Computational Mechanics 26 (2000) 497–505 © Springer-Verlag 2000

Applications of a reduction method for reanalysis to nonlinear dynamic analysis of framed structures

L.-J. Leu, C.-H. Tsou

Abstract This paper is concerned with the nonlinear dynamic analysis of framed structures using a reduction method recently proposed by the authors. The reduction method is originally devised for structural static reanalysis and has been applied in optimal design of structures to speed up the design process. For nonlinear dynamic analysis of framed structures, the incremental or iterative equations of motion can be transformed into an algebraic system of equations if appropriate integration methods such as Newmark's method are used to integrate the equations of motion. The resulting algebraic system, referred to as the effective system in this paper, changes during the simulation for a nonlinear dynamic problem. Therefore, from the point of view of solving systems of equations, a nonlinear dynamic problem is very similar to an optimal design problem in that the system of equations changes for both types of problems. Hence, any reanalysis technique can be readily applied to carry out a nonlinear dynamic analysis of structures. As demonstrated from the presented numerical examples, the response obtained by the adopted reduction method is as accurate as that obtained by the Cholesky method, and as estimated from the operation counts involved in the method, it is more efficient than the Cholesky method when the half-band width is greater than about 50.

1 Introduction

To evaluate the ultimate state of critical structures subjected to severe dynamic loads, a nonlinear dynamic analysis is usually called for. An accurate modeling of practical structures often requires a large number of degrees of freedom. Regardless of the rapid advances in computer hardware, nonlinear dynamic analyses of large scale structures are still very time consuming. Several approaches have been developed to reduce the time of solving such problems. Among them, the reduction method is a popular one. The main idea of this method is to transform the original large system of equations to a small reduced system, thereby reducing the computational effort.

The reduction of the degrees of freedom for the reduction method is achieved first by expressing the system unknowns as a linear combination of a small number of basis vectors using a set of generalized coordinates. The reduced system of equations governing the generalized coordinates can then be obtained through the Rayleigh–Ritz procedure. Two types of basis vectors are commonly used. One is the eigenvectors or mode shapes of the corresponding dynamic system; the other is called Ritz vectors or load dependent vectors. The above two types of basis vectors have been applied to both linear and nonlinear dynamic analysis of structures.

The reduction method employing the eigenvectors is the classical modal superposition method. To account for the contribution of higher modes, two variants of the modal superposition method have been proposed. These include the static correction method (SCM) and the mode acceleration method (MAM); both have been shown to be equivalent (Léger and Wilson, 1988). As for the Ritz vectors, they were proposed by Wilson et al. (1982) as an alternative to the MAM, where a Gram Schmidt orthogonalization procedure was employed to generate a set of Ritz vectors that automatically account for the static correction terms. Note that Nour-Omid and Clough (1984) used the Lanczos vectors as the basis vectors and they showed that the Ritz vectors were identical to the Lanczos vectors with full reorthogonalization.

Applications of reduction methods adopting the above two types of basis vectors to linear problems are straightforward. However, their uses in nonlinear problems need due consideration to reflect the continuously changing characteristics of nonlinear systems. Several variants have been proposed in the literature. A very good review was given by Léger (1993) and Kapania and Byun (1993). According to Léger (1993), two main types of approaches can be distinguished. One approach is referred to as the tangent spectrum method, where the basis vectors are updated according to the tangent system matrices. For example, the works of Nickell (1976), Gillies and Shepherd (1983), and Mohraz et al. (1991) belong to this category. Another related work is that of Noor (1981), where initial and steady state eigenvectors are used for the case of step loading on spherical shells and arches. The other popular approach is the pseudo-force method, in which the basis vectors, obtained purely using the linear system matrices, are not updated during the simulation and the effects of nonlinearities are taken as pseudo-forces on the right side

Received 23 March 2000

L.-J. Leu, C.-H. Tsou
Department of Civil Engineering, National Taiwan University, Taipei 10617, Taiwan
e-mail: ljleu@ce.ntu.edu.tw

This research was sponsored by the National Science Council of the Republic of China under Grant Number NSC 86-2221-E-002-076.

of the equations of motion. For example, Shah et al. (1979), Clough and Wilson (1979), Muscolino (1990), and Vukazich et al. (1996) presented results on nonlinear dynamic analysis of structures obtained using this approach.

In addition to eigenvectors and Ritz vectors, their derivatives with respect to the generalized coordinates have also been used as basis vectors. For example, Idelsohn and Cardona (1985a) adopted the dominant eigenvectors of the tangent stiffness matrix along with their derivatives; Idelsohn and Cardona (1985b), and Chang and Engblom (1991) used Ritz vectors and their derivatives. The combination of eigenvectors or Ritz vectors and their derivatives has shown to enhance the convergence rate and accuracy of solutions. However, extra efforts are required to obtain the derivatives.

This paper presents an application of the reduction method recently proposed by Leu and Huang (2000) in nonlinear dynamic analysis of framed structures. Originally developed for structural static reanalysis, this reduction method has been shown to be efficient in the application of optimal design of trusses. For nonlinear dynamic analysis of framed structures, the incremental or iterative equations of motion can be discretized into an algebraic system of equations, if proper integration schemes are employed to integrate the equations of motion. The resulting algebraic system changes during the simulation for a nonlinear dynamic problem in general. This is similar to what occurs in optimal design problems. Therefore, any static reanalysis technique can be applied to perform nonlinear dynamic analysis as well. Making use of this viewpoint, this paper employs the reduction method proposed by Leu and Huang (2000) for reanalysis to carry out nonlinear dynamic analyses of framed structures.

