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Lower Bounds for Asian Options

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Abstract

There are two types of Asian options, fixed-strike and floating-strike, in the literature. We give lower bounds on the values of both fixed-strike and floating-strike Asian options in continuous case. Good lower bounds for both options have been derived earlier by Rogers & Shi (1995) and Thompson (1998). But the lower bound derived by Thompson assumes a maturity of one year. This thesis extends Thompson's version of the lower bound to the case of general maturities. Numerical experiments are performed to confirm the extreme accuracy of the lower bound.

Chapter 1

Introduction

1.1 Background

Fixed-strike Asian (call) options are options whose payoff depends on the average price of the underlying asset during at least some part of the life of the option. The payoff from a fixed-strike Asian call is $\max(0\ ,\ S_{ave}-K)=\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+$, where S_{ave} is the average value of the underlying asset calculated over a predetermined averaging period, $K>0$ is called the strike price or the exercise price, and T is the maturity date. We assume that the asset price follows a geometric Browning motion $S_t = S\exp(\sigma B_t + \alpha t)$, where $\alpha = r - \frac{1}{2}\sigma^2$ is a constant, r is the risk-free interest rate (assumed to be a constant), S is the stock price, σ is the volatility, and B_t is a Browning motion.

Another type of Asian (call) option is the floating-strike Asian (call) option. The payoff is $\max(0\ ,\ S_{ave}-S_T)=\left(\frac{1}{T}\int_0^T S_t dt - S_T\right)^+$, where S_T is the stock price at the maturity date.

Exact analytic formulas for Asian options don't exist. This is primarily due to the fact that the arithmetic average of a set of lognormal random variables has a distribution that is largely intractable. Then several approaches to the problem of valuing Asian options have been put forward in the literature:

1. Monte-carlo simulation: Boyle (1977) and Kemna & Vorst (1990) and Corwin & Boyle & Tan (1996).
2. Binomial tree method: Hull & White (1993) and Neave & Turnbull (1993).
3. Convolution method: Carverhill & Clewlow (1990).
4. Triple integral: Yor (1992) and Geman & Yor(1993).
5. Partial derivative equation (PDE): Dewynne & Wilmott (1995) and Alziary & Decamps & Koehl (1997).
6. Fast fourier transform (FFT): Caverhill and Clewlow (1992) and Benhamou (2002).
7. Approximation method: Turnbull & Wakeman (1991) and Levy (1992) and Vorst (1992) and Milevsky & Posner (1998) and Curran (1992) and Rogers & Shi (1995) and Thompson (1998).

In this thesis, we consider analytical approximation methods. They are easier to evaluate.

Thompson (1998) assumes $T = 1$ (year). In this thesis, we extend his result to the general case of $T = \tau$ years.

1.2 Structures of the Thesis

There are five chapters in this thesis. In Chapter 2, we introduce some useful Mathematical Preliminaries for later analysis. In Chapter 3, we introduce lower bounds to the case of general maturities. In Chapter 4, we present the numerical results. Conclusions and future work are in Chapter 5.

Chapter 2

Mathematical Preliminaries

2.1 Correlation Matrices

Like most of the approximation formulas in the literature, we need the correlation matrix between $\frac{1}{T}\int_0^T B_s ds$ and B_t.

Theorem 2.1

The correlation matrix between $\frac{1}{T}\int_0^T B_s ds$ and B_t equals

$$\begin{bmatrix} \mathrm{Cov}(B_t\ ,\ B_t) & \mathrm{Cov}\left(B_t\ ,\ \frac{1}{T}\int_0^T B_s ds\right) \\ \mathrm{Cov}\left(B_t\ ,\ \frac{1}{T}\int_0^T B_s ds\right) & \mathrm{Cov}\left(\frac{1}{T}\int_0^T B_s ds\ ,\ \frac{1}{T}\int_0^T B_s ds\right) \end{bmatrix} = \begin{bmatrix} t & \frac{t}{T}\left(T-\frac{t}{2}\right) \\ \frac{t}{T}\left(T-\frac{t}{2}\right) & \frac{T}{3} \end{bmatrix}$$

where $0 < t < T$.

Proof:

(i) $\mathrm{Cov}(B_t\ ,\ B_t) = \mathrm{Var}(B_t) = t$ by the definition of Brownian motion.

(ii) $\mathrm{Cov}\left(B_t\ ,\ \frac{1}{T}\int_0^T B_s ds\right) = E\left(B_t \cdot \frac{1}{T}\int_0^T B_s ds\right) - E(B_t)E\left(\frac{1}{T}\int_0^T B_s ds\right) = E\left(\frac{1}{T}\int_0^T B_t \cdot B_s ds\right)$

$$= \frac{1}{T}\int_0^T E(B_t \cdot B_s)\, ds = \frac{1}{T}\int_0^t E(B_t \cdot B_s)\, ds + \frac{1}{T}\int_t^T E(B_t \cdot B_s)\, ds$$

$$= \frac{1}{T}\int_0^t s\ ds + \frac{1}{T}\int_t^T t\ ds = \frac{1}{T}\left(\frac{t^2}{2} + t(T-t)\right) = \frac{t}{T}\left(T - \frac{t}{2}\right).$$

(iii) $\mathrm{Cov}\left(\frac{1}{T}\int_0^T B_s ds\ ,\ \frac{1}{T}\int_0^T B_s ds\right)=\mathrm{Var}\left(\frac{1}{T}\int_0^T B_s ds\right)=\frac{1}{T^2}\mathrm{Var}\left(\int_0^T B_s ds\right).$

By Hoel, Port, and Stone (1986),

$$\mathrm{Var}\left(\int_a^b f'(t)(B_t-B_a)\ dt\right)=\int_a^b \left(f(t)-f(b)\right)^2\ dt\ .$$

So $\frac{1}{T^2}\mathrm{Var}\left(\int_0^T B_s ds\right)=\frac{1}{T^2}\int_0^T (s-T)^2 ds=\frac{T}{3}$.

