(1-\Gamma_i S_{ii}^{(k)}\right) \end{bmatrix} \tag{A.2}$$

$\det\left[R^{(3)'}\right]$ can also be shown equal to zero.

Note that $\left[R^{(3)}\right]$ has a similar form as $\left[R_{3\times 3}\right]$ given in (6), but they are not equal. The elements in $\left[R^{(3)}\right]$ are from the $(n-1)$-port scattering matrices, whereas the elements of $\left[R_{3\times 3}\right]$ are from the two-port scattering matrices. However, for an n-port network, $\left[R^{(n)}\right]=\left[R_{n\times n}\right]$.

Step 2: Assuming $\det\left[R^{(k-1)}\right]=0$, prove that $\det\left[R^{(k)}\right]=0$.

See (A.3), shown at the bottom of this page. Since $\det\left[R^{(3)''}\right]=0$, the first row can be written as a linear combination of the second and third rows.

Similarly, $\left[R^{(k)}\right]$ is written as (A.4), shown at the bottom of this page.

The last two rows can be rewritten into a new form by using $\det\left[R^{(3)''}\right]=0$, then (A.4) becomes (A.5), shown at the bottom of the following page, where c is a constant. One can express this determinant by extracting the last column to give $\det\left[R^{(k)}\right]=c\left(1-\Gamma_1 S_{11}^{(k)}\right)(-1)^k\det\left[R^{(k-1)}\right]=0$ since $\det\left[R^{(k-1)}\right]=0$. Therefore, based on steps 1 and 2 of the induction method, $\det\left[R^{(n)}\right]=\det\left[R_{n\times n}\right]=0$.

B. Proof of the Required Minimal Number of Known Terminators to be Two

Without the loss of generality, two known terminators are connected at ports 1 and 2 with their reflection coefficients as Γ_1

$$\left[R^{(3)''}\right]=\begin{bmatrix} 1-\Gamma_{k-1} S_{k-1,\,k-1}^{(1)} & -\left(1-\Gamma_1 S_{11}^{(k-1)}\right) & 0 \\ 0 & 1-\Gamma_k S_{kk}^{(k-1)} & -\left(1-\Gamma_{k-1} S_{k-1,\,k-1}^{(k)}\right) \\ 1-\Gamma_k S_{kk}^{(1)} & 0 & -\left(1-\Gamma_1 S_{11}^{(k)}\right) \end{bmatrix} \tag{A.3}$$

$$\left[R^{(k)}\right]=\begin{bmatrix} 1-\Gamma_2 S_{22}^{(1)} & -\left(1-\Gamma_1 S_{11}^{(2)}\right) & 0 & \cdots & 0 & 0 \\ 0 & 1-\Gamma_3 S_{33}^{(2)} & -\left(1-\Gamma_2 S_{22}^{(3)}\right) & \cdots & 0 & 0 \\ 0 & 0 & 1-\Gamma_4 S_{44}^{(3)} & \cdots & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 & & 1-\Gamma_k S_{kk}^{(k-1)} & -\left(1-\Gamma_{k-1} S_{k-1,\,k-1}^{(k)}\right) \\ 1-\Gamma_k S_{kk}^{(k)} & 0 & 0 & \cdots & 0 & -\left(1-\Gamma_1 S_{11}^{(k)}\right) \end{bmatrix} \tag{A.4}$$

and Γ_2. Det $\left[R^{(3)}\right]=0$ of (A.1) gives a linear equation with the order of one for Γ_3, therefore, Γ_3 can be calculated. Similarly, the reflection coefficients of other terminators can be calculated using $\det\left[R^{(3)'}\right]=0$ of (A.2) by properly selecting the ports i, j, and k. Therefore, the minimal number of known terminators is two.

