On the Complexity of the Ritchken Sankarasubramanian Interest Rate Model

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Chapter 1

Introduction

Most term structure models exist for pricing interest-rate contingent claims. The popular one presented by Heath, Jarrow, and Morton (HJM) (1992) is a powerful model. In the HJM model, the only inputs needed to construct the term structure are the initial yield curve and the volatility structure for all forward rates. Its advantage is that risk preference assumptions and estimation of the drift term are not needed. Because of fewer inputs required of the model, the stochastic process of the model could cover general classes of stochastic processes. Due to fewer restrictions on the term structure, the HJM model may be non-Markovian, and contingent claims are difficult to price with the lattice method. This is because when pricing derivatives of non-Markovian processes with the lattice method, it is needed to preserve all the information on whole paths, and since evolving nodes usually do not combine, the lattice grows exponentially with the number of time periods. The algorithm with exponentially growth rate would suffer in both time and memory resources. To solve this problem and make values of contingent claims computable under the HJM model, some conditions and new parameters would be imposed. Based on the HJM models, Ritchken and Sankarasubramanian (RS) develop a family of models for pricing American options on interest rate derivatives. In the RS model, the term structure is described by a two-state Markovian process, and for this sake the volatility structure has to be restricted. The restrictions on the volatility structure are on forward rates instead of spot rates. It still allows general classes of volatility structures. The algorithms developed by Ritchken and Sankarasubramanian (1995) for the RS model are more efficient than the original HJM model. Because of the Markovian property for the RS model, forward induction can move further with only the information at the time rather than the past whole information. Although the algorithm is more efficient in practice, it still faces the problem of increased computational complexity with increasing number of time periods. In fact, the algorithm still grows exponentially for small time partition Δt, as we will show in the thesis.

The purpose of this thesis is to illustrate how the lattice of the RS algorithm grows with the number of time periods under the proportional RS model with a flat forward rate curve. Our finding is that the RS algorithm grows exponentially under particular assumptions for small time partition Δt, or, equivalently, large n. In the paper of Cakici and Zhu (2001), the algorithm based on the RS algorithm is simplified without "mean tracking." We mean that the tree's growth is centered around its mean. The thesis shows how their algorithm explodes exponentially too. After showing the growth rate of the lattice by the mathematical approach, this thesis will provide numerical examples on the RS algorithm and compare the numerical results with our theoretical results. The numerical results confirm the theoretical results that the lattice explodes exponentially for sufficiently large n. For example, under parameters $\sigma = 0.25$, $T = 5$, $\kappa = 0.02$, and $r_0 = 0.04$ (to be defined later), the algorithm works fine for about $n \leq 310$. Once n is larger, the total number of nodes will grow exponentially large beyond computer memory capacity.

The thesis proceeds as follows. Chapter 2 describes the HJM and RS models. In chapter 3, our theoretical results on the RS algorithm will be presented. In chapter 4, we will give some numerical results and check if our assumptions made in theoretical results are sustainable. Then chapter 5 summarizes the thesis.

Chapter 2

Some Backgrounds

2.1 The HJM Model

In the HJM model, $f(t,T)$ denotes the forward rate at time t for instantaneous and riskless rate at time T. The forward rate $f(t,T)$ follows the stochastic process

$$df(t,T) = \mu_f(t,T)dt + \sigma_f(t,T)d\omega(t),$$

where $\mu_f(t,T)$ and $\sigma_f(t,T)$ are the drift and volatility terms which may depend on the past history of the Wiener processes $\omega(t)$. In particular, the short rate $r(t)$ equals $f(t,t)$. The volatility term $\sigma_f(t,T)$ could be chosen arbitrarily and once it is chosen, the drift term $\mu_f(t,T)$ is uniquely determined by

$$\mu_f(t,T) = \sigma_f(t,T)\int_t^T \sigma_f(t,s)ds.$$

The price of a pure discount bond $P(t,T)$ at time t with maturity date T equals

$$P(t,T) = e^{-\int_t^T f(t,s)ds}.$$

Let $g(0)$ denote the value of a European claim at date 0 with payoff at date s. The fair price of the claim is

$$g(0) = E_0[e^{-\int_0^s r(t)dt} g(s)].$$

An arbitrary selection of the volatility term $\sigma_f(t,T)$ makes the model include various term structure models as special cases. But this generality leads to problems on pricing in practice, and no efficient numerical methods exist for the HJM model. Ritchken and Sankarasubramanian place particular restrictions on the volatility term and eliminate some pricing problems.

2.2 The RS Model

The restriction on the volatility term $\sigma_f(t,T)$ is given by

$$\sigma_f(t,T) = \sigma_f(t,t)k(t,T),$$

where

$$k(t,T) = e^{-\int_t^T \kappa(x)dx},$$

$\sigma_f(t,t)$ is volatility of the short rate $r(t)$, and $\kappa(x)$ is a deterministic function given exogenously. No particular restrictions are imposed on the short rate volatility $\sigma_f(t,t)$. The restriction on $\sigma_f(t,T)$ does sacrifice some freedom for the volatility term, but it turns out that the term structure can be represented by a two-state Markovian model. Define $\phi(t)$ to represent the accumulated variance for the forward rate up to date t and it equals

$$\phi(t) = \int_0^t \sigma_f^2(u,t)du = \int_0^t \sigma_f^2(u,u)k^2(u,t)du.$$

Then the two-state Markovian model for the RS model follows

$$\begin{aligned} dr(t) &= \mu(r,\phi,t)dt + \sigma_f(t,t)d\omega(t), \\ d\phi(t) &= [\sigma_f^2(t,t) - 2\kappa(t)\phi(t)]dt, \end{aligned}$$

where

$$\mu(r,\phi,t) = \kappa[f(0,t) - r(t)] + \phi(t) + \frac{d}{dt}f(0,t)$$

Under the RS model, the value of pure discount bond and contingent claim on the term structure turns out to be

$$P(t,T) = (\frac{P(0,T)}{P(0,t)})e^{-\beta(t,T)(r(t)-f(0,t))-\frac{1}{2}\beta^2(t,T)\phi(t)},$$

$$g(0) = E_{r,\phi}[e^{\int_0^s r(t)dt}g(s)],$$

where

$$\beta(t,T) = \int_t^T k(t,u)du.$$

2.3 The RS Algorithm

In this section, we introduce the RS algorithm for the proportional model with flat forward rate curve. One family of the RS model is given by setting

$$\sigma_f(t,t) = \sigma[r(t)]^\gamma$$

as the volatility term for constants $\sigma, \gamma \geq 0$. Particularly, $\gamma = 1$ is the proportional model. In the next step, we use another transformation $Y(t)$ in order to form a constant volatility process by Nelson and Ramaswamy (1990),

$$Y(t) = \int \frac{1}{\sigma r(t)} dr(t) = ln[r(t)]/\sigma.$$

That is

$$r(t) = e^{\sigma Y(t)}.$$

And the process $Y(t)$ follows

$$dY(t) = m(Y,\phi,t)dt + d\omega(t),$$

where

$$m(Y,\phi,t) = \frac{1}{\sigma}[\nu(Y,\phi,t) - \frac{1}{2}\sigma^2],$$

and

$$\nu(Y,\phi,t) = \frac{\kappa}{e^{\sigma Y(t)}}[f(0,t) - e^{\sigma Y(t)}] + \frac{\phi(t)}{e^{\sigma Y(t)}}.$$

Let $\kappa(t)$ be a constant κ. The process of $Y(t)$ is then with constant volatility 1 and drift $m(Y, \phi, t)$. After introduction to the settings of the model, we begin the procedure to establish the lattice of the RS algorithm.