2 Equations of motion and corresponding effective systems of equations

The dynamic equilibrium equations for a general nonlinear structural system at time t can be expressed as

$$\mathbf{M}_t\ddot{\mathbf{U}}_t + \mathbf{C}_t\dot{\mathbf{U}}_t + \mathbf{R}_t = \mathbf{P}_t \tag{1}$$

where $\mathbf{M}_t$ and $\mathbf{C}_t$ are the mass and damping matrices at time t, respectively; $\ddot{\mathbf{U}}_t$ and $\dot{\mathbf{U}}_t$ are the acceleration and velocity vectors, respectively; $\mathbf{R}_t$ is the nonlinear restoring force vector; and $\mathbf{P}_t$ is the applied load vector. The counterpart equations at time $t+\Delta t$ are

$$\mathbf{M}_{t+\Delta t}\ddot{\mathbf{U}}_{t+\Delta t} + \mathbf{C}_{t+t\Delta}\dot{\mathbf{U}}_{t+\Delta t} + \mathbf{R}_{t+\Delta t} = \mathbf{P}_{t+\Delta t} \tag{2}$$

The linearized incremental equations of motion can be obtained by subtracting Eq. (1) from Eq. (2) and then making use of the linearized relation: $\Delta\mathbf{R} = \mathbf{K}_t\Delta\mathbf{U}$, where $\mathbf{K}_t$ is the tangent stiffness matrix, $\Delta\mathbf{U} = \mathbf{U}_{t+\Delta t} - \mathbf{U}_t$, and $\Delta\mathbf{R} = \mathbf{R}_{t+\Delta t} - \mathbf{R}_t$. The resulting equations take the form:

$$\mathbf{M}_{t+\Delta t}\Delta\ddot{\mathbf{U}} + \mathbf{C}_{t+t\Delta}\Delta\dot{\mathbf{U}} + \mathbf{K}_t\Delta\mathbf{U} = \Delta\mathbf{P} - \Delta\mathbf{M}\ddot{\mathbf{U}}_t - \Delta\mathbf{C}\dot{\mathbf{U}}_t \tag{3}$$

The constant average acceleration scheme ($\beta = 0.25$ and $\gamma = 0.5$) of Newmark's method (Newmark, 1959) is employed to obtain the corresponding effective system:

$$\hat{\mathbf{K}}\Delta\mathbf{U} = \Delta\hat{\mathbf{P}} \tag{4}$$

The explicit expressions for $\hat{\mathbf{K}}$ and $\Delta\hat{\mathbf{P}}$ can be found in Olsson (1985) with the following two slight modifications. First, the term $\mathbf{K}_{t+\Delta t}$ in $\hat{\mathbf{K}}$ in that paper shall be replaced by $\mathbf{K}_t$. Second, the term $\Delta\mathbf{K}\mathbf{U}_t$ in $\Delta\hat{\mathbf{P}}$ shall be dropped.

The solution to the linearized system will not satisfy the dynamic equilibrium equations. To this end, iterations are needed, which are governed by

$$\mathbf{M}_{t+\Delta t}\delta\ddot{\mathbf{U}}^i + \mathbf{C}_{t+\Delta t}\delta\dot{\mathbf{U}}^i + \mathbf{K}^i_{t+\Delta t}\delta\mathbf{U}^i = \mathbf{E}^i \tag{5}$$

In Eq. (5) $\delta\mathbf{U}^i$, $\delta\dot{\mathbf{U}}^i$ and $\delta\ddot{\mathbf{U}}^i$ are, respectively, the displacement, velocity and acceleration increments associated with the ith iteration; $\mathbf{K}^i_{t+\Delta t}$ denotes the tangential stiffness matrix at the current state; and $\mathbf{E}^i$ is the residual or unbalanced force vector corresponding to Eq. (2) evaluated at the current state, i.e.,

$$\mathbf{E}^i = \mathbf{P}_{t+\Delta t} - \mathbf{M}_{t+\Delta t}\ddot{\mathbf{U}}^i_{t+\Delta t} - \mathbf{C}_{t+\Delta t}\dot{\mathbf{U}}^i_{t+\Delta t} - \mathbf{R}^i_{t+\Delta t} \tag{6}$$

Note that for $i = 1$, $\mathbf{U}^1_{t+\Delta t} = \mathbf{U}_{t+\Delta t}$, $\dot{\mathbf{U}}^1_{t+\Delta t} = \dot{\mathbf{U}}_{t+\Delta t}$, $\ddot{\mathbf{U}}^1_{t+\Delta t} = \ddot{\mathbf{U}}_{t+\Delta t}$, and $\mathbf{R}^1_{t+\Delta t} = \mathbf{R}_{t+\Delta t}$, i.e., solutions of the incremental step.

The corresponding effective system of Eq. (5) can be written as:

$$\hat{\mathbf{K}}^i\delta\mathbf{U}^i = \mathbf{E}^i \tag{7}$$

where $\hat{\mathbf{K}}^i$ has the same form as $\hat{\mathbf{K}}$ with $\mathbf{K}_t$ there being replaced by $\mathbf{K}^i_{t+\Delta t}$.

After solving Eq. (7), the state vectors can be updated as follows: $\mathbf{U}^{i+1}_{t+\Delta t} = \mathbf{U}^i_{t+\Delta t} + \delta\mathbf{U}^i$; $\dot{\mathbf{U}}^{i+1}_{t+\Delta t} = \dot{\mathbf{U}}^i_{t+\Delta t} + \delta\dot{\mathbf{U}}^i$; and $\ddot{\mathbf{U}}^{i+1}_{t+\Delta t} = \ddot{\mathbf{U}}^i_{t+\Delta t} + \delta\ddot{\mathbf{U}}^i$. The associated restoring force vector $\mathbf{R}^{i+1}_{t+\Delta t}$ can then be evaluated using appropriate constitutive laws. The iterative process is continued until

$$E_\mathrm{r} \equiv \frac{||\mathbf{R}^{i+1}_{t+\Delta t}||}{P_\mathrm{r}} < e_\mathrm{r} \tag{8}$$

where P_r is the magnitude of a specified reference load and e_r is the tolerance of E_r.

3 Reduction method for structural reanalysis

In Sects. 3.1–3.4, the reduction method proposed by Leu and Huang (2000) is briefly reviewed for completeness of this paper. In order to measure the efficiency of the reduction method, operation counts involved in the method are discussed in Sect. 3.5; these are not available in Leu and Huang (2000).

3.1 Reduced system

Define a reference system as the system whose solution has been obtained exactly using the Cholesky method. This is the system to which later analyses using the reduction technique are referred. At the beginning of a design process, the reference system is the one corresponding to the original design. Let the reference system be expressed as

$$\mathbf{K}_0\mathbf{U}_0 = \mathbf{P}_0 \tag{9}$$

where $\mathbf{K}_0, \mathbf{U}_0$, and $\mathbf{P}_0$ are the stiffness matrix, displacement, and load vectors, respectively. In this study $\mathbf{K}_0$ is stored in banded format.