C. Accuracy of Reconstructed Scattering Parameters

The accuracy of the reconstructed four-port scattering matrix of the "NR" network is discussed in the following by expressing CS_{ij} and MS_{ij} as $CS_{ij}=S_{ij}+ST_C$ and $MS_{ij}=S^a_{ij}+ST_M$, where S^a_{ij} is the actual scattering parameter of the "NR" network. ST_C and ST_M are the spurious components due to the reflection from imperfect termination to CS_{ij} and MS_{ij}, respectively. One can then use the difference of CS_{ij} and MS_{ij} to estimate the reconstructed S_{ij} accuracy given by

$$S_{ij}-S^a_{ij}=(CS_{ij}-MS_{ij})-(ST_C-ST_M). \quad \text{(A.6)}$$

In (A.6), the values of mean and standard deviation of $CS_{ij}-MS_{ij}$ are first calculated. Since ST_C and ST_M are at least -20 dB below S_{ij} for all 16 scattering parameters, one can assume the mean and standard deviation for $S_{ij}-S^a_{ij}$ are equal to those for $CS_{ij}-MS_{ij}$.

This assumption is then validated by expanding ST_C and ST_M. Taking S_{14}, for example, the difference between $S_{14}-S^a_{14}$ and $CS_{14}-MS_{14}$ can be expressed as

$$(CS_{14}-MS_{14})-(S_{14}-S^a_{14})\cong\frac{1}{\Gamma_3}\left(S_{34}S_{13}-S^a_{34}S^a_{13}\right) \quad \text{(A.7)}$$

where Γ_3 is the reflection coefficient of the 50-Ω load connected at port 3. $(1/\Gamma_3)S_{34}S_{13}$ and $(1/\Gamma_3)S^a_{34}S^a_{13}$ are the dominant terms in ST_C and ST_M, respectively. In (A.7), $S_{34}S_{13}-S^a_{34}S^a_{13}$ can be rewritten as

$$S_{34}S_{13}-S^a_{34}S^a_{13}=S_{34}p_1+S_{13}p_2-p_1p_2 \quad \text{(A.8)}$$

where $p_1=S_{13}-S^a_{13}$ and $p_2=S_{34}-S^a_{34}$. Since the mean and variance of p_1 and p_2 are known, the mean and variance of $S_{34}S_{13}-S^a_{34}S^a_{13}$ can then be calculated. The calculated results show that the mean and variance differences between $S_{ij}-S^a_{ij}$ and $CS_{ij}-MS_{ij}$ are quite small, as assumed. Results of the estimated values of the mean and standard deviation for $S_{ij}-S^a_{ij}$ are listed in Table II.

Since the standard deviation gives the root mean square distance between S^a_{ij} and S_{ij}, one can then use the mean value of $\left|S^a_{ij}\right|$ and the calculated standard deviation to estimate the magnitude and phase errors. As the error vector is in the same direction as S^a_{ij}, it has the largest magnitude error. On the other hand, as the error vector is perpendicular to S^a_{ij}, it has the largest phase error. With the value of S^a_{ij} assumed to be equal to S_{ij}, the calculated mean value of $\left|S^a_{ij}\right|$, magnitude, and phase errors are listed in Table II. Note only eight typical terms are given. It shows that all the magnitude and phase errors are approximately less than 0.18 dB and 1.2°, except for S_{12} and S_{42}. The larger errors of these two parameters may be due to their magnitudes being significantly smaller than others.

TABLE II
CALCULATED RESULTS OF MEAN ABSOLUTE VALUE $|S_{ij}|$, ABSOLUTE MEAN VALUE $|\mu|$, STANDARD DEVIATION σ, MAGNITUDE ERROR, AND PHASE ERROR OF S_{ij}

S_{ij}	S_{11}	S_{22}	S_{12}	S_{21}
mean of $\|S_{ij}\|$	0.369	0.229	0.066	0.346
$\|\mu\|$	0.12×10^{-3}	0.52×10^{-3}	0.18×10^{-3}	0.36×10^{-3}
σ	5.79×10^{-3}	2.20×10^{-3}	2.65×10^{-3}	7.41×10^{-3}
magnitude error (dB)	0.135	0.083	0.344	0.184
phase error (degree)	0.899	0.550	2.317	1.228