Theorem 2.2

The correlation matrix between $\frac{1}{T}\int_0^T B_s ds-B_T$ and B_t equals

$$\begin{bmatrix} \mathrm{Cov}(B_t\ ,\ B_t) & \mathrm{Cov}\left(B_t\ ,\ \frac{1}{T}\int_0^T B_s ds-B_T\right) \\ \mathrm{Cov}\left(B_t\ ,\ \frac{1}{T}\int_0^T B_s ds-B_T\right) & \mathrm{Cov}\left(\frac{1}{T}\int_0^T B_s ds-B_T\ ,\ \frac{1}{T}\int_0^T B_s ds-B_T\right)\end{bmatrix}=\begin{bmatrix} t & \frac{-t^2}{2T} \\ \frac{-t^2}{2T} & \frac{T}{3}\end{bmatrix}$$

where $0<t<T$.

Proof:

(i) $\mathrm{Cov}\left(B_t\ ,\ \frac{1}{T}\int_0^T B_s ds-B_T\right)=E\left(B_t\cdot\left(\frac{1}{T}\int_0^T B_s ds-B_T\right)\right)-E(B_t)E\left(\frac{1}{T}\int_0^T B_s ds-B_T\right)$

$$=E\left(B_t\cdot\left(\frac{1}{T}\int_0^T B_s ds-B_T\right)\right)=\frac{1}{T}\int_0^T E(B_t\cdot B_s)\,ds-t=\frac{t}{T}(T-\frac{t}{2})-t=\frac{-t^2}{2T}\ .$$

(ii)

$$\mathrm{Cov}\left(\frac{1}{T}\int_0^T B_s ds-B_T\ ,\ \frac{1}{T}\int_0^T B_s ds-B_T\right)=\mathrm{Var}\left(\frac{1}{T}\int_0^T B_s ds-B_T\right)=\frac{1}{T^2}\mathrm{Var}\left(\int_0^T B_s ds-B_T\right)$$

$$=\frac{1}{T^2}\left\{\mathrm{Var}\left(\int_0^T B_s ds\right)+\mathrm{Var}(B_T)-2\mathrm{Cov}\left(\int_0^T B_s ds\ ,\ B_T\right)\right\}=\frac{1}{T^2}\left\{\frac{T^3}{3}+T-2\frac{T}{2}\right\}=\frac{T}{3}\ .$$

2.2 Basic Statistical Properties

Definitions

Let $-\infty < \mu_x < \infty$, $-\infty < \mu_y < \infty$, $0 < \sigma_x$, $0 < \sigma_y$ and $-1 < r < 1$ be real numbers. The bivariate normal PDF (probability density function) with means μ_x and μ_y, variances σ_x^2 and σ_y^2, and correlation ρ is the bivariate PDF given by

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(\frac{-1}{2(1-\rho^2)} \left\{ \left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 \right\} \right)$$

for $-\infty < x < \infty$ and $-\infty < y < \infty$, and uaually denoted by $\phi(\mu_x, \mu_x, \sigma_x^2, \sigma_y^2, \rho)$

Theorem 2.3

If bivariate normal random variable $(X, Y) \sim \phi(\mu_x, \mu_x, \sigma_x^2, \sigma_y^2, \rho)$, the conditional distribution of X given $Y = y$ is normal with mean $\mu_x + \frac{\rho\sigma_x}{\sigma_y}(y - \mu_x)$ and variance $\sigma_x^2(1-\rho^2)$.

Proof: See Sheldon Ross (1998).

By Theorem 2.3, if $X = B_t$ and $Y = \frac{1}{T}\int_0^T B_s ds$, then the conditional distribution of B_t given $\frac{1}{T}\int_0^T B_s ds = z$ is normal with mean $\frac{3t(T - t/2)z}{T^2}$ and variance $t-\frac{3t^2}{T^3}\left(T-\frac{t}{2}\right)^2$, and the conditional distribution of B_t given $\frac{1}{T}\int_0^T B_s ds - B_T = x$ is normal with mean $\frac{-3t^2 x}{2T^2}$ and variance $t-\frac{3t^4}{4T^3}$.

Definitions

The MGF (moment generating function) $M_X(A)$ of the random variable X is defined for all real values of A by

$$M_X(A)=E\left[e^{AX}\right]=\int_{-\infty}^{\infty} e^{Ax} f(x)dx \quad \text{if } X \text{ is contimuous with density } f(x).$$

Theorem 2.4

The MGF $M_X(A)$ of the random variable $X \sim \phi(\mu\ ,\ \sigma^2)$ is

$M_X(A)=E\left(e^{AX}\right)=\exp\left(\mu A+\frac{1}{2}\sigma^2 A^2\right)$ for all real value of A .

Proof: See Sheldon Ross (1998).

Theorem 2.5

Suppose we are given two random variables X, Y with $X \sim \phi(\mu_x , \sigma_x^2)$ and $Y \sim \phi(\mu_y , \sigma_y^2)$. Then $E\left(e^X I(Y>0)\right)=e^{u_x+\frac{1}{2}\sigma_x^2}\Phi\left(\frac{\mu_y + c}{\sigma_y}\right)$, where $\Phi(\cdot)$ is the normal distribution function , $I(\cdot)$ is the indicator function and $c=\mathrm{Cov}(X\ ,\ Y)$.

Proof: See Appendix.