Let the system of a typical intermediate design be written as

$$\mathbf{KU} = (\mathbf{K}_0 + \Delta\mathbf{K})(\mathbf{U}_0 + \Delta\mathbf{U}) = (\mathbf{P}_0 + \Delta\mathbf{P}) = \mathbf{P} \tag{10}$$

Employing the reduced basis technique, the unknowns of Eq. (10) can be approximated as

$$\mathbf{U}_{n\times 1} \approx \tilde{\mathbf{U}}_{n\times 1} = \sum_{i=1}^{m} z_i \phi_i = \Phi_{n\times m}\mathbf{z}_{m\times 1} \tag{11}$$

The reduced system corresponding to Eq. (10) can then be expressed as

$$\Phi^{\mathrm{T}}\mathbf{K}\Phi\mathbf{z} = \Phi^{\mathrm{T}}\mathbf{P} \tag{12}$$

When using Eq. (11) the question arises: How many basis vectors should be used to achieve a given level of accuracy? One direct approach is to increase the number of basis vectors by one each time until a defined error measure is smaller than a preset tolerance. For example, an error measure E_p may be defined as

$$E_p = \frac{||\mathbf{K}\tilde{\mathbf{U}} - \mathbf{P}||}{||\mathbf{P}||} \tag{13}$$

It takes a fairly large computational effort to evaluate Eq. (13) though. A more convenient error measure will be discussed in Sect. 3.2.

3.2
Uncoupled reduced system and convergence criteria

Instead of using ϕ_i directly, a Gram–Schmidt orthonormalization procedure, as listed in Table 1, is employed to generate a new set of basis vectors ψ_i from ϕ_i such that

$$\psi_i^{\mathrm{T}}\mathbf{K}\psi_j = \delta_{ij} \quad \text{or} \quad \Psi^{\mathrm{T}}\mathbf{K}\Psi = \mathbf{I} \tag{14}$$

where δ_{ij} is the Kronecker delta and $\mathbf{I}$ is the $s \times s$ identity matrix.

If ψ_i's are adopted as the basis vectors, the reduced system stated in Eq. (12) becomes

$$\mathbf{z} = \Psi^{\mathrm{T}}\mathbf{P} \quad \text{or} \quad z_i = \psi_i^{\mathrm{T}}\mathbf{P} \tag{15}$$

Clearly, the reduced system is uncoupled. Then, from Eq. (11),

$$\tilde{\mathbf{U}} = \sum_{i=1}^{m} z_i\psi_i = \sum_{i=1}^{m}(\psi_i^{\mathrm{T}}\mathbf{P})\psi_i \tag{16}$$

A computation-inexpensive criterion for adaptively determining the required number of basis vectors is proposed as follows:

$$E_z = \frac{|z_m|}{\sum_{i=1}^{m}|z_i|} < e_z \tag{17}$$

Table 1. Generation of orthonormal basis vectors with respect to **K**

(a) if $i = 1$,	$\psi_1 = \phi_1/(\phi_1^{\mathrm{T}}\mathbf{K}\phi_1)^{1/2}$
(b) if $i > 1$,	$\overline{\psi}_i = \phi_i - \sum_{j=1}^{i-1}(\phi_i^{\mathrm{T}}\mathbf{K}\psi_j)\psi_j$
	$\psi_i = \overline{\psi}_i/(\overline{\psi}_i^{\mathrm{T}}\mathbf{K}\overline{\psi}_i)^{1/2}$

where e_z is a given tolerance associated with E_z. This criterion is proper because the contribution of the last term, $z_m\psi_m$, to $\tilde{\mathbf{U}}$ becomes very small when $\tilde{\mathbf{U}}$ converges and also the magnitudes of $||\psi_i||$ are of about the same order as implied by Eq. (14).

3.3
Generation of the original basis vectors

As for the selection of the original basis vectors ϕ_i, Kirsch's method (Kirsch, 1991) is followed. Therefore,

$$\Phi = [\phi_1, \phi_2, \phi_3, \ldots] = [\mathbf{U}_p, \mathbf{B}\mathbf{U}_p, \mathbf{B}^2\mathbf{U}_p, \ldots] \tag{18}$$

where

$$\mathbf{B} = \mathbf{K}_0^{-1}\Delta\mathbf{K} \tag{19}$$

$$\mathbf{U}_p = \mathbf{K}_0^{-1}\mathbf{P} \tag{20}$$

Generating ϕ_i according to Eq. (18) is computation-inexpensive. This is because

$$\phi_i = \mathbf{B}\phi_{i-1} \ \ (i > 1) \tag{21}$$

With the definition of **B** given in Eq. (19), this becomes

$$\phi_i = \mathbf{K}^{-1}\Delta\mathbf{K}\phi_{i-1} \quad \text{or} \quad \mathbf{K}_0\phi_i = \Delta\mathbf{K}\phi_{i-1} \ \ (i > 1) \tag{22}$$

Given the factored form of $\mathbf{K}_0$, the calculation of ϕ_i $(i > 1)$ using Eq. (22) involves only the multiplication of $\Delta\mathbf{K}$ and ϕ_{i-1}, followed by a backward and a forward substitution.

3.4
Summary of the reanalysis procedure

From the above discussions, the reanalysis procedure using the adopted reduction method can be summarized in Table 2. The values $e_p = 0.001$ and $e_z = 0.01$ adopted in the numerical examples presented later are suggested according to our numerical experience. Note that the corresponding value of E_p is about 10^{-3} to 10^{-4} when $e_z = 0.01$ is used (Leu and Huang, 1997).

Note that in this study the maximum number of basis vectors allowed is 10. If convergence has not been achieved upon using 10 basis vectors, the system currently being solved using the reanalysis method needs to be solved again using the Cholesky method. The system is then taken as the new reference system, i.e., the system described by Eq. (9). It is very possible that in this case the changes in the design variables between the current design and the reference design are relatively large. Hence, if the reference system is updated, latter reanalyses may require fewer basis vectors. The maximum number of 10 is a good choice from our experience.