S_{ij}	S_{14}	S_{41}	S_{24}	S_{42}
mean of $\|S_{ij}\|$	0.629	0.629	0.339	0.064
$\|\mu\|$	0.22×10^{-3}	0.18×10^{-3}	0.13×10^{-3}	0.08×10^{-3}
σ	3.67×10^{-3}	4.88×10^{-3}	3.42×10^{-3}	1.54×10^{-3}
magnitude error (dB)	0.051	0.067	0.087	0.205
phase error (degree)	0.334	0.444	0.577	1.365

D. Accuracy Comparison of Three PRMs

This section gives a comparison of the accuracy of type-I, type-II [15], and type-III PRMs in this paper. The accuracy of reconstructed scattering parameters is expressed as the magnitude and phase errors. Since the errors are normalized to the

$$\det\left[R^{(k)}\right]=c\times\det\begin{bmatrix} 1-\Gamma_2S^{(1)}_{22} & -\left(1-\Gamma_1S^{(2)}_{11}\right) & 0 & \cdots & 0 & 0\\ 0 & 1-\Gamma_3S^{(2)}_{33} & -\left(1-\Gamma_2S^{(3)}_{22}\right) & \cdots & 0 & 0\\ 0 & 0 & 1-\Gamma_4S^{(3)}_{44} & \cdots & 0 & 0\\ \vdots & \vdots & & & \vdots & \vdots\\ 1-\Gamma_{k-1}S^{(1)}_{k-1,\,k-1} & 0 & 0 & & -\left(1-\Gamma_1S^{(k-1)}_{11}\right) & 0\\ 1-\Gamma_kS^{(1)}_{kk} & 0 & 0 & \cdots & 0 & -\left(1-\Gamma_1S^{(k)}_{11}\right)\end{bmatrix} \quad \text{(A.5)}$$

TABLE III
CALCULATED RESULTS OF MAGNITUDE AND PHASE ERRORS OF THE RECONSTRUCTED SCATTERING PARAMETERS USING THREE PRMs

S_{ij}		S_{11}	S_{22}	S_{12}	S_{21}
Type I PRM	mag. error (dB)	0.032	0.652	0.086	0.027
	phase error (degree)	0.212	4.46	0.579	0.178
Type II PRM	mag. error (dB)	0.047	0.223	0.067	0.028
	phase error (degree)	0.309	1.489	0.441	0.183
Type III PRM	mag. error (dB)	0.135	0.083	0.344	0.184
	phase error (degree)	0.899	0.550	2.317	1.228

S_{ij}		S_{13}	S_{31}	S_{24}	S_{42}
Type I PRM	mag. error (dB)	0.070	0.021	0.104	0.248
	phase error (degree)	0.464	1.398	0.688	1.661
Type II PRM	mag. error (dB)	0.126	0.432	0.112	0.416
	phase error (degree)	0.836	2.920	0.745	2.805
Type III PRM	mag. error (dB)	0.166	0.167	0.087	0.205
	phase error (degree)	1.107	1.107	0.577	1.365

mean values in type-III PRMs or typical values of the respective scattering parameters in type-I and type-II PRMs, the comparison can be made upon these numbers even when the DUTs are not the same. Table III lists the results by adopting Table II of this paper and [15, Table II].

As shown in Table III, S_{11}, S_{21}, S_{13}, and S_{31} of the type-I PRM, S_{12} of type-II PRM, and S_{22}, S_{24}, and S_{42} of type-III PRM have the most accurate results. To explain these accuracy characteristics, one may note that, for the reconstruction of a four-port scattering matrix using a reduced-port vector network analyzer, different numbers of measurement are performed in these three PRMs.