Theorem 2.6

For any random variable X with density $f_X(x)$, we have

(i) $E[(S_t - K)I(X > \gamma)] = E[S_t - K; X > \gamma]$;

(ii) $\frac{\partial}{\partial \lambda}\frac{1}{T}\int_0^T E(S_t - K; X > \gamma)\,dt = \frac{1}{T}\int_0^T E(S_t - K \mid X = \gamma)(-f_X(x))dt$.

Proof: See Appendix.

Chapter 3

Lower Bounds

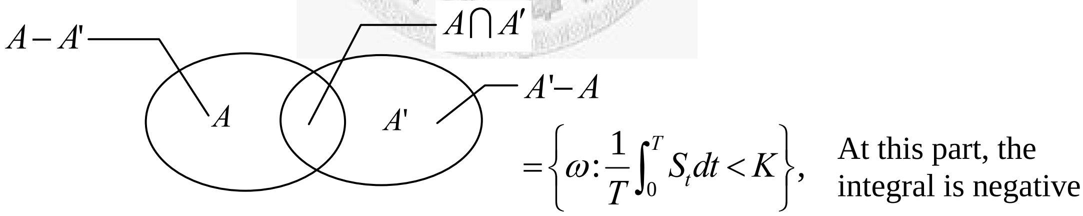
3.1 Fixed-Strike Asian Options

Let $A=\left\{\omega:\frac{1}{T}\int_0^T S_t dt > K\right\}$. Then

$$E\left\{\left(\frac{1}{T}\int_0^T S_t dt-K\right)^+\right\}=E\left\{\left(\frac{1}{T}\int_0^T S_t dt-K\right)I(A)\right\}=E\left\{\left(\frac{1}{T}\int_0^T (S_t-K)\,dt\right)I(A)\right\}$$

$$=E\left\{\left(\frac{1}{T}\int_0^T (S_t-K)I(A)\,dt\right)\right\}=\frac{1}{T}\int_0^T E(S_t-K)I(A)\,dt\geq\frac{1}{T}\int_0^T E(S_t-K)I(A')dt$$

If we replace event A with another event A', we no longer have equality (see the illustration below).



We will use $A'=\left\{\omega:\frac{1}{T}\int_0^T B_t dt > \gamma\right\}$.

We now determine the value of γ that maximizes $\frac{1}{T}\int_0^T E\left(S_t-K\right)I\left(A'\right)dt$. Note for any random variable X with density $f_X(x)$,

$$\frac{\partial}{\partial\lambda}\frac{1}{T}\int_0^T E\left(S_t-K;X>\gamma\right)\,dt=\frac{1}{T}\int_0^T E\left(S_t-K\mid X=\gamma\right)\left(-f_X(x)\right)dt\,.$$

Thus the optimal value of γ, γ^*, satisfies

$$\frac{1}{T}\int_0^T E(S_t - K \mid X = \gamma)\,dt = K .$$

With our choice of $X = \frac{1}{T}\int_0^T B_t dt$, we conclude that

$$\frac{1}{T}\int_0^T S\exp\left\{\frac{3t(T-t/2)\gamma^*\sigma}{T^2} + \alpha t + \frac{\sigma^2}{2}\left(t - \frac{3t^2}{T^3}\left(T - \frac{t}{2}\right)^2\right)\right\}dt = K$$

which determines γ^* uniquely. We now have the bound

$$V_{fixed} \geq e^{-T\times r}\left\{\frac{1}{T}\int_0^T E\left[\left(Se^{\sigma B_t + \alpha t} - K\right)I\left(\frac{1}{T}\int_0^T B_s ds > \gamma^*\right)\right]dt\right\}.$$

It remains to calculate the expectation. Fix $t \in (0, T)$ and let $N_1 = \sigma B_t + \alpha t + \log S$ and $N_2 = \frac{1}{T}\int_0^T B_t dt - \gamma^*$. Write $u_i = E(N_i), \sigma_i^2 = \mathrm{Var}(N_i)$ and $c = \mathrm{Cov}(N_1, N_2)$. Then

$$E\left[\left(e^{N_1} - K\right)I\left(N_2 > 0\right)\right] = e^{u_1 + \frac{1}{2}\sigma_1^2}\Phi\left(\frac{u_2 + c}{\sigma_2}\right) - K\Phi\left(\frac{u_2}{\sigma_2}\right)$$

where Φ is the normal distribution function. With $u_1 = \alpha t + \log S$, $u_2 = -\gamma^*$, $\sigma_1^2 = \sigma^2 t$, $\sigma_2^2 = \frac{T}{3}$, $c = \sigma\frac{t}{T}(T - \frac{t}{2})$, we have

$$V_{fixed} \geq e^{-T\times r}\left\{\frac{1}{T}\int_0^T Se^{\alpha t + \frac{1}{2}\sigma^2 t}\Phi\left(\frac{-\gamma^* + \sigma(t/\mathrm{T})(T - t/2)}{\sqrt{T/3}}\right)dt - K\Phi\left(\frac{-\gamma^*}{\sqrt{T/3}}\right)\right\}.$$

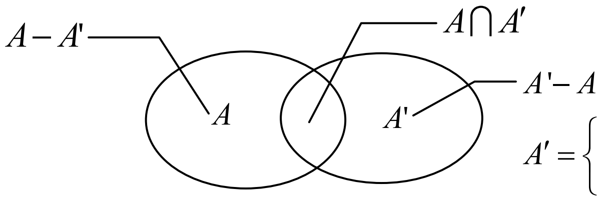
3.2 Floating- Strike Asian Options

Let $A = \left\{\omega : \frac{1}{T}\int_0^T S_t dt > S_T\right\}$. Then

$$E\left\{\left(\frac{1}{T}\int_0^T S_t dt - S_T\right)^+\right\} = E\left\{\left(\frac{1}{T}\int_0^T S_t dt - S_T\right)I(A)\right\} = E\left\{\left(\frac{1}{T}\int_0^T (S_t - S_T)\,dt\right)I(A)\right\}$$

$$= E\left(\frac{1}{T}\int_0^T (S_t - S_T)I(A)\,dt\right) = \frac{1}{T}\int_0^T E(S_t - S_T)I(A)dt \geq \frac{1}{T}\int_0^T E(S_t - S_T)I(A')dt .$$

If we replace event A with another event A', we no longer have equality (see the illustration below).