The tree begins at time 0 with a pair of data (y_0, ϕ_0) with parameters r_0, κ, σ and ϕ_0. At each time i, assume a node with data of (y_i, ϕ_i) which includes its past information up to time 0. In the next time $i+1$, (y_i, ϕ_i) moves up to $(y^+_{i+1}, \phi^+_{i+1})$ or down to $(y^-_{i+1}, \phi^-_{i+1})$ given by

$$\begin{aligned} y^+_{i+1} &= y_i + (J_i + 1)\sqrt{\Delta t}, \\ y^-_{i+1} &= y_i + (J_i - 1)\sqrt{\Delta t}, \end{aligned}$$

and

$$\phi^+_{i+1} = \phi^-_{i+1} = \phi_i + (\sigma^2 r_i^2 - 2\kappa\phi_i)\Delta t,$$

where J_i is an integer satisfying

$$m_i\sqrt{\Delta t} + 1 \geq J_i \geq m_i\sqrt{\Delta t} - 1,$$

where

$$m_i = \frac{\kappa(r_0 - r_i) + \phi_i}{\sigma r_i} - \frac{\sigma}{2}.$$

Figure 2.1 shows that the evolving nodes at time $i+1$ of the RS algorithm are centered at J_i steps from y_i, and the up node y^+_{i+1} and the down node y^-_{i+1} are J_i+1 and J_i-1 steps from y_i, respectively.

Let $p_i = p(y_i, \phi_i)$ be the branching probability of the up jump for the node. Then p must satisfy

$$p_i(y^+_{i+1} - y_i) + (1-p)(y_i - y^-_{i+1}) = m_i\Delta t,$$

or

$$p_i = \frac{m_i\Delta t + (y_i - y^-_{i+1})}{(y^+_{i+1} - y^-_{i+1})}$$

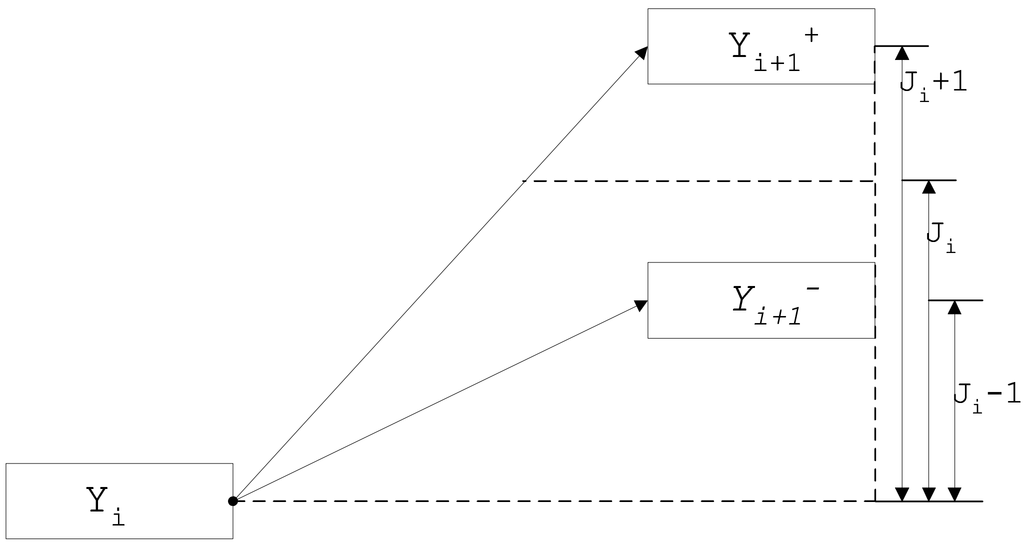

Figure 2.1: EVOLVING NODES FOR THE RS ALGORITHM.

to ensure that the mean of $Y(t)$ matches the true drift $m(t)$. For probability to lie between 0 and 1, J_i is chosen to satisfy

$$\begin{aligned}
0 &\leq \frac{m_i\Delta t + (y_i - y_{i+1}^-)}{(y_{i+1}^+ - y_{i+1}^-)} \leq 1, \\
0 &\leq \frac{m_i\Delta t + (1-J_i)\sqrt{\Delta t}}{2\sqrt{\Delta t}} \leq 1, \\
-m_i\Delta t &\leq (1-J_i)\sqrt{\Delta t} \leq 2\sqrt{\Delta t} - m_i\Delta t, \\
-m_i\sqrt{\Delta t} - 1 &\leq -J_i \leq 1 - m_i\sqrt{\Delta t}, \\
m_i\sqrt{\Delta t} - 1 &\leq J_i \leq m_i\sqrt{\Delta t} + 1.
\end{aligned}$$

At any time t, there may be many different ϕ values corresponding to the same value of y since every ϕ value has its unique path to y. We only keep track of two paths which yield maximum and minimum values of ϕ without keeping all values of ϕ and replace original ϕ values with linear interpolation method. For example, a node at time i is with the same value of y_i but different values of ϕ_i. We only keep its maximum and minimum values denoted by ϕ_i^{max} and ϕ_i^{min}. Then we partition the node into m equally divided points with $\phi_i(k)$, $k = 1, 2, \ldots, m$ as

$$\phi_i^{max} = \phi_i(1) > \phi_i(2) > \cdots > \phi_i(m) = \phi_i^{min}.$$

Every node with a value of y_i and m different values of $\phi_i(k)$ generates $2m$ results for the time $i+1$ since a pair of (y_i, ϕ_i) moves either up or down. $2m$ new paths move into different nodes at the time $i+1$ and the same process proceeds for the nodes till time to maturity. The example is shown in Figure 2.2. Path 1 of an up move from $(y_i, \phi_i(1))$ yields neither maximum nor minimum ϕ value at node y_{i+1}^{+} and the ϕ value is ignored. Path 2 of a down move from $(y_i, \phi_i(1))$ yields maximum ϕ value at the node y_{i+1}^{-} and the value will be preserved as $\phi_{i+1}^{max} = \phi_{i+1}(1)$ for the node y_{i+1}^{-}. Path 3 of an up move from $(y_i, \phi_i(m))$ yields minimum ϕ value at the node y_{i+1}^{+} and the value will be preserved as $\phi_{i+1}^{min} = \phi_{i+1}(m)$ for the node y_{i+1}^{+}.

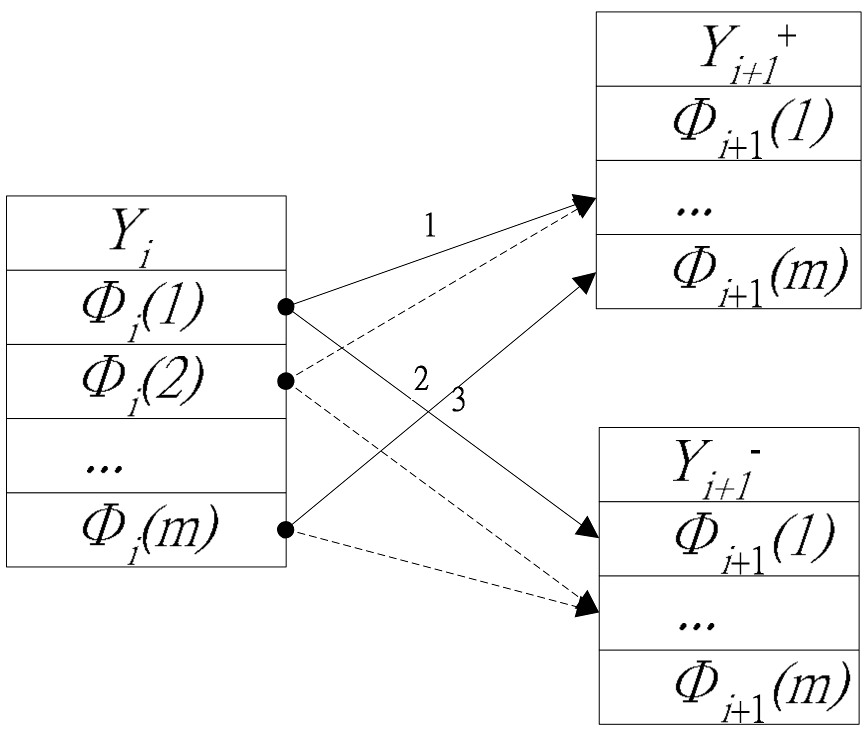

Figure 2.2: PARTITIONS OF Φ VALUES.