Table 2. Reanalysis procedure

1. Initialization: Calculate $\mathbf{U}_p(= \mathbf{K}_0^{-1}\mathbf{P})$; let $i = 1$ and the initial solution $\tilde{\mathbf{U}}_0 = \mathbf{0}$
2. Calculate ϕ_i according to Eq. (21)
3. Orthonormalize ϕ_i according to Table 1 to obtain ψ_i
4. Calculate z_i using Eq. (15)
5. Update solution: $\tilde{\mathbf{U}}_i = \tilde{\mathbf{U}}_{i-1} + z_i\psi_i$
6. Check convergence:
   Compute E_p using Eq. (13) if $i = 1$ and E_z using Eq. (17) if $i > 1$; check whether they are less than the specified tolerances e_p and e_z, respectively. If yes, stop; otherwise let $i = i + 1$ and go to step 2

3.5 Operation counts

The operation counts (OCs), considering only multiplications and divisions required for the orthonormalization procedure given in Table 1, are listed in Table 3, in which n and b denote, respectively, the order and half-band width (including the diagonal elements) of the system. Moreover, the OCs required for the reanalysis procedure of Table 2 are listed in Table 4.

From Table 4, the total operation counts (TOC) for the present reduction method adopting an m-term approximation would be

$$\begin{aligned} \mathrm{TOC} &= 6nb + 4n + 1 + \sum_{i=2}^{m} [6nb + n + 2ni + 1] \\ &= m^2 n + 2mn(3b + 1) + n + m \end{aligned} \tag{23}$$

If the Cholesky method is employed, the total operation counts including those involved in the factorization of $\mathbf{K}$, one backward and one forward substitution would be $n(b-1)(b+2)/2 + 2nb$ (Schwarz, 1989). Let's define the theoretical speed up S_t as the ratio of the operation counts using the Cholesky method to those using the present reduction method; that is,

$$\begin{aligned} S_t &= \frac{0.5n(b-1)(b+2) + 2nb}{m^2 n + 2mn(3b+1) + n + m} \\ &\approx \frac{b^2 + 5b - 2}{2m^2 + 4m(3b+1) + 2} \end{aligned} \tag{24}$$

Table 3. Operation counts (OCs) required for the Gram–Schmidt orthonormalization procedure

Orthogonalization steps	OCs
1. $\mathbf{K}\psi_j, j = 1, i-1$	$n(i-1)$
2. $\phi_i^{\mathrm{T}}(\mathbf{K}\psi_j), j = 1, i-1$	n
3. $\psi_i = \phi_i - \sum_{j=1}^{i-1}(\phi_i^{\mathrm{T}}\mathbf{K}\psi_j)\psi_j$	$n(i-1)$
Subtotal	$n + 2n(i-1)$
Normalization steps	
1. $\mathbf{K}\bar{\psi}_i$	$n(2b-1)$
2. $\bar{\psi}_i^{\mathrm{T}}(\mathbf{K}\bar{\psi}_i)$	n
3. $\psi_i = \bar{\psi}_i/(\bar{\psi}_i^{\mathrm{T}}\mathbf{K}\bar{\psi}_i)^{1/2}$	n
Subtotal	$2nb + n$
Total OCs:	
$i = 1: 2nb + n$ (orthogonalization is not needed);	
$i > 1: 2nb + 2ni$	

Table 4. Operation counts required for the adopted reduction method

Steps	if $i = 1$	if $i > 1$
1	$2nb$	0
2	0	$4nb - n$
3	$2nb + n$	$2nb + 2ni$
4	n	n
5	n	n
6	$2nb + n + 1$	1
Sum	$6nb + 4n + 1$	$6nb + n + 2ni + 1$

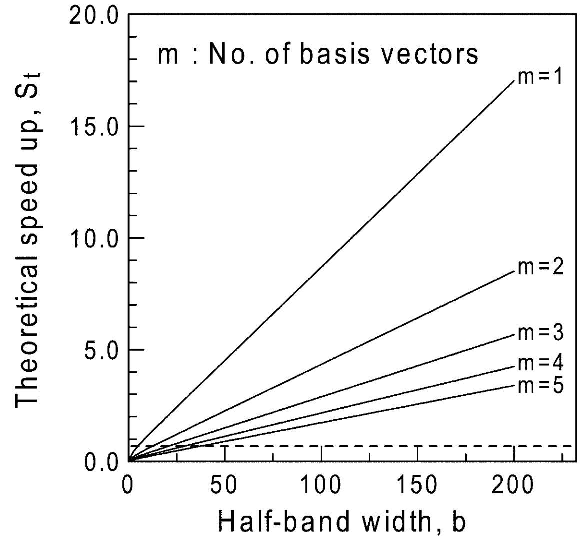

Fig. 1. Theoretical speed-up versus the half-band width for different numbers of basis vectors

From Eq. (24), S_t is independent of n. Figure 1 shows the relation between S_t and the half-band width b for different values of m. Clearly, the present reduction method becomes efficient as b gets larger. For example, if $m = 2$, then $S_t \geq 1$ when $b \geq 20$; $m = 3$, then $S_t > 1$ when $b \geq 33$; if $m = 4$, then $S_t > 1$ when $b \geq 45$; and if $m = 5$, then $S_t > 1$ when $b \geq 57$. Note that from the numerical examples presented later and those presented in Leu and Huang (2000), the value of m ranges from 2 to 5 mostly when the convergence criterion of Eq. (17) is satisfied. In addition, this range is independent of n, which is a main advantage of the method. Therefore, in general, when the half-band width is greater than about 50, the reduction method will be more efficient than the Cholesky method; in addition, in this case since $b^2 \gg b$, Eq. (24) can be further approximated by

$$S_t = \frac{b}{12m} \tag{25}$$

4 Application of the reduction method in nonlinear dynamic analysis

4.1 Similarity between reanalysis and nonlinear dynamic problems

The focus now is turned to the solutions of the effective systems stated in Eqs. (4) and (7) using the reduction method discussed in Sect. 3. As the dynamic problems considered in this study are nonlinear, these two systems evolve during the simulation in general; this is very similar to what occurs in design optimization problems. That is, the systems change for both types of problems although the causes are very different; one is due to the change in design variables and the other is due to nonlinearities. Therefore, if a reference system similar to the system corresponding to the original design in optimal design problems can be determined, the reduction method for reanalysis can be used to solve Eqs. (4) and (7).

Naturally, the system associated with the first incremental step of the simulation, i.e., the initial linear system, is chosen as the reference system. Hence, the Cholesky method is applied at the very beginning to solve the effective system corresponding to the first incremental step. Other incremental systems of Eq. (4) and all iterative systems of Eq. (7) are solved by the reduction method unless the number of basis vectors is increased up to 10 while the convergence criterion of Eq. (17) has not been satisfied. In this case, as discussed in the last paragraph of Sect. 3.4, the system currently being solved needs to be solved again by the Cholesky method and this system is taken to be the reference system from now on. From the above discussion, it is clear that the basis vectors in the reduction method need to be updated at each incremental and iterative steps in general. Therefore, it seems that the application of the reduction method to nonlinear dynamic analysis of structures is not computationally efficient. However, this is not true because the adopted reduction method requires very few basis vectors to obtain accurate solutions. Typically, the average number of basis vectors is around 3 for such kind of application irrespective of the number of the degrees of freedom of the system, as will be seen in the numerical examples presented in Sect. 5.