$A' = \left\{ \omega : \frac{1}{T}\int_0^T B_t dt - B_T < \gamma \right\}$ At this part, the integral is negative

We will use $A' = \left\{ \omega : \frac{1}{T}\int_0^T B_t dt - B_T > \gamma \right\}$.

We now determine the value of γ that maximizes $\frac{1}{T}\int_0^T E(S_t - S_T) I(A')\, dt$. Note for any random variable X with density $f_X(x)$,

$$\frac{\partial}{\partial \lambda}\frac{1}{T}\int_0^T E\left(S_t - S_T ; X > \gamma\right)\, dt = \frac{1}{T}\int_0^T E\left(S_t - S_T \mid X = \gamma\right)\left(-f_X(x)\right) dt .$$

Thus the optimal value of γ, γ^*, satisfies

$$\frac{1}{T}\int_0^T E\left(S_t - S_T \mid X = \gamma^*\right) dt = \frac{1}{T}\int_0^T E\left(S_T \mid X = \gamma^*\right) dt .$$

With our choice of $X = \frac{1}{T}\int_0^T B_t dt - B_T$, we conclude that

$$\frac{1}{T}\int_0^T S \exp\left\{ \frac{3(-t^2)\gamma^* \sigma}{2T^2} + \alpha t + \frac{\sigma^2}{2}\left(t - \frac{3t^4}{4T^3} \right) \right\} dt = S \exp\left(\alpha t - \frac{3\sigma\gamma^*}{2} + \frac{T\sigma^2}{8} \right)$$

which determines γ^* uniquely. We now have the bound

$$V_{floating} \geq e^{-T \times \rho} \left\{ \frac{1}{T}\int_0^T E\left[\left(S e^{\sigma B_t + \alpha t} - S_T \right) I\left(\frac{1}{T}\int_0^T B_s ds - B_T > \gamma^* \right) \right] dt \right\} .$$

It remains to calculate the expectation. Fix $t \in (0,T)$ and let $N_1 = \sigma B_t + \alpha t + \log S$ and

$N_2 = \frac{1}{T}\int_0^T B_t dt - B_T - \gamma^*$. Write $u_i = E(N_i)$, $\sigma_i^2 = \mathrm{Var}(N_i)$, and $c = \mathrm{Cov}(N_1 , N_2)$.

Note that $E\left[e^{N_1} I(N_2 > 0)\right] = e^{u_1 + \frac{1}{2}\sigma_1^2} \Phi\left(\frac{u_2 + c}{\sigma_2}\right)$,

where Φ is the normal distribution function. With $u_1 = \alpha t + \log S$, $u_2 = -\gamma^*$,

$\sigma_1^2 = \sigma^2 t$, $\sigma_2^2 = \frac{T}{3}$, $c = \sigma \frac{t}{T}\left(T - \frac{t}{2}\right) - \sigma t$, we have

$$V_{floating} \geq \frac{e^{-T \times r}}{T}\left\{\int_0^T Se^{\alpha t + \frac{1}{2}\sigma^2 t}\Phi\left(\frac{-\gamma^* + \sigma(t/\mathrm{T})(T - t/2) - \sigma t}{\sqrt{T/3}}\right) - Se^{\alpha T + \frac{1}{2}\sigma^2 T}\Phi\left(\frac{-\gamma^* - \sigma T/2}{\sqrt{T/3}}\right) dt\right\}$$

.

Chapter 4

Numerical Results

4.1 Numerical Results

Case 1: $S = 100$; $X = 100$; $r = 0.1$; $\sigma = 0.1$; $\tau = 0.25$				
Algorithms				
	Hull-White	PDE	Hsu & Lyuu	Ours
n	Option value	Option value	Option value	Lower bound
50	1.8486	1.8478	1.8714720	-
100	1.8501	1.8492	1.9095930	-
200	1.8508	1.8503	1.8891953	-
400	1.8512	1.8509	1.8703678	-
∞	1.8516	1.8514	1.8515402	1.85158801
Case 2: $S = 100$; $X = 100$; $r = 0.1$; $\sigma = 0.5$; $\tau = 5$				
Algorithms				
	Hull-White	PDE	Hsu & Lyuu	Ours
n	Option value	Option value	Option value	Lower bound
50	28.3899	28.3573	28.3893142	-
100	28.3972	28.3842	28.3973455	-
200	28.4011	28.3952	28.4013633	-
400	28.4031	28.4003	28.4032833	-
∞	28.4051	28.4054	28.4052033	28.3641004

Table 1: Comparison with the Hull-White and PDE methods and Hsu and Lyuu (2005). The parameters are from Tables 3 and 4 of Forsyth et al. (2002) and Table 1 of Hsu and Lyuu (2005). The numbers quoted for the Hull-White are based on calculations using the finest grids. The "∞" row lists the extrapolated option values.