2.4 The Cakici-Zhu Algorithm

In the Cakici-Zhu algorithm, the process of Y is simplified to

$$\begin{aligned} y_{i+1}^{+} &= y_i + J\sqrt{\Delta t} \\ y_{i+1}^{-} &= y_i - J\sqrt{\Delta t}, \end{aligned}$$

where J is chosen to have probability value

$$p_i = \frac{m_i \Delta t + (y_i - y_{i+1}^{-})}{(y_{i+1}^{+} - y_{i+1}^{-})} = \frac{m_i \Delta t + J\sqrt{\Delta t}}{2J\sqrt{\Delta t}}$$

with $[0, 1]$. Procedure for the lattice is the same with the RS algorithm. Figure 2.3 shows that the evolving nodes for the Cakici-Zhu algorithm are centered to the original node y_i and equally expand with J_i steps up to y_{i+1}^{+} and J_i steps down to y_{i+1}^{-}.

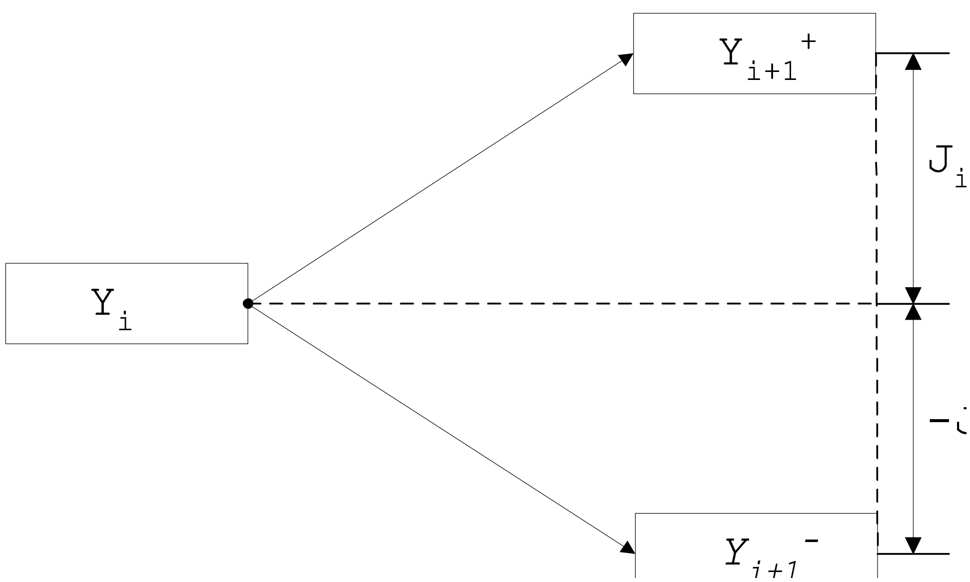

Figure 2.3: EVOLVING NODES FOR THE CAKICI-ZHU ALGORITHM.

Chapter 3

Theoretical Results

3.1 The Explosion of Cakici-Zhu Algorithm

Let (y_i, Φ_i) denote a node at time i, which depends on the path reaching the node. Every node generates m values of m_i respect to pairs of $(y_i, \phi_i(k))$, $k = 1, \ldots, m$, to determine (y_{i+1}, Φ_{i+1}) in the next time increment. Given (y_i, ϕ_i),

$$y_{i+1}^{+} = y_i + J_i\sqrt{\Delta t}, \tag{3.1}$$

$$y_{i+1}^{-} = y_i - J_i\sqrt{\Delta t}, \tag{3.2}$$

$$\phi_{i+1} = \phi_i + [\sigma^2 r_i^2 - 2\kappa\phi_i]\Delta t, \tag{3.3}$$

where $r_i = e^{\sigma y_i}$ and J_i is chosen to satisfy the valid probability value $p(y_i, \phi_i)$,

$$p(y_i, \phi_i) = p_i = \frac{m_i\Delta t + J_i\sqrt{\Delta t}}{2J_i\sqrt{\Delta t}},$$

where

$$m_i = \frac{\kappa(r_0 - r_i) + \phi_i}{\sigma r_i} - \frac{\sigma}{2}.$$

Steps to obtain y_{i+1} are:

1. Use (y_i, ϕ_i) to compute m_i.

2. Use m_i to compute J_i under the restriction $0 \leq p_i \leq 1$.

3. Use equations (3.1) and (3.2) to derive y_{i+1}.

From equation (3.3),

$$\begin{aligned}\phi_{i+1} &= \phi_i + [\sigma^2 r_i^2 - 2\kappa\phi_i]\Delta t \\ &= [1 - 2\kappa\Delta t]\phi_i + \sigma^2 r_i^2 \Delta t.\end{aligned}$$

Let $a = 1 - 2\kappa\Delta t > 0$ and $\phi_0 = 0$. Then

$$\begin{aligned}\phi_{i+1} &= a\phi_i + \sigma^2 r_i^2 \Delta t \\ &= a(a\phi_{i-1} + \sigma^2 r_{i-1}^2 \Delta t) + \sigma^2 r_i^2 \Delta t \\ &= a^2\phi_{i-1} + a\sigma^2 r_{i-1}^2 \Delta t + \sigma^2 r_i^2 \Delta t \\ &= a^{i+1}\phi_0 + \sum_{j=0}^{i} a^{i-j} r_j^2 \sigma^2 \Delta t \\ &= \sum_{j=0}^{i} a^{i-j} r_j^2 \sigma^2 \Delta t.\end{aligned}$$