It is emphasized that since an equilibrium check as stated in Eq. (8) is performed during the simulation, the accuracy of the solutions obtained from the reduction and Cholesky methods is the same as long as the same error tolerance e_r is adopted in Eq. (8). However, the reduction method may require more iterations than the Cholesky method in order to achieve this error tolerance because the reduction method solves Eqs. (4) and (7) approximately; this will be seen in the numerical examples presented later.

It is worth mentioning that following the same idea a reanalysis technique employing the reduction method can be used to carry out a geometric nonlinear static analysis of structures as well. However, in order to deal with the case where the system matrix is not positive-definite arising from limit points that occur in such an analysis, an extension of the present reduction method is needed. This is because the present reduction method is only applicable to the case where the system matrix is positive definite. The extension has been made in Leu and Huang (1998). Moreover, the extension to the case where the system matrix is unsymmetric can be seen in Leu (1999).

4.2
Comparison of present and conventional reduction methods

It is interesting to compare the present reduction method with the conventional reduction methods in the application of nonlinear dynamic analysis. In the present approach, the reduction method is employed to solve the effective systems obtained from integrating the equations of motion. Because an equilibrium check as stated in Eq. (8) is performed in this study as discussed in Sect. 4.1, the accuracy of the solutions obtained using the reduction method and using the Cholesky method is comparable.

For the conventional reduction methods, employing either the eigenvectors or Ritz vectors, reduction techniques are applied directly to the equations of motion, i.e., Eqs. (3) and (5), to yield reduced dynamic systems. For example, considering the solution of Eq. (3), one would assume $\Delta\mathbf{U}(t) = \sum_{i=1}^{m} z_i(t)\phi_i$. The associated reduced dynamic system can be obtained similarly and is given by

$$(\mathbf{M}_{t+\Delta t})_r\ddot{\mathbf{z}} + (\mathbf{C}_{t+\Delta t})_r\dot{\mathbf{z}} + (\mathbf{K}_t)_r\mathbf{z} = (\Delta\mathbf{P})_r \tag{26}$$

where $(\mathbf{M}_{t+\Delta t})_r = \Phi^{\mathrm{T}}\mathbf{M}_{t+\Delta t}\Phi$; $(\mathbf{C}_{t+\Delta t})_r = \Phi^{\mathrm{T}}\mathbf{C}_{t+\Delta t}\Phi$; $(\mathbf{K}_t)_r = \Phi^{\mathrm{T}}\mathbf{K}_t\Phi$; and $(\Delta\mathbf{P})_r = \Phi^{\mathrm{T}}(\Delta\mathbf{P} - \Delta\mathbf{M}\ddot{\mathbf{U}}_t - \Delta\mathbf{C}\dot{\mathbf{U}}_t)$. The reduced dynamic system can be solved by standard methods with less computational effort compared with the original dynamic system. The accuracy of the solution to Eq. (26) depends, to some extent, on the number of basis vectors used and also on the types of basis vectors chosen. In nonlinear dynamic analysis of structures, the basis vectors may include the derivatives of the eigenvectors and Ritz vectors with respect to the generalized coordinates. Moreover, the required number of basis vectors often increases as the order of the system increases. However, as mentioned in 3.5, the required number of basis vectors is independent of the order of the system for the adopted reduction method.

5
Numerical examples

Three examples are used to demonstrate the efficiency and accuracy of the adopted reduction method in the application of nonlinear dynamic analysis of framed structures. As can be seen from the expression for $\hat{\mathbf{K}}$, the change in the effective stiffness matrix may result from the changes in $\mathbf{K}_t, \mathbf{C}_{t+\Delta t}, \mathbf{M}_{t+\Delta t}$, Δt, or a combination of them. The first example is concerned with the problem of varying the mass matrix, where a vehicle moving along a simply supported beam is investigated. The second and third examples deal with the case where the stiffness matrix changes during the simulation. Moreover, the CPU time is reported for only the third example because the half-band width of its system matrix is large. Finally, in each example, the Cholesky method is also used to solve the effective systems of Eqs. (4) and (7). The dynamic responses thus obtained are found to be of the same accuracy as those obtained using the reduction method as explained in the third paragraph of Sect. 4.1, and are, therefore, not reported. Note that the value of e_r in Eq. (8) is taken as 10^{-6} for all examples.

5.1
Beam with moving mass

Consider a beam with a moving mass as depicted in Fig. 2. Some data for the beam are as follows. The mass

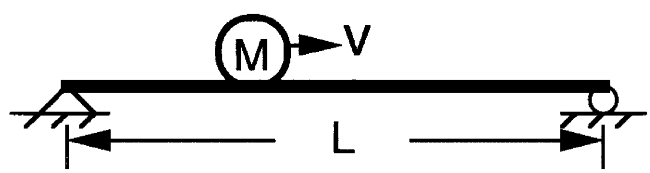

Fig. 2. A simply supported beam with a moving mass

per unit length $\bar{m} = 18,000$ kg/m; length $L = 50$ m; mass $M_b = \bar{m}L = 9 \times 10^5$ kg; and flexural rigidity $EI = 2 \times 10^{11}$ N m^2. The moving mass M equals $0.5M_b$ and moves to the right at a constant velocity of 27.78 m/s starting from the left end of the beam. No damping is assumed for the beam. In addition, the beam is discretized into 20 beam elements and its mass matrix is modeled using the consistent mass matrix. The time increment Δt equals 0.01 s and therefore a total of 180 time increments are needed for the whole simulation. The magnitude of the reference load P_r in Eq. (8) is taken as the weight of the beam, being equal to 8.83×10^6 N.