X	σ	r	Exact	AA2	AA3	Hsu & Lyuu	Ours
95	0.05	0.05	7.1777275	7.1777244	7.1777279	7.178812	7.17772612
100			2.7161745	2.7161755	2.7161744	2.715613	2.71616846
105			0.3372614	0.3372601	0.3372614	0.338863	0.33723147
95		0.09	8.8088392	8.8088441	8.8088397	8.808717	8.80883886
100			4.3082350	4.3082253	4.3082331	4.309247	4.30823109
105			0.9583841	0.9583838	0.9583841	0.960068	0.95833085
95		0.15	11.0940944	11.0940964	11.0940943	11.093903	11.0940944
100			6.7943550	6.7943510	6.7943553	6.795678	6.79435363
105			2.7444531	2.7444538	2.7444531	2.743798	2.74440648
90	0.10	0.05	11.9510927	11.9509331	11.9510871	11.951610	11.9510757
100			3.6413864	3.6414032	3.6413875	3.642325	3.64134347
110			0.3312030	0.3312563	0.3311968	0.331348	0.33107348
90		0.09	13.3851974	13.3851165	13.3852048	13.385563	13.3851902
100			4.9151167	4.9151388	4.9151177	4.914254	4.91507534
110			0.6302713	0.6302538	0.6302717	0.629843	0.63006361
90		0.15	15.3987687	15.3988062	15.3987860	15.398885	15.3987669
100			7.0277081	7.0276544	7.0277022	7.027385	7.02767816
110			1.4136149	1.4136013	1.4136161	1.414953	1.41328566
90	0.20	0.05	12.5959916	12.5957894	12.5959304	12.596052	12.5956015
100			5.7630881	5.7631987	5.7631187	5.763664	5.76270847
110			1.9898945	1.9894855	1.9899382	1.989962	1.98924214
90		0.09	13.8314996	13.8307782	13.8313482	13.831604	13.8312204
100			6.7773481	6.7775756	6.7773833	6.777748	6.77699941
110			2.5462209	2.5459150	2.5462598	2.546397	2.54545882
90		0.15	15.6417575	15.6401370	15.6414533	15.641911	15.6415977
100			8.4088330	8.4091957	8.4088744	8.408966	8.40851852
110			3.5556100	3.5554997	3.5556415	3.556094	3.55468744
90	0.30	0.05	13.9538233	13.9555691	13.9540973	13.953937	13.9524219
100			7.9456288	7.9459286	7.9458549	7.945918	7.94435709
110			4.0717942	4.0702869	4.0720881	4.071945	4.07011502
90		0.09	14.9839595	14.9854235	14.9841522	14.984037	14.9827819
100			8.8287588	8.8294164	8.8289978	8.829033	8.82754823
110			4.6967089	4.6956764	4.6969698	4.696895	4.69490183
90		0.15	16.5129113	16.5133090	16.5128376	16.512963	16.512024
100			10.2098305	10.2110681	10.2101058	10.210039	10.2087236
110			5.7301225	5.7296982	5.7303567	5.730357	5.72816084

Table 2: Comparison with Zhang (2001, 2003) and Hsu and Lyuu (2005). The parameters are from Table 1 of Zhang (2003) and Table 2 of Hsu and Lyuu (2005). The options are calls with $S = 100$ and $\tau = 1$.

X	σ	Exact	AA2	AA3	Hsu & Lyuu	Ours
95	0.05	8.8088392	8.80884	8.80884	8.808717	8.80883886
100		4.3082350	4.30823	4.30823	4.309247	4.30823109
105		0.9583841	0.95838	0.95838	0.960068	0.95833085
95	0.1	8.9118509	8.91171	8.91184	8.912238	8.91183603
100		4.9151167	4.91514	4.91512	4.914254	4.91507534
105		2.0700634	2.07006	2.07006	2.072473	2.06992974
95	0.2	9.9956567	9.99597	9.99569	9.995661	9.99536215
100		6.7773481	6.77758	6.77738	6.777748	6.77699941
105		4.2965626	4.29643	4.29649	4.297021	4.29594093
95	0.3	11.6558858	11.65747	11.65618	11.656062	11.6547575
100		8.8287588	8.82942	8.82900	8.829033	8.82754823
105		6.5177905	6.51763	6.51802	6.518063	6.51635508
95	0.4	13.5107083	13.51426	13.51182	13.510861	13.5078924
100		10.9237708	10.92507	10.92474	10.923943	10.9208908
105		8.7299362	8.72936	8.73089	8.730102	8.72680424
95	0.5	15.4427163	15.44890	15.44587	15.442822	15.4370694
100		13.0281555	13.03015	13.03107	13.028271	13.0225321
105		10.9296247	10.92800	10.93253	10.929736	10.9237503
95	0.6	-	-	-	17.406402	17.396428
100		-	-	-	15.128426	15.118595
105		-	-	-	13.113874	13.1038552
95	0.8	-	-	-	21.349949	21.3261438
100		-	-	-	19.288780	19.2655176
105		-	-	-	17.423935	17.4008033
95	1.0	-	-	-	25.252051	25.2052379
100		-	-	-	23.367535	23.3219514
105		-	-	-	21.638238	21.5933927

Table 3: Comparison with Zhang (2001, 2003) and Hsu and Lyuu (2005) with a wide range of volatilities. The parameters are from Table 2 of Zhang (2003) and Table 3 of Hsu and Lyuu (2005). The options are calls with $S = 100$, $r = 0.09$, and $\tau = 1$.