We assume a path with $i-1$ up moves and 1 down move from (y_0, ϕ_0). We will show that the path grows exponentially in order to satisfy the valid probability. Hence the Cakici-Zhu tree will explode with the path. Now,

$$r_{i-1} = r_{i-2} e^{J_{i-2}\sigma\sqrt{\Delta t}} \geq r_{i-2} e^{\sigma\sqrt{\Delta t}} \geq r_0 e^{(i-1)\sigma\sqrt{\Delta t}}$$

and

$$r_i = r_{i-1} e^{-J_{i-1}\sigma\sqrt{\Delta t}} \leq r_{i-1} e^{-\sigma\sqrt{\Delta t}}$$

since $J_j \geq 1$ for any time j. Hence probability p_i must satisfy

$$\begin{aligned}p_i &= \frac{(\frac{\kappa(r_0 - r_i) + \phi_i}{\sigma r_i} - \frac{\sigma}{2})\Delta t + J_i\sqrt{\Delta t}}{2J_i\sqrt{\Delta t}} \\ &\geq \frac{(\frac{\phi_i}{\sigma r_i} - \frac{\kappa}{\sigma} - \frac{\sigma}{2})\Delta t}{2J_i\sqrt{\Delta t}} + \frac{1}{2} \\ &= \frac{\Delta t}{2J_i\sqrt{\Delta t}}\left(\frac{\sigma\Delta t\sum_{j=0}^{i-1} a^{i-1-j} r_j^2}{r_i}\right) - \frac{\Delta t(\frac{\kappa}{\sigma} - \frac{\sigma}{2})}{2J\sqrt{\Delta t}} + \frac{1}{2} \\ &\geq \frac{\sqrt{\Delta t}}{2J_i}\left(\frac{\sigma\Delta t r_{i-1}^2}{r_i}\right) - \frac{\Delta t(\frac{\kappa}{\sigma} - \frac{\sigma}{2})}{2J_i\sqrt{\Delta t}} + \frac{1}{2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sqrt{\Delta t}}{2J_i}(\frac{\sigma \Delta t r_{i-1}}{r_i/r_{i-1}}) - \frac{\Delta t(\frac{\kappa}{\sigma} - \frac{\sigma}{2})}{2J_i\sqrt{\Delta t}} + \frac{1}{2} \\
&\geq \frac{\sqrt{\Delta t}}{2J_i}(\sigma r_0 \Delta t e^{i\sigma\sqrt{\Delta t}}) - \frac{\sqrt{\Delta t}(\frac{\kappa}{\sigma} - \frac{\sigma}{2})}{2J} + \frac{1}{2} \\
&= \frac{\sqrt{\Delta t}}{2J_i}[\sigma r_0 \Delta t e^{i\sigma\sqrt{\Delta t}} - (\frac{\kappa}{\sigma} - \frac{\sigma}{2})] + \frac{1}{2}.
\end{aligned}
$$

For $p_i \leq 1$,

$$J_i \geq \sigma(\Delta t)^{1.5} r_0 e^{i\sigma\sqrt{\Delta t}} - \sqrt{\Delta t}(\frac{\kappa}{\sigma} - \frac{\sigma}{2}).$$

This means at time $i+1$, the distance of the top node from the root is at least

$$\sigma(\Delta t)^{1.5} r_0 e^{i\sigma\sqrt{\Delta t}} - \sqrt{\Delta t}(\frac{\kappa}{\sigma} - \frac{\sigma}{2})$$

and since all the nodes are located within, the size of the tree grows exponentially in i.

3.2 The Explosion of the RS Algorithm

In the RS algorithm, y_{i+1} follows the mean m_i of (y_i, ϕ_i) presented by

$$y^+_{i+1} = y_i + (J_i + 1)\sqrt{\Delta t},$$

$$y^-_{i+1} = y_i + (J_i - 1)\sqrt{\Delta t},$$

where J_i is an integer satisfying

$$m_i\sqrt{\Delta t} + 1 \geq J_i \geq m_i\sqrt{\Delta t} - 1$$

for $r_i = e^{\sigma y_i}$ and

$$m_i = \frac{\kappa(r_0 - r_i) + \phi_i}{\sigma r_i} - \frac{\sigma}{2}.$$

For ϕ_{i+1}, the equation follows the previous section,

$$\phi_{i+1} = \phi_i + [\sigma^2 r_i^2 - 2\kappa\phi_i]\Delta t = \sum_{j=0}^{i} a^{i-j} r_j^2 \sigma^2 \Delta t.$$

In this section, we show that under some conditions the RS tree will grow exponentially. After showing the growth rate of the RS tree, we will show that the conditions can be achieved if n is large enough. Suppose the path with i up moves satisfies the following conditions: (1) There exists a large number N, $N \leq n$, such that for $n \geq i \geq N$, $\sum_{j=0}^{i} J_j > 0$, and (2) we claim parameters κ and σ satisfy

$$(\frac{\kappa}{\sigma} + \frac{\sigma}{2})\sqrt{\Delta t} < 1.$$

Then

$$m_i\sqrt{\Delta t} = (\frac{\kappa(r_0 - r_i) + \phi_i}{\sigma r_i} - \frac{\sigma}{2})\sqrt{\Delta t} \geq (-\frac{\kappa}{\sigma} - \frac{\sigma}{2})\sqrt{\Delta t} > -1,$$

and $J_i \geq 0$ if we choose $J_i = \lfloor (m_i\sqrt{\Delta t}) \rfloor$. The second condition can be true with increasing n. Because σ and κ are given constants, we let $c = (\frac{\kappa}{\sigma} + \frac{\sigma}{2})$ in the later derivation for convenience. The value of m_i for the path at time i satisfies

$$\begin{aligned} m_i &= \frac{\kappa(r_0 - r_i) + \phi_i}{\sigma r_i} - \frac{\sigma}{2} \\ &\geq \frac{\phi_i}{\sigma r_i} - (\frac{\sigma}{2} + \frac{\kappa}{\sigma}) \\ &= \frac{a\phi_{i-1}}{\sigma r_i} - c + \frac{\sigma \Delta t r_{i-1}^2}{r_i} \\ &\geq \frac{\sigma \Delta t r_{i-1}}{r_i / r_{i-1}} - c \\ &= \frac{r_0 \sigma \Delta t e^{\sigma \sum_{j=0}^{i-2}(J_j+1)\sqrt{\Delta t}}}{e^{\sigma(J_{i-1}+1)\sqrt{\Delta t}}} - c \\ &= r_0 \sigma \Delta t e^{\sigma(i-2+\sum_{j=0}^{i-2} J_j - J_{i-1})\sqrt{\Delta t}} - c. \end{aligned}$$

If $\sum_{j=0}^{i-2} J_j - J_{i-1} \geq 0$ for any i, then $m_i > r_0\sigma\Delta t e^{(i-2)\sigma\sqrt{\Delta t}} - c$. And hence

$$y_{i+1} = y_0 + \sum_{j=0}^{i}(J_j + 1)\sqrt{\Delta t} \geq (J_i + 1)\sqrt{\Delta t} \geq m_i \Delta t > r_0 \sigma (\Delta t)^2 e^{\sigma(i-2)\sqrt{\Delta t}} - c\Delta t$$

grows exponentially in i. Otherwise, if $\sum_{j=0}^{i-1} J_j - J_i \leq 0$ in some time interval $N \leq$

$i \leq b$,[1] we have

$$J_b \geq \sum_{j=0}^{b-1} J_j = \sum_{j=0}^{b-2} J_j + J_{b-1} \geq \sum_{j=0}^{b-2} J_j + \sum_{j=0}^{b-2} J_j = 2\sum_{j=0}^{b-2} J_j$$
$$= 2\sum_{j=0}^{b-3} J_j + 2J_{b-2} \geq 4\sum_{j=0}^{b-3} J_j \geq \cdots \geq 2^{(b-N-1)} \sum_{j=0}^{N} J_j.$$

Then

$$y_{b+1} = y_0 + (\sum_{j=0}^{b} J_j + 1)\sqrt{\Delta t} = y_0 + (\sum_{j=0}^{b-1} J_j + 1)\sqrt{\Delta t} + J_b\sqrt{\Delta t} \geq 2^{(b-N-1)}(\sum_{j=0}^{N} J_j)\sqrt{\Delta t}.$$

Since $\sum_{j=0}^{N} J_j > 0$ is given at time N and can be considered as a constant after time N, y_b grows exponentially from time N.