The midpoint displacement response of the beam is shown in Fig. 3, where the semi-analytical solution is obtained by a modal superposition method (a series solution) given in Yau (1996); a similar method was also reported in Akin and Mofid (1989). Clearly, the accuracy of the reduction method is verified. The total number of iterative steps associated with Eq. (7) is 179 for the whole simulation. Therefore the total number of solving the effective systems is 359 (=180 incremental steps plus 179 iterative steps), in which only the effective system associated with the first incremental step is solved by the Cholesky method and the other 358 by the reduction method. Out of the 358 solvings, there is no one requiring more than four basis vectors. The statistics about the number of basis vectors used, m, are as follows: one effective system is solved with $m = 1$, eight with $m = 2$, 348 with $m = 3$, and one with $m = 4$. Thus the average number of basis vectors used, denoted by m_{av}, equals 2.97 $(=(1 \times 1 + 2 \times 8 + 3 \times 348 + 4 \times 1)/358)$. The small value of m_{av} indicates that the basis vectors used in the reduction method are of good quality. Note that if the Cholesky method is used as the solver, the number of iterations is still 179.

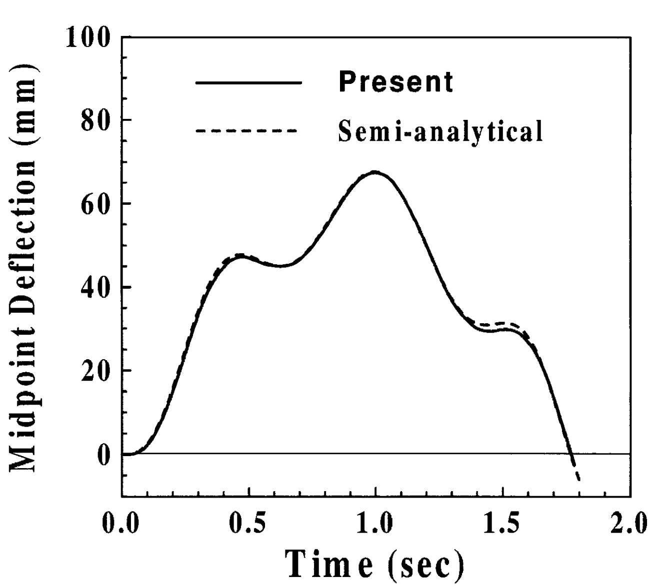

Fig. 3. Comparisons of the midpoint displacement response of the beam with a moving mass

5.2
25-story elastic–plastic shear building

Figure 4 shows a 25-story shear building (Léger, 1993). For each story, the relation between the story shear R and the story drift Δu is assumed to be bilinear with the kinematic hardening rule as depicted in Fig. 5. In the figure, $K_1 = \alpha K_0$, where a value of 0.1 is used for the hardening parameter α; the values of the elastic stiffness K_0 and yield strength R_y for each story are given in Table 5. Each story mass $m_i (i = 1, \ldots, 25)$ equals 10^5 kg. Also, Rayleigh damping is assumed; that is, $\mathbf{C} = a_0\mathbf{M} + a_1\mathbf{K}_0$, where $\mathbf{M}$ and $\mathbf{K}_0$ are the mass and initial stiffness matrices, respectively. If a damping ratio of 5% is adopted for the first two modes, the values of a_0 and a_1 can be obtained to be

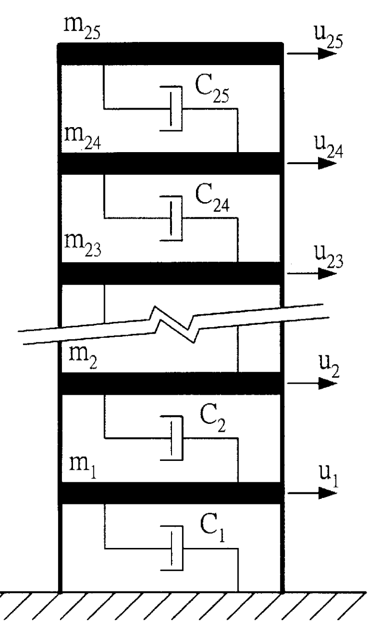

Fig. 4. 25-story shear building

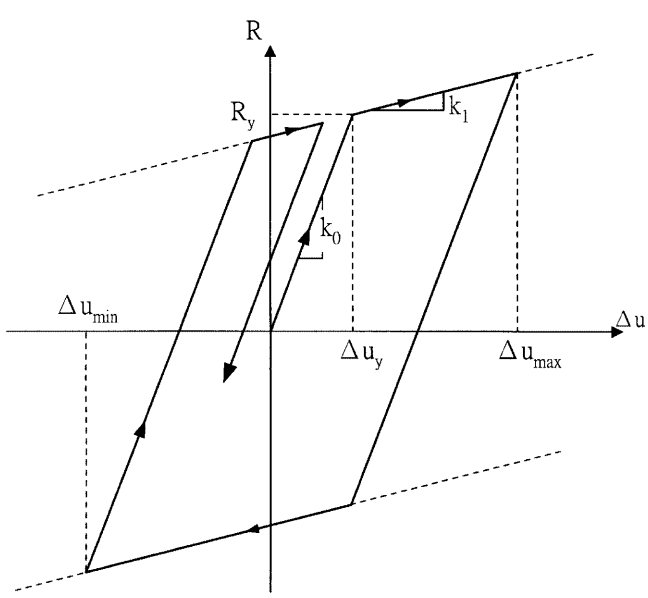

Fig. 5. Bilinear force–deformation relationship with kinematic hardening

Table 5. Elastic stiffness and yield strength

Level	K_0 (kN/m)	R_y (kN)
25-story shear building		
1	232000	1513
2–4	220400	1509
5–7	197200	1474
8–10	174000	1405
11–13	150800	1301
14–16	127600	1163
17–19	104400	991
20–22	81200	783
23–25	58000	541

0.182 and 0.011, respectively. Note that the first and second natural frequencies are 2.516 rad/s and 6.560 rad/s, respectively, for this building when all members remain elastic.

The shear building is subjected to the ground motion of the first 20 s of the S00E component of the 1940 El Centro earthquake. If $\Delta t = 0.005$ s is adopted, 4000 time increments are needed for the simulation. The value of P_r in Eq. (8) is taken as the weight of the building. Figure 6 shows the time history of the roof displacement u_{25} obtained by the reduction method and by the commercial program DRAIN-2D (1973). The good agreement between the two results demonstrates the accuracy of the reduction method. Figure 7 compares the ductility obtained from these two methods. The ductility, referring to Fig. 5, is defined as the ratio of the larger of the absolute values of $\Delta u_{\max}$ and $\Delta u_{\min}$ to Δu_y. From Fig. 7, close agreement between the two results can also be seen again.