From [15, Table I], 13 two-port scattering matrices are measured for the type-I PRM. Among them, nine matrices are measured between ports 1–3, three are measured between ports 1 and 2, and one is measured between ports 2–4. For the type-II PRM, ten two-port scattering matrices are measured with six matrices measured between ports 1–3, three measured between ports 1 and 2, and one measured between ports 2–4. For the type-III PRM, all possible two-port connections are measured. As shown in Table I, six two-port scattering matrices are measured. One may observe that, for the scattering parameters that are directly and frequently measured, they are given with better accuracy. This also indicates that the type-III PRM has the smallest range of maximum and minimum values for the magnitude and phase errors. As shown in Table III, the maximum and minimum values of the magnitude error for type-I, type-II, and type-III PRMs are given as $(0.652, 0.021)$, $(0.432, 0.028)$, and $(0.344, 0.083)$, respectively.

REFERENCES

[1] R. A. Speciale, "Multi-port network analyzers meeting the design need," *Microwave Syst. News.*, vol. 10, no. 6, pp. 67–89, June 1980.

[2] A. Ferrero, U. Pisani, and K. J. Kerwin, "A new implementation of a multi-port automatic network analyzer," *IEEE Trans. Microwave Theory Tech.*, vol. 40, pp. 2078–2085, Nov. 1992.

[3] A. Ferrero, F. Sanpietro, and U. Pisani, "Accurate coaxial standard verification by multiport vector network analyzer," in *IEEE MTT-S Int. Microwave Symp. Dig.*, May 1994, pp. 1365–1368.

[4] A. Ferrero and F. Sanpietro, "A simplified algorithm for leaky network analyzer calibration," *IEEE Microwave Guided Wave Lett.*, vol. 5, pp. 119–121, Apr. 1995.

[5] R. A. Speciale, "A generalization of the TSD network-analyzer calibration procedure, covering n-port scattering matrix measurements," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-25, pp. 1100–1115, Dec. 1977.

[6] J. V. Butler *et al.*, "16-term error model and calibration procedure for on-wafer network analysis measurements," *IEEE Trans. Microwave Theory Tech.*, vol. 39, pp. 2211–2217, Dec. 1991.

[7] H. Van Hamme and M. Vanden Bossche, "Flexible vector analyzer calibration with accuracy bounds using an 8-term or 16-term error correction model," *IEEE Trans. Microwave Theory Tech.*, vol. 42, pp. 976–987, June 1994.

[8] H. Heuerman and B. Schiek, "Results of NWA measurements with leakage errors corrected with the TMS-15 term procedures," in *IEEE MTT-S Int. Microwave Symp. Dig.*, May 1994, pp. 1361–1364.

[9] J. C. Tippet and R. A. Speciale, "A rigorous technique for measuring the scattering matrix of a multi-port device with a two-port network analyzer," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-30, pp. 661–666, May 1982.

[10] H. Dropkin, "Comments on 'A rigorous technique for measuring the scattering matrix of a multi-port device with a two-port network analyzer'," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-31, pp. 79–81, Jan. 1983.

[11] S. Sercu and L. Martens, "Characterizing n-port packages and interconnections with a 2-port network analyzer," in *IEEE 6th Electrical Performance of Electronic Packaging Topical Meeting Dig.*, Oct. 1997, pp. 163–166.

[12] D. Woods, "Multi-port-network analysis by matrix renormalization employing voltage-wave S-parameters with complex normalization," *Proc. Inst. Elect. Eng.*, vol. 124, no. 3, pp. 198–204, Mar. 1977.

[13] D. F. Williams and D. K. Walker, "In-line multiport calibration," in *51st ARFTG Dig.*, June 1998, pp. 88–90.

[14] W. Lin and C. Ruan, "Measurement and calibration of a universal six-port network analyzer," *IEEE Trans. Microwave Theory Tech.*, vol. 37, pp. 734–742, Apr. 1989.

[15] H. C. Lu and T. H. Chu, "Port reduction methods for scattering matrix measurement of an N-port network," *IEEE Trans. Microwave Theory Tech.*, vol. 48, pp. 959–968, June 2000.

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