The next question is if the number N exists. For N to exist, there must be at least one J_i and $J_i \geq 1$ in the path to satisfy $\sum_{j=0}^{i} J_j > 0$. If there is no such J_i that all $J_i = 0$ in the path, from

$$m_i > r_0\sigma\Delta t e^{(i-2+\sum_{j=0}^{i-2} J_j - J_{i-1})\sigma\sqrt{\Delta t}} - c = r_0\sigma\Delta t e^{(i-2)\sigma\sqrt{\Delta t}} - c,$$

m_i is bounded below. Let n be an even number and take $N = n/2$. The above inequality implies

$$m_N > r_0\sigma\Delta t e^{(n/2-2)\sigma\sqrt{\Delta t}} - c,$$

which shows that there exists an $m_N\sqrt{\Delta t} \geq 1$ with increasing n and thus $\sum_{j=0}^{N} J_j > 0$. We conclude that if n is sufficiently large, there exists a number N and for $i \geq N$, the tree grows exponentially. Particularly, we take $N = n$ and derive the number $n \equiv n^*$ to satisfy the inequality

$$m_N\sqrt{\Delta t} > (r_0\sigma\Delta t e^{(n^*-2)\sigma\sqrt{\Delta t}} - c)\sqrt{\Delta t} \geq 1.$$

[1] Here we claim b could extend to maturity date with the property $\sum_{j=0}^{i-1} J_j - J_i \leq 0$. If not that at the inequality does not sustain at some time t^*, we use the result of case $\sum_{j=0}^{i-2} J_j - J_{i-1} \geq 0$ and y_{t^*} will be exponentially large in t^*.

The RS tree will grow exponentially for some time if time periods n is larger than n^*. To simplify the inequality, let $\Delta t < 1$ and modify the inequality subject to our requirements $m_N\sqrt{\Delta t} \geq 1$. The inequality is then

$$\begin{aligned} r_0\sigma(\Delta t)^{1.5}e^{n\sigma\sqrt{\Delta t}} &> (1+c), \\ e^{(\sigma\sqrt{T})\sqrt{n}} &> [\frac{(1+c)}{r_0\sigma T^{1.5}}]n^{1.5}. \end{aligned}$$

Let both sides of inequality take logarithm and we have

$$(\sigma\sqrt{T})\sqrt{n} - 1.5\ln(n) > \lambda,$$

where $\lambda = \ln[\frac{(1+c)}{r_0\sigma T^{1.5}}]$. In the inequality $(\sigma\sqrt{T})\sqrt{n} - 1.5\ln(n) > \lambda$, although the terms σ and T are also in λ, the coefficient of $\sqrt{n}$, $\sigma\sqrt{T}$, seems to have the most important effect about the value of n that if the value of $\sigma\sqrt{T}$ is larger, a smaller n is required to satisfy the inequality and thus the tree will explode for smaller n. We will examine this conjecture in the next chapter.

Chapter 4

Numerical Results

In the previous chapter, we show that the process of Y grows exponentially with sufficiently large n. In this chapter, the numerical results will be presented to examine the growth rate of the lattice. Under the parameter values illustrated in RS's paper, the tree seems to grow linearly for small n. For larger n, the tree grows linearly in the early time periods but the nodes increase in tremendous speed in the rest of the time, and the tree explodes soon as time goes on.

The lattice is established by the nodes in the form of (y_i, ϕ_i) for $0 \leq i \leq n$, and the value of each y_i equals $y_0 + (Z)\sqrt{\Delta t}$ for some integer Z from

$$y_{i+1}^{+} = y_i + (J_i + 1)\sqrt{\Delta t},$$

$$y_{i+1}^{-} = y_i + (J_i - 1)\sqrt{\Delta t},$$

for integer J. At a particular time i, there will be maximum value and minimum value of y_i for all possible paths from bottom node to top node. Maximum value and minimum value of y_i are represented separately by y_i^{max} and y_i^{min}. Since y_i only takes value on $y_0 + (Z)\sqrt{\Delta t}$ for integers Z, the total number of different values of y_i at time i is at most $(y_i^{max} - y_i^{min})/\sqrt{\Delta t} + 1$. We call the number $(y_i^{max} - y_i^{min})/\sqrt{\Delta t} + 1$ the total number of nodes at time i. Note that all possible nodes from y_i^{min} to y_i^{max} are enumerated step by step in the lattice regardless of the existence of the corresponding path. For example, letting $i = 0$ and $J_0 = 0$, since there is only one node at time 0, we have

$$y_1^{max} = y_1^+ = y_0 + \sqrt{\Delta t} > y_0 - \sqrt{\Delta t} = y_1^- = y_1^{min},$$

so the total number of nodes at time 1 is 3. We illustrate the example in Figure 4.1 that three nodes are enumerated. But the nodes are not all reached. The node of $y_1 = y_0 + 0\sqrt{\Delta t}$, the middle node in the figure, is never reached by any of the paths, and the number of reachable nodes in the example is 2. Although all nodes from y_i^{min} to y_i^{max} are enumerated, only reachable nodes proceed to evolve next nodes for the time $i + 1$, and the other nodes except the reachable ones are skipped. Our measurement of the growth rate of the lattice is to see the total number of nodes with increasing time i. Since J is an integer satisfying

$$m_i\sqrt{\Delta t} + 1 \geq J \geq m_i\sqrt{\Delta t} - 1,$$

one choice of $J_i = \lfloor (m_i\sqrt{\Delta t}) \rfloor$ is made in our algorithm.

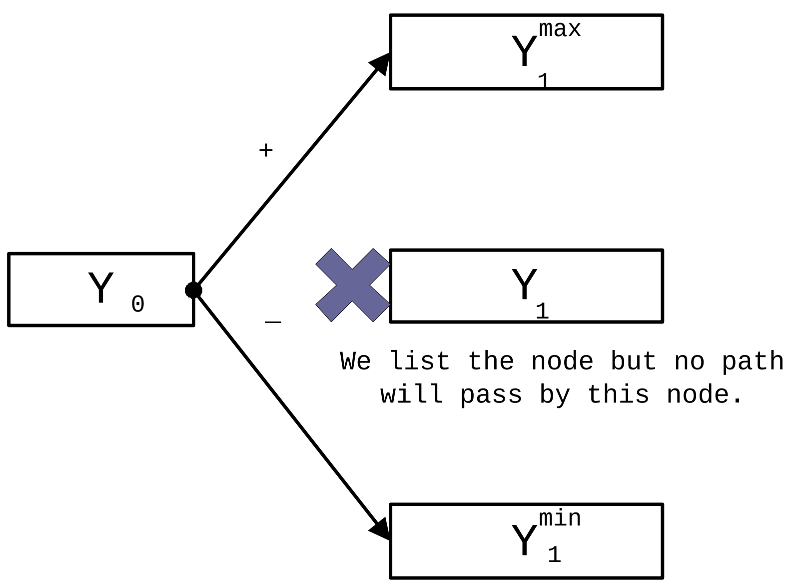

Figure 4.1: Total number of nodes and the number of reachable nodes.

In the following example, we use different n's to compare the number of nodes with increasing time i between small n and large n with other parameters unchanged.