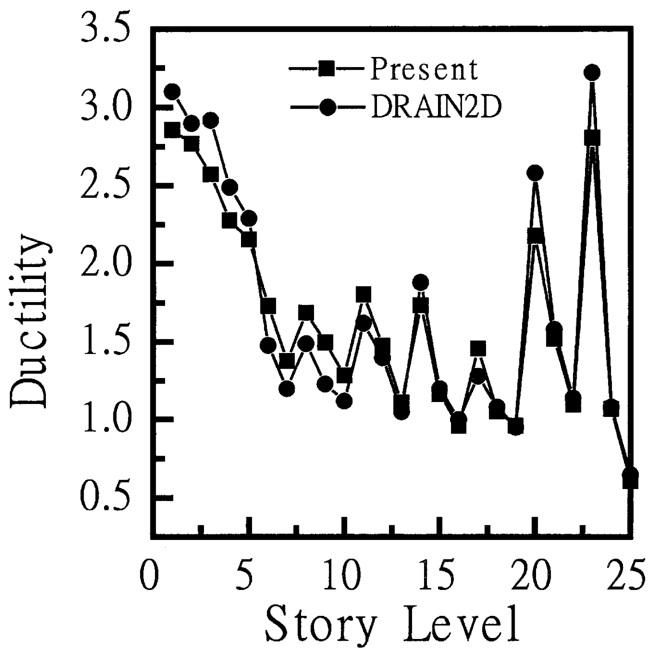

Fig. 7. Comparisons of story ductility

The total number of iterative steps is 1884 for the whole simulation when the reduction method is used, while the number is 1409 if the Cholesky method is employed. The reason is that the reduction method solves the systems of equations approximately. Therefore, to satisfy the convergence condition stated in Eq. (8), more iterative steps may be needed. The performance of the reduction method can also be evaluated from m_{av}. For this example, there are 5883 $(=4000 + 1884 - 1)$ effective systems that are solved by the reduction method, among which 4169 effective systems are solved with $m = 1$, 481 with $m = 2$, 1000 with $m = 3$, 212 with $m = 4$, and 21 with $m = 5$. The average is $m_{av} = 1.54$. Why m_{av} is so small can be explained with the aid of Fig. 8, which shows the evolution of the fundamental

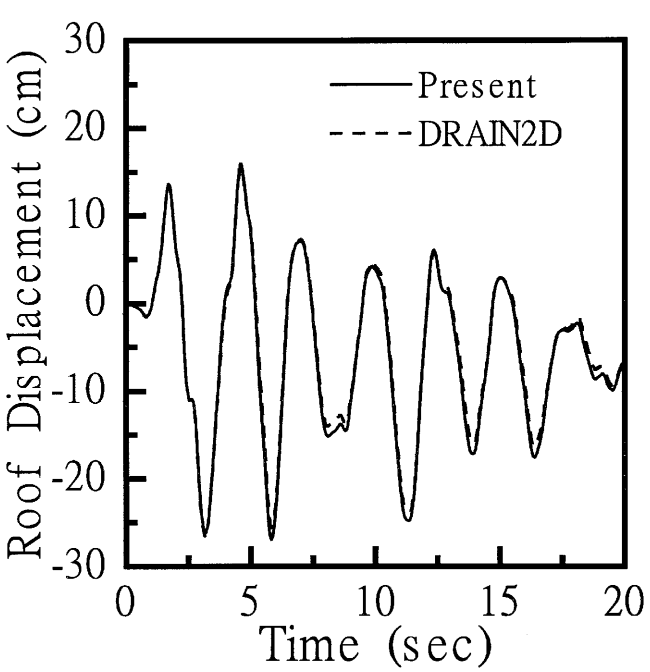

Fig. 6. Comparisons of the roof displacement response

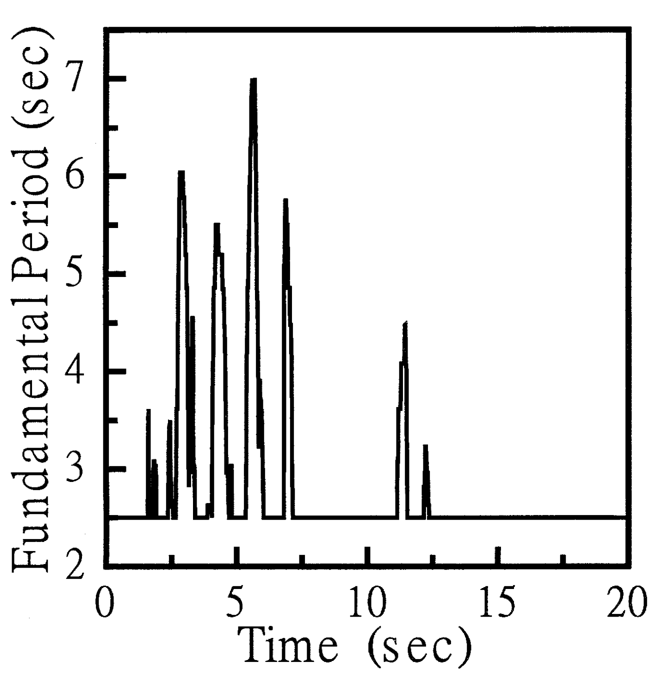

Fig. 8. Evolution of fundamental period

period obtained from the present method. It is known from this figure that the building is in an elastic-plastic state for a short time and in an elastic state mostly. When the building is elastic, the required number of basis vectors is one with the present reduction method as can be seen from Eqs. (20) and (18). This is why the number corresponding to $m = 1$ is so large and therefore a small m_{av} is obtained.

5.3
Geometric nonlinear analysis of 156-member space dome

Figure 9 shows the topology and geometry of a shallow geodesic dome, where $a = 6$ m, $f = 0.6$ m, and $l = 0.75$ m. It has a total of 61 nodes, whose coordinates in meters are generated according to $X^2 + Y^2 + (Z + 7.2)^2 = 60.84$. The 24 boundary nodes are fixed against both translations and rotations. The space dome is modeled using 156 space frame elements, one element per member, with a total of 222 degrees of freedom. The half-band width of the system matrix is 180 if the nodes are numbered from the inner to the outer hexagons as depicted in Fig. 9. The following material and sectional properties are used in the numerical simulation: Young's modulus $E = 6.895 \times 10^{10}$ N/m^2; shear modulus $G = 2.652 \times 10^{10}$ N/m^2; density $\rho = 2400$ kg/m^3; cross-sectional area $A = 6.5 \times 10^{-4}$ m^2; moments of inertia $I_y = I_z = 3.362 \times 10^{-8}$ m^4; and torsional constant $J = 6.724 \times 10^{-8}$ m^4.