X	σ	Exact	TE6	Hsu & Lyuu	Ours
95	0.05	15.1162646	15.11626	15.116230	15.1162644
100		11.3036080	11.30360	11.304034	11.3036045
105		7.5533233	7.55335	7.554073	7.55327778
95	0.1	15.2138005	15.21396	15.213921	15.2137608
100		11.6376573	11.63798	11.637813	11.637525
105		8.3912219	8.39140	8.391189	8.39083318
95	0.2	16.6372081	16.63942	16.637276	16.6361089
100		13.7669267	13.76770	13.767043	13.7654757
105		11.2198706	11.21879	11.220047	11.217842
95	0.3	19.0231619	19.02652	19.023236	19.0185667
100		16.5861236	16.58509	16.586222	16.5810236
105		14.3929780	14.38751	14.393083	14.3870805
95	0.4	21.7409242	21.74461	21.740973	21.7291244
100		19.5882516	19.58355	19.588307	19.5759378
105		17.6254416	17.61269	17.625501	17.6123103
95	0.5	24.5718705	24.57740	24.571913	24.5479028
100		22.6307858	22.62276	22.630828	22.6065085
105		20.8431853	20.82213	20.843226	20.8182163
95	0.6	-	-	27.425278	27.3828984
100		-	-	25.655297	25.612978
105		-	-	24.013011	23.9703437
95	0.8	-	-	33.031740	32.9288767
100		-	-	31.535716	31.434324
105		-	-	30.133505	30.0330626
95	1.0	-	-	38.361352	38.1586628
100		-	-	37.085174	36.8860543
105		-	-	35.881483	35.685358

Table 4: Comparison with Ju (2002) and Zhang (2001) and Hsu and Lyuu (2005) . The exact values are based on Zhang (2001) and quoted from Table 7 of Zhang (2001). Ju's Taylor expansion method is denoted as TE6. The parameters are from Table 2 of Ju (2002) and Table 7 of Zhang (2001) and Table 4 of Hsu and Lyuu (2005). The options are calls with $S = 100$, $r = 0.09$, and $\tau = 3$.

		τ = 1					τ = 3				
X	σ	Exact	PDE1	PDE2	Hsu & Lyuu	Ours	Exact	PDE1	PDE2	Hsu & Lyuu	Ours
95	0.05	8.8088392	8.8088241	8.8088241	8.808717	8.80883886	15.1162646	15.1162526	15.1162526	15.116230	15.1162644
100		4.3082350	4.3080602	4.3080602	4.309247	4.30823109	11.3036080	11.3035792	11.3035792	11.304034	11.3036045
105		0.9583841	0.9583277	0.9583277	0.960068	0.95833085	7.5533233	7.5531978	7.5531978	7.554073	7.55327778
95	0.1	8.9118509	8.9118054	8.9118054	8.912238	8.91183603	15.2138005	15.2137661	15.2137661	15.213921	15.2137608
100		4.9151167	4.9150253	4.9150253	4.914254	4.91507534	11.6376573	11.6376011	11.6376011	11.637813	11.637525
105		2.0700634	2.0700251	2.0700251	2.072473	2.06992974	8.3912219	8.3911498	8.3911498	8.391189	8.39083318
95	0.2	9.9956567	9.9956323	9.9956323	9.995661	9.99536215	16.6372081	16.6371770	16.6371770	16.637276	16.6361089
100		6.7773481	6.7773279	6.7773279	6.777748	6.77699941	13.7669267	13.7668950	13.7668950	13.767043	13.7654757
105		4.2965626	4.2964614	4.2964614	4.297021	4.29594093	11.2198706	11.2198412	11.2198412	11.220047	11.217842
95	0.3	11.6558858	11.6558892	11.6558892	11.656062	11.6547575	19.0231619	19.0230953	19.0231388	19.023236	19.0185667
100		8.8287588	8.8287699	8.8287699	8.829033	8.82754823	16.5861236	16.5860134	16.5861083	16.586222	16.5810236
105		6.5177905	6.5178134	6.5178134	6.518063	6.51635508	14.3929780	14.3927638	14.3929591	14.393083	14.3870805
95	0.4	13.5107083	13.5107373	13.5107373	13.510861	13.5078924	21.7409242	21.7359140	21.7409067	21.740973	21.7291244
100		10.9237708	10.9238047	10.9238049	10.923943	10.9208908	19.5882516	19.5801909	19.5882367	19.588307	19.5759378
105		8.7299362	8.7299785	8.7299789	8.730102	8.72680424	17.6254416	17.6129231	17.6254290	17.625501	17.6123103
95	0.5	15.4427163	15.4427436	15.4427631	15.442822	15.4370694	24.5718705	24.5164835	24.5718583	24.571913	24.5479028
100		13.0281555	13.0281668	13.0282104	13.028271	13.0225321	22.6307858	22.5534589	22.6307744	22.630828	22.6065085
105		10.9296247	10.9295940	10.9296853	10.929736	10.9237503	20.8431853	20.7378307	20.8431724	20.843226	20.8182163
95	0.6	-	17.4057119	17.4063840	17.406402	17.396428	-	27.1922830	27.4252385	27.425278	27.3828984
100		-	15.1272033	15.1284092	15.128426	15.118595	-	25.3547907	25.6552489	25.655297	25.612978
105		-	13.1117954	13.1138637	13.113874	13.1038552	-	23.6323908	24.0129680	24.013011	23.9703437
95	0.8	-	21.3206229	21.3500057	21.349949	21.3261438	-	31.8446547	33.0316957	33.031740	32.9288767
100		-	19.2465024	19.2888389	19.288780	19.2655176	-	30.1240393	31.5356736	31.535716	31.434324
105		-	17.3646285	17.4239955	17.423935	17.4008033	-	28.4742281	30.1334450	30.133505	30.0330626
95	1.0	-	25.0465250	25.2521580	25.252051	25.2052379	-	35.4451734	38.3595938	38.361352	38.1586628
100		-	23.1006194	23.3676388	23.367535	23.3219514	-	33.7509030	37.0830464	37.085174	36.8860543
105		-	21.2980435	21.6383464	21.638238	21.5933927	-	32.1020703	35.8789184	35.881483	35.685358

Table 5: Comparison with the one-dimensional PDE method of $Ve\tilde{c}e\tilde{r}$ (2001) and Hsu and Lyuu (2005).PDE1 is based on the 100×2000 grid over $[0,\tau]\times[-1,1]$ .PDE2 is based on the 100×10000 grid over $[0,\tau]\times[-1,9]$. The parameters and numerical data for Exact and Hsu & Lyuu are from Tables 3 and 4. The numerical data for PDE1 and PDE2 are from Hsu (2005).The options are calls with $S = 100$ and $r = 0.09$.