Other parameters needed are given as $r_0 = 0.04$, $\sigma = 0.3$, $m = 10$, $\kappa = 0.02$ and time to maturity $T = 5$. Figure 4.2 uses $n = 100$, and the total number of nodes seems to increase linearly with the forward direction of time. We check the values of J_i in the path with all up moves from the beginning time and find that the values of J_i are all 0. From theoretical results, we know that the tree with sufficiently large n yields $J_i > 0$ for some node at time i and the tree will grow exponentially. In the first example of $n = 100$, it seems that the condition to generate exponentially many nodes does not happen, and the number $n = 100$ may be too small to yield an N in theoretical results. So we try to increase time periods n. If n rises to a larger number, say $n = 200$, in Figure 4.3, the lattice grows very fast after 170th time point. We show the total number of nodes from the 170th to the nth time points in Figure 4.4 and the logarithm value of the total number of nodes in Figure 4.5. It is obvious that the tree expands so fast during the time before the end of the process that the total number of nodes is nearly explosive even after the number takes logarithm. Again, we check the process of J_i from the path with all up moves. The value of J_i equals 1 from time $i = 163$ and the value is nondecreasing after the node. This means that the 163th time point can be considered as a choice of the number N in theoretical results, and the tree indeed grows exponentially after that in numerical results. Furthermore, if the number of time periods n is larger than 210, the tree grows with too many nodes and runs out of memory before the end of the nth time period.

We present another example of different parameter values of $r_0 = 0.04$, $\sigma = 0.25$, $m = 10$, $\kappa = 0.02$ and time to maturity $T = 5$ in Figures 4.6 and 4.7 with $n = 200$ and $n = 300$, and only the parameter σ changes to 0.25 in this example as a comparison with the previous example. The graphs of this example are similar to previous ones. We still see that the tree grows almost linearly for $n = 200$ and grows exponentially for $n = 300$ although the number of n is larger than the example of $\sigma = 0.3$ for the tree to grow exponentially. The time i for $J_i \geq 1$ of the path with all up moves in the case of $n = 200$ is 196 and in the case of $n = 300$, the time i is 252. So our theoretical results are verified to be consistent with the numerical results, that if the number of time periods n continues to increase, the RS tree will grow exponentially large.

The Table 4.1 shows the value of n for the lattice to run beyond memory capacity with different values of σ and time to maturity T. The results from the table are as follows. If volatility parameter σ becomes larger with other parameter unchanged, the tree explodes sooner than with a small σ. If time to maturity T extends to a farther date, the tree also explodes sooner. From the inequality $(\sigma\sqrt{T})\sqrt{n} - 1.5\ln(n) > \lambda$ in theoretical results, numerical results also support the conjecture that if values of σ and T are larger, the tree can only work under relatively smaller n.

n	$\sigma = 0.15$	$\sigma = 0.2$	$\sigma = 0.25$	$\sigma = 0.3$	$\sigma = 0.4$
$T = 3$	2160	1090	650	420	220
$T = 5$	1060	530	310	220	110
$T = 10$	390	200	120	80	50

Table 4.1 DIFFERENT VALUES OF σ AND T RELATIVE TO n FOR THE ALGORITHM OUT OF MEMORY WITH $r_0 = 0.04$, $m = 10$ AND $\kappa = 0.02$.

We go on to test the effects on the tree with different values of mean reversion parameter κ. In Figure 4.8 and Figure 4.9, we use the parameter $\kappa = 0.10$ and 0.30 respectively, and compare it with Figure 4.7 with $\kappa = 0.02$. We find that the larger value of κ indeed lowers the growth rate of the tree, but the tree still grows exponentially if n increase further in Figure 4.10.

Another question is the difference between the total number of nodes and the number of reachable nodes. Table 4.2 shows that if number of partitions m for Φ value increases, the total number of nodes seems unchanged but the number of reachable nodes increases and the difference between the two numbers decreases. So our claim that the total number of nodes grows exponentially can extend to that the number of reachable nodes grows exponentially.

Number of partitions of Φ	$m = 50$	$m = 10$	$m = 2$
The total number of nodes	793	793	793
The number of reachable nodes	781	714	681

Table 4.2 The total number of nodes and the number of reachable nodes relative to number of partitions of Φ at maturity date $i = n = 200$ for $r_0 = 0.04,\ \sigma = 0.3,\ m = 10,\ T = 5$ and $\kappa = 0.02$.

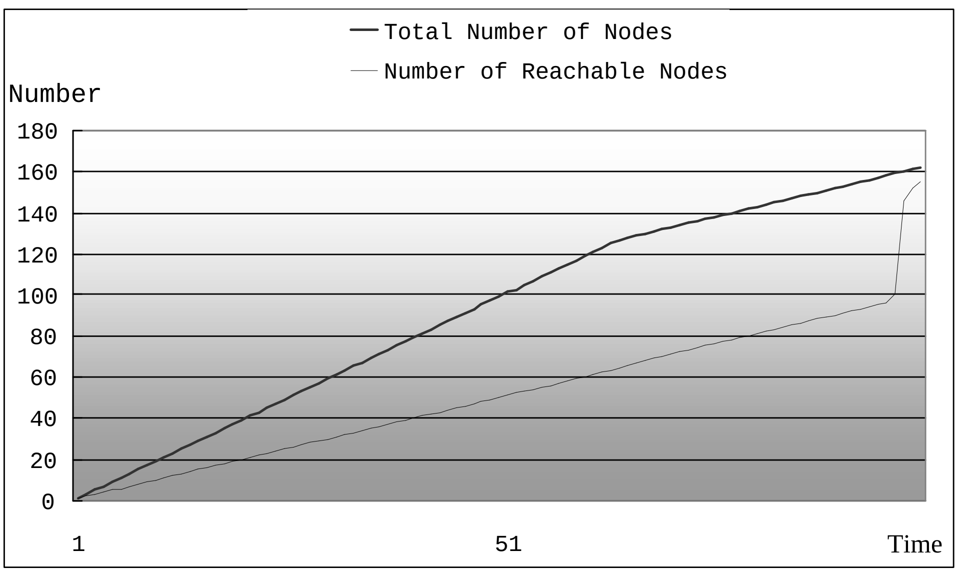

Figure 4.2: The number of nodes for $n = 100,\ T = 5,\ r_0 = 0.04,\ \sigma = 0.3,$ $m = 10$ and $\kappa = 0.02$.

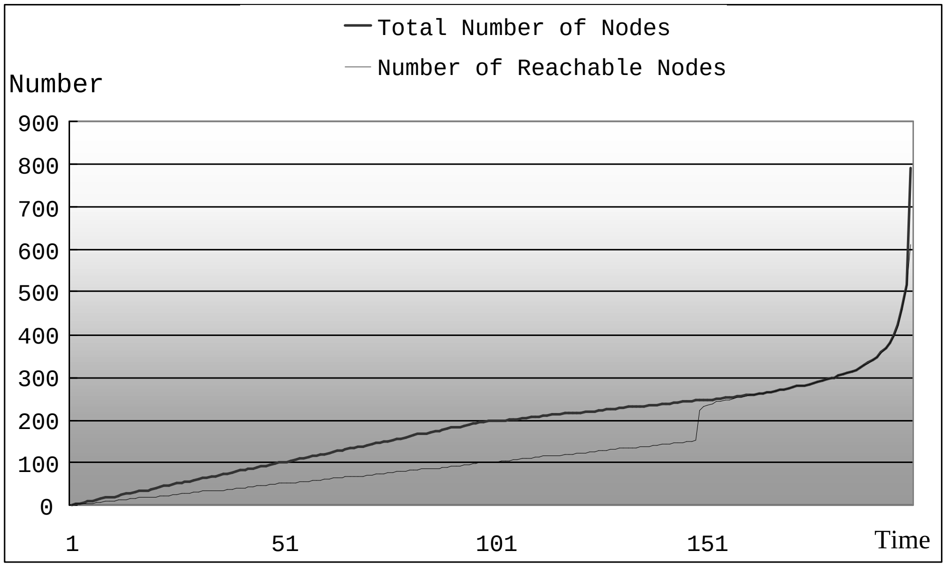

Figure 4.3: The number of nodes for $n = 200$, $T = 5$, $r_0 = 0.04$, $\sigma = 0.3$, $m = 10$ and $\kappa = 0.02$.

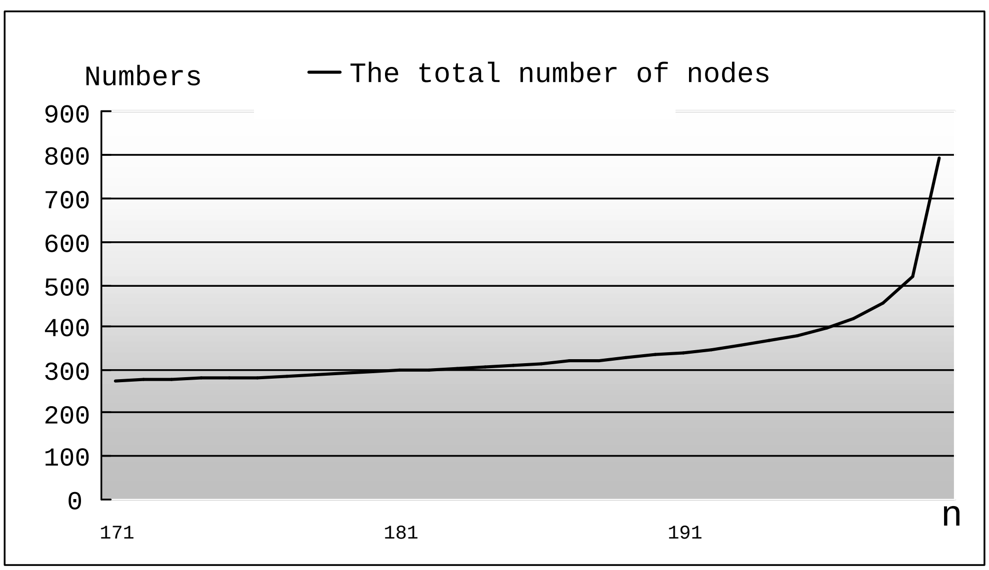

Figure 4.4: The number of nodes for $n = 200$,$T = 5$, $r_0 = 0.04$, $\sigma = 0.3$, $m = 10$ and $\kappa = 0.02$ (Part of the tree with exponential growth rate).

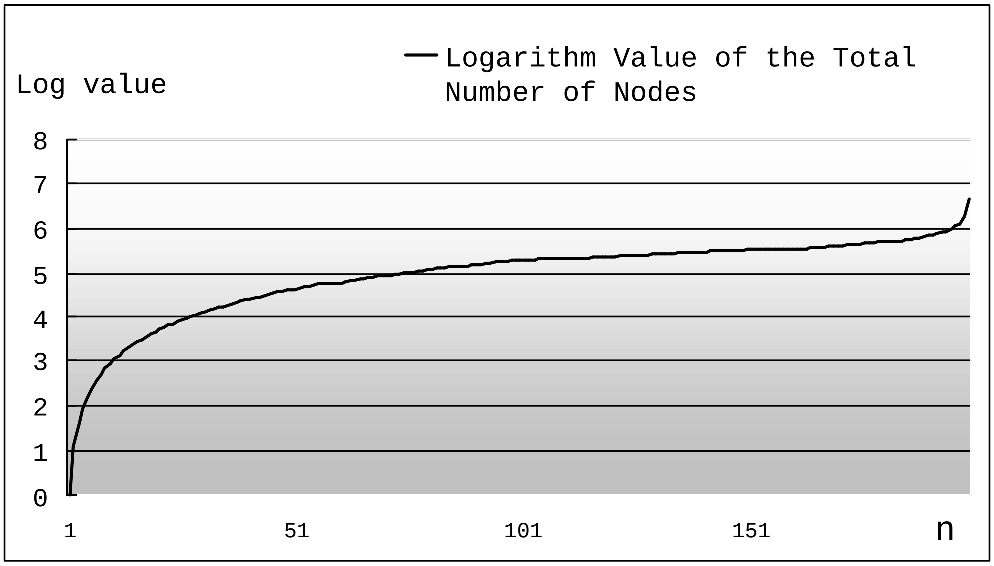

Figure 4.5: LOGARITHM VALUE OF TOTAL NUMBER OF NODES FOR $n = 200$, $T = 5$, $r_0 = 0.04$, $\sigma = 0.3$, $m = 10$ AND $\kappa = 0.02$.

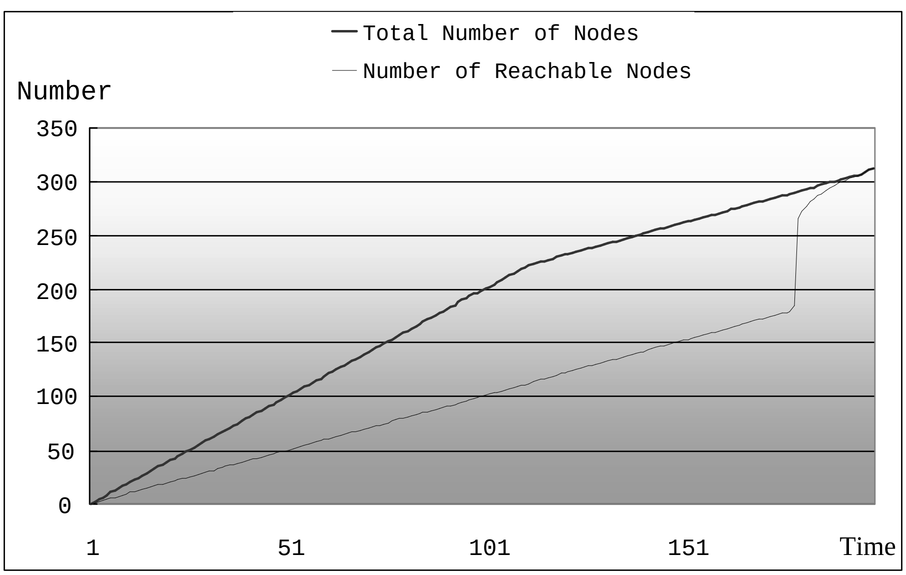

Figure 4.6: THE NUMBER OF NODES FOR $n = 200$, $T = 5$, $r_0 = 0.04$, $\sigma = 0.25$, $m = 10$ AND $\kappa = 0.02$.

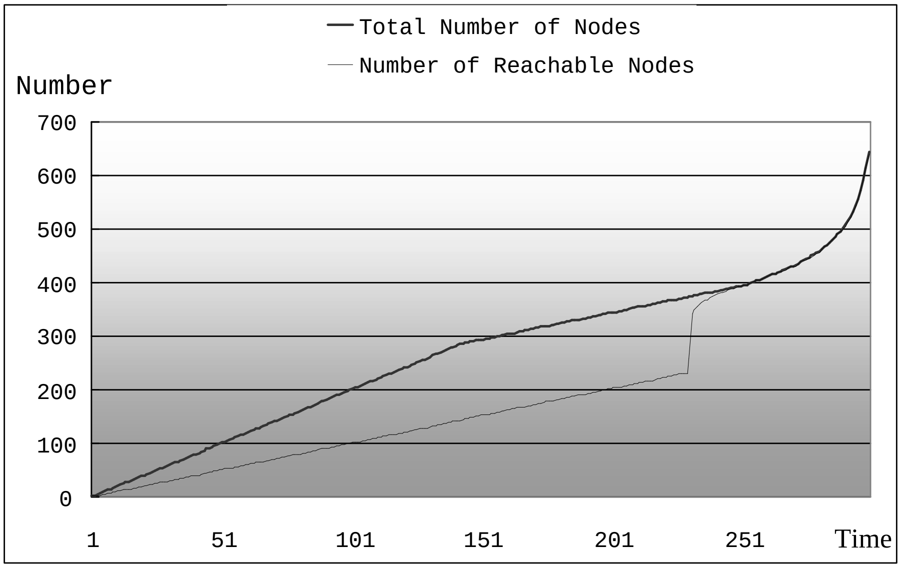

Figure 4.7: THE NUMBER OF NODES FOR $n = 300$, $T = 5$, $r_0 = 0.04$, $\sigma = 0.25$, $m = 10$ AND $\kappa = 0.02$.