X	σ	Lower Bound	Hsu & Lyuu	Fusai	Exact	Monte Carlo	Ours
95	0.05	8.8088	8.808717	8.80885	8.8088392	8.81	8.80883886
100		4.3082	4.309246	4.30824	4.3082350	4.31	4.30823109
105		0.9583	0.960069	0.95839	0.9583841	0.95	0.95833085
95	0.10	8.9118	8.912238	8.91185	8.9118509	8.91	8.91183603
100		4.9150	4.914254	4.91512	4.9151167	4.91	4.91507534
105		2.0699	2.072473	2.07007	2.0700634	2.06	2.06992974
90	0.30	14.9827	14.984037	14.98396	-	14.96	14.9827819
100		8.8275	8.829033	8.82876	8.8287588	8.81	8.82754823
110		4.6949	4.696895	4.69671	-	4.68	4.69490183
90	0.50	18.1829	18.188933	18.18885	-	18.14	18.1829569
100		13.0225	13.028271	13.02816	13.0281555	12.98	13.0225321
110		9.1179	9.124414	9.12432	-	9.10	9.11794987
90	0.60	-	19.964542	-	-	19.94	19.9541628
100		-	15.128426	-	-	15.13	15.118595
110		-	11.342769	-	-	11.36	11.3322821
90	0.80	-	23.622784	-	-	23.61	23.5980253
100		-	19.288780	-	-	19.33	19.2655176
110		-	15.739790	-	-	15.74	15.7164077
90	1.00	-	27.305012	-	-	27.25	27.256476
100		-	23.367535	-	-	23.36	23.3219514
110		-	20.051542	-	-	20.03	20.0069488

Table 6: Comparison with the lower bounds of Rogers and Shi (1995) and Hsu and Lyuu (2005). The parameters, lower bounds, and Monte Carlo results for $\sigma \leq 0.5$ are from Table 3 of Rogers and Shi (1995) and Table 1 of Thompson (1999). The Monte Carlo results for $\sigma \geq 0.5$ are based on 2×10^6 simulation paths. The options are calls with $S = 100$, $r = 0.09$, and $\tau = 1$. Hsu & Lyuu algorithm's computed option values and the exact option values are from Table 3. Fusai is based on the data in Table 3 of Fusai (2004) computed using the most computing times. The two boxed numbers are lower than the lower bounds of Rogers and Shi.

X	σ	Lower Bound	Hsu & Lyuu	Hsu & Lyuu (revised)	Exact	Ours
95	0.05	8.8088	8.808717	8.808855	8.8088392	8.80883886
100		4.3082	4.309246	4.308307	4.3082350	4.30823109
105		0.9583	0.001136	0.958552	0.9583841	0.95833085
95	0.10	8.9118	8.912238	8.912392	8.9118509	8.91183603
100		4.9150	4.914254	4.916203	4.9151167	4.91507534
105		2.0699	2.072473	2.071247	2.0700634	2.06992974

Table 7: Comparison with the lower bounds of Rogers and Shi (1995) and Hsu and Lyuu (2005) using the revised algorithm. The data and parameters are from Table 6 except that the revised algorithm incorporates tighter running-sum ranges.

Sigma σ	Interest Rate r	Roger & Shi Lower	Ours
0.1	0.05	1.2454	1.24541
	0.09	0.6992	0.699247
	0.15	0.2516	0.251641
0.2	0.05	3.4044	3.40441
	0.09	2.6216	2.62164
	0.15	1.7098	1.70982
0.3	0.05	5.6246	5.62469
	0.09	4.7382	4.73822
	0.15	3.6085	3.60852

Table 8: Comparison with the lower bounds of Rogers and Shi (1995) on foating-strike Asian option prices for $S = 100$ with an expiry time of 1 year.

Chapter 5

Conclusions and Future Work

We extend Thompson (1998) a lower bound to price fixed-strike and floating-strike Asian options. The results can be summarized as follows:

- When volatility is small, the result of our approximation A' approaches the exact result with the difference starting from ten-thousandths place.
- When volatility is large, the result of our approximation A' approaches the exact result with the difference starting from one-hundredth place.
- We can get lower bounds efficiently.
- As stated above, our lower bound deteriorates somewhat when the volatility is large. A better approximate lower bound may be possible by replacing approximation A' with a different one. That may be a course for future work .
- If stock price does not follow a generalized Browning motion, we cannot use this approximation formula. We should try to seek for another approximation formula under this situation.

Appendix

In this appendix, we will prove two theorems.

Proof for Theorem 2.5 Two random variable X, Y with $X \sim \phi(\mu_x, \sigma_x^2)$ and $Y \sim \phi(\mu_y, \sigma_y^2)$ then $E(e^X I(Y>0)) = e^{u_x + \frac{1}{2}\sigma_x^2}\Phi\left(\frac{\mu_y + c}{\sigma_y}\right)$ where $\Phi(\cdot)$ is the normal distribution function, $I(\cdot)$ is the indicator function and $c = Cov(X, Y)$.