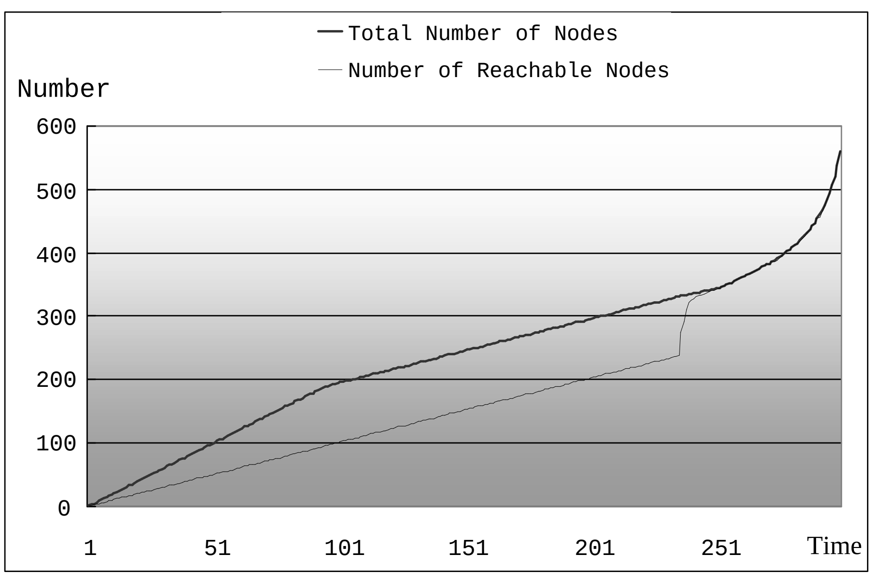

Figure 4.8: THE NUMBER OF NODES FOR $n = 300$, $T = 5$, $r_0 = 0.04$, $\sigma = 0.25$, $m = 10$ AND $\kappa = 0.10$.

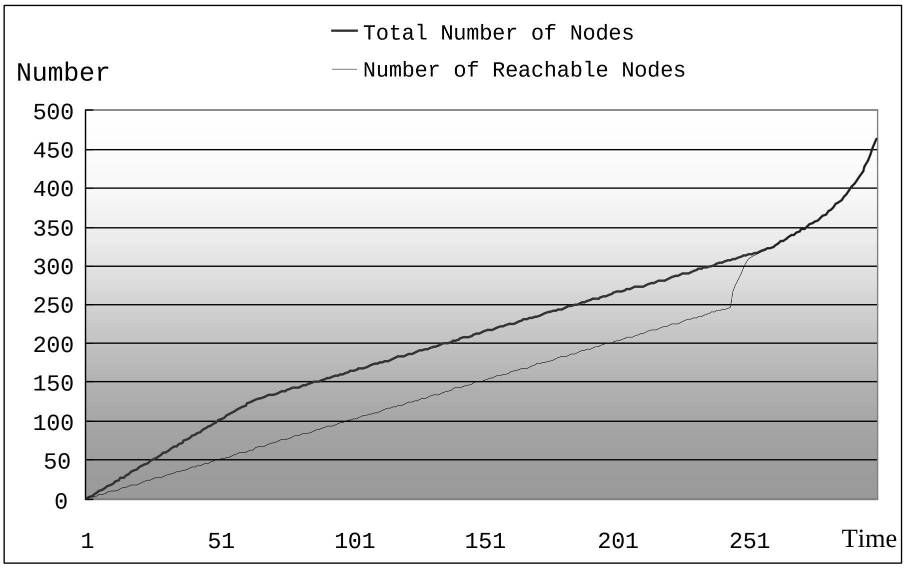

Figure 4.9: THE NUMBER OF NODES FOR $n = 300$, $T = 5$, $r_0 = 0.04$, $\sigma = 0.25$, $m = 10$ AND $\kappa = 0.30$.

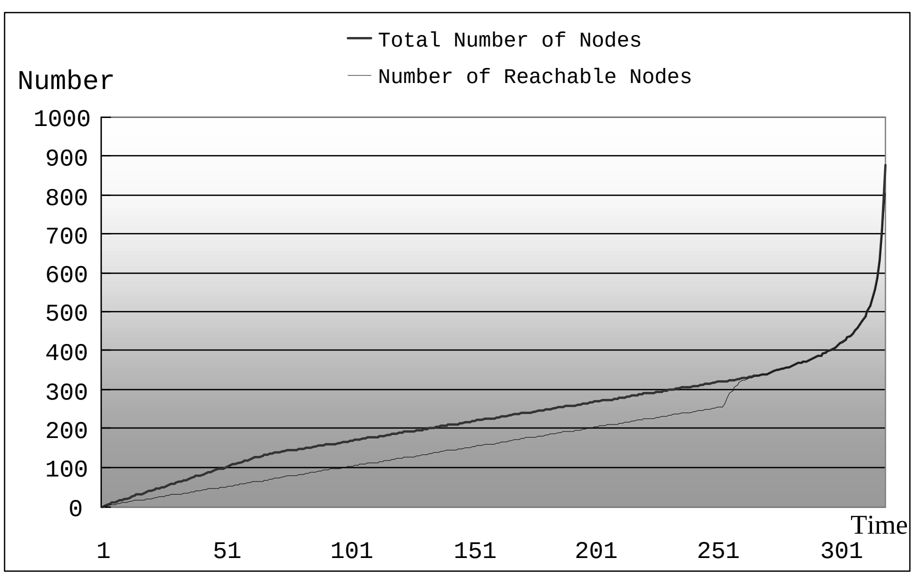

Figure 4.10: THE NUMBER OF NODES FOR $n = 320$, $T = 5$, $r_0 = 0.04$, $\sigma = 0.25$, $m = 10$ AND $\kappa = 0.30$.

Chapter 5

Conclusions

The HJM model is a powerful model that it can include many models as special cases. If there also exists an efficient and accurate algorithm to evolve the lattice of the HJM model, we will be able to value the derivatives on the term structure with various applications of the HJM model. The RS model avoids some difficulties of the HJM model that non-Markovian process is hard to be established with lattice method. However, our thesis shows that the RS algorithm on the RS model still has limitations. From theoretical results, we know that the RS tree grows exponentially under some conditions. If n is sufficiently large, the conditions finally kick in, and the RS tree grows exponentially. Numerical results also reveal similar results, and further, once the conditions are satisfied, the tree size explodes beyond memory capacity in a short time. Although the RS algorithm is more efficient that the tree can be linear for some time periods, it is still limited in how small the time partition Δt is. We conclude that the RS algorithm on the RS model does not solve the fundamental problem that the lattice of the HJM model suffer from exponential explosion.

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