Proof:

By definition of Expectation,

$$E(e^X I(Y>0)) = \int_{-\infty}^{\infty}\int_0^{\infty} \frac{e^x}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}} dydx.$$

Let $u = \frac{x-\mu_x}{\sigma_x}$, $v = \frac{y-\mu_y}{\sigma_y}$. Then $dxdy = \sigma_x\sigma_y dudv$, and we can get

$$= \int_{-\infty}^{\infty}\int_{-\mu_y/\sigma_y}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{u\sigma_x + \mu_x} \cdot e^{\frac{-1}{2(1-\rho^2)}\{u^2 - 2\rho uv + v^2\}} dvdu$$

$$= \frac{e^{\mu_x}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\mu_y/\sigma_y}^{\infty} e^{u\sigma_x} \cdot e^{\frac{-1}{2(1-\rho^2)}\{(u-\rho v)^2 + v^2(1-\rho^2)\}} dvdu$$

$$= \frac{e^{\mu_x}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\mu_y/\sigma_y}^{\infty} e^{u\sigma_x} \cdot e^{\frac{(u-\rho v)^2}{-2(1-\rho^2)} + \frac{v^2}{-2}} dvdu.$$

We change the variables again: $w = \frac{u-\rho v}{\sqrt{1-\rho^2}}$. Then $dw = \frac{du}{\sqrt{1-\rho^2}}$, and we get

$$= \frac{e^{\mu_x}}{2\pi} \int_{-\mu_y/\sigma_y}^{\infty}\int_{-\infty}^{\infty} e^{\frac{-w^2}{2} + \sigma_x\sqrt{1-\rho^2}\cdot w} \cdot e^{\frac{-v^2}{2} + \sigma_x\rho\cdot v} dwdv$$

$$= \frac{e^{\mu_x + 0.5\cdot\sigma_x^2}}{\sqrt{2\pi}} \int_{-\mu_y/\sigma_y}^{\infty} e^{-\frac{1}{2}(v-\sigma_x\rho)^2} dv \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\left(w-\sigma_x\sqrt{1-\rho^2}\right)^2}{2}} dw$$

Given that $f(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(w-\sigma_x\sqrt{1-\rho^2}\right)^2}{2}}$ is a PDF, then $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\left(w-\sigma_x\sqrt{1-\rho^2}\right)^2}{2}} dw = 1$, and

the above equals

$$= \frac{e^{\mu_x + 0.5\cdot\sigma_x^2}}{\sqrt{2\pi}} \int_{-\mu_y/\sigma_y}^{\infty} e^{-\frac{1}{2}(v-\sigma_x\rho)^2} dv$$

We can change variables: $k = v - \sigma_x\rho$. Then $dk = dv$, we can reduce above equation

$$= \frac{e^{\mu_x + 0.5\cdot\sigma_x^2}}{\sqrt{2\pi}} \int_{\frac{-\mu_y}{\sigma_y}-\sigma_x\rho}^{\infty} e^{-\frac{1}{2}k^2} dk = \frac{e^{\mu_x + 0.5\cdot\sigma_x^2}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\mu_y}{\sigma_y}+\sigma_x\rho} e^{-\frac{1}{2}k^2} dk$$

(by the symmetry property of normal distribution)

$$= e^{\mu_x + 0.5\cdot\sigma_x^2} \times \Phi\left(\frac{\mu_y}{\sigma_y} + \sigma_x\rho\right) = e^{\mu_x + 0.5\cdot\sigma_x^2} \times \Phi\left(\frac{\mu_y + c}{\sigma_y}\right) .$$

Proof for Theorem 2.6

For any random variable X with density $f_X(x)$, we have

(i) $E[(S_t - K)I(X > \gamma)] = E[S_t - K; X > \gamma]$

(ii) $\frac{\partial}{\partial\lambda}\frac{1}{T}\int_0^T E(S_t - K; X > \gamma)\, dt = \frac{1}{T}\int_0^T E(S_t - K \mid X = \gamma)(- f_X(x))dt$

Proof:

(i) $[(S_t - K)I(X > \gamma)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (S_t - K)I(X > \gamma) f_{B_t, X}(B_t, X) dX dB_t$

$$= \int_{-\infty}^{\infty}\int_{r}^{\infty} (S_t - K) f_{B_t, X}(B_t, X) dX dB_t = E[S_t - K; X > \gamma]$$

(ii) By the definition of expectation,

$$\frac{\partial}{\partial\lambda}\frac{1}{T}\int_0^T E(S_t - K; X > \gamma)\,dt = \frac{\partial}{\partial\gamma}\frac{1}{T}\int_0^T\int_{-\infty}^{\infty}\int_{\gamma}^{\infty}(S_t - K)f_{B_t,X}(B_t, X)\,dX\,dB_t\,dt\,.$$

We can exchange the integral and the partial derivative to get

$$= \frac{1}{T}\int_0^T\int_{-\infty}^{\infty}\frac{\partial}{\partial\gamma}\int_{\gamma}^{\infty}(S_t - K)f_{B_t,X}(B_t, X)\,dX\,dB_t\,dt$$

$$= \frac{1}{T}\int_0^T\int_{-\infty}^{\infty}\frac{\partial}{\partial\gamma}\int_{\gamma}^{\infty}(S_t - K)f_{B_t,X}(B_t, X)\,dX\,dB_t\,dt\,.$$

By Leibnitz’s Rule,

$$= \frac{1}{T}\int_0^T\int_{-\infty}^{\infty}-(S_t - K)f_{B_t,X}(B_t, \gamma)\,dB_t\,dt$$

$$= \frac{1}{T}\int_0^T\int_{-\infty}^{\infty}-(S_t - K)f_{B_t,X}(B_t, \gamma)\cdot\frac{f_X(\gamma)}{f_X(\gamma)}\,dB_t\,dt$$

$$= \frac{1}{T}\int_0^T\int_{-\infty}^{\infty}-(S_t - K)f_{B_t,X}(B_t \mid \gamma)\cdot f_X(\gamma)\,dB_t\,dt\,.$$

By the definition of expectation, the above equals

$$\frac{1}{T}\int_0^T E(S_t - K \mid X = \gamma)(-f_X(x))\,dt\,.$$

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