Suppose that the dome is subjected to a concentrated force $P(t) = 6000 \sin(200t)$ N downward at the apex. Note that the fundamental period is obtained numerically to be 0.016 s for this dome. The nonlinear dynamic responses with the consideration of geometric nonlinearity are obtained by three approaches: ABAQUS (1998), the present approach with the Cholesky method and with the reduction method. In the present approach, the effects of geometric nonlinearity are taken into account by the use of the geometric stiffness matrix, which is available in Yang and Kuo (1994). Also, the consistent mass matrix is used and zero damping is assumed. The time increment is 0.001 s and thus a total of 150 incremental steps are used for the simulation time of 0.15 s. The value of the reference load P_r in Eq. (8) is 6000 N. The nonlinear responses at the apex together with the linear response are plotted in Fig. 10, from which close agreement between the present and ABAQUS results can be seen. Moreover, the nonlinear dynamic responses are quite different from the linear response.

In the whole simulation time of 0.15 s, there are 329, and 309 iterative steps for the reduction method and for the Cholesky method, respectively. Therefore, a total of 478 (=150 + 329 − 1)effective systems are solved by the reduction method in the present approach. The statistics about the number of basis vectors used are as follows: one effective system is solved with $m = 1$, three with $m = 2$, 367 with $m = 3$, 90 with $m = 4$, and 17 with $m = 5$. Thus the average number of basis vectors used m_{av} equals 3.25, again quite small. The efficiency of the reduction method, compared with that of the Cholesky method, can be evaluated by counting the CPU time. With the reduction method, the total CPU time, counted on a PC-486 DX2-66, is 1526 seconds, among which 481.7 s are spent in solving the effective systems. With the Cholesky method, the total CPU time is 3109.4 s and 2120 s are used in solving the effective systems. Therefore, the overall speed-up is 2.04 (=3109.4/1526), and the speed-up in terms of solving the effective systems is 4.4 (=2120/481.7). It is interesting to know that the theoretical speed-up is 4.72 according to Eq. (24) and is 4.62 according to Eq. (25) given that

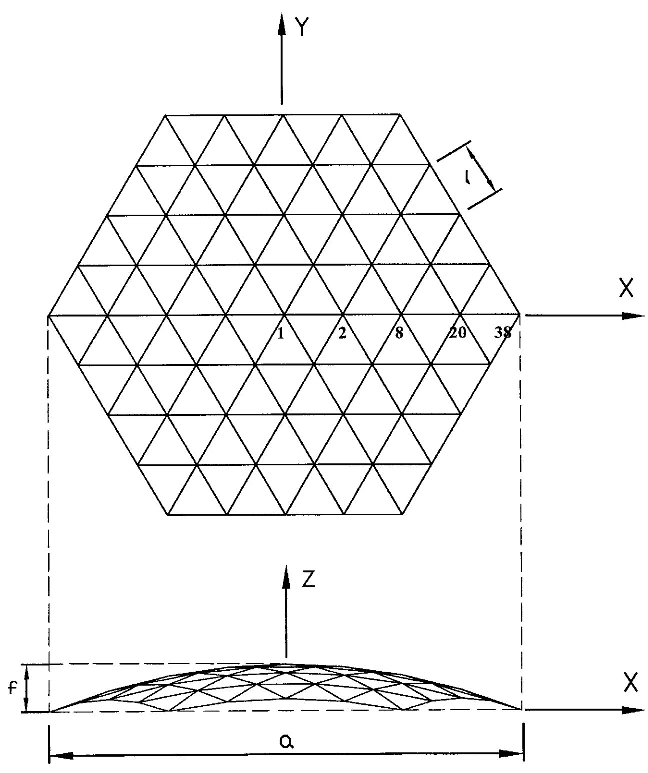

Fig. 9. 156-member space dome

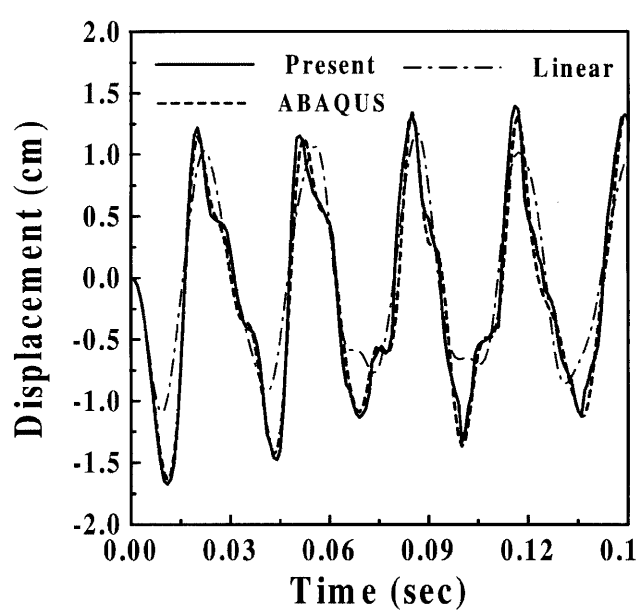

Fig. 10. Comparisons of the vertical displacement response at the apex

$b = 180$ and $m = m_{av} = 3.25$. Clearly, both equations predict very well; especially Eq. (25) is very simple and is recommended for estimating the speed-up when the present reduction method is used.

6
Conclusions

In this paper, the reduction method proposed by Leu and Huang (2000) has been used to carry out nonlinear dynamic analysis of framed structures for the first time. In the present approach, the reduction method is employed to solve the effective system resulted from the integration of the incremental or iterative equations of motion using Newmark's method. Of course, the reduction method is still applicable when other integration methods are employed.

The present approach has several merits as follows. First, the accuracy of the dynamic response obtained by the reduction method and by the Cholesky method is the same if a check on dynamic equilibrium such as that stated in Eq. (8) is performed. Second, the required number of basis vectors is quite few, which ranges from 2 to 5 in general. Third, the required number of basis vectors for each increment or iteration can be determined adaptively and efficiently through the use of the convergence criterion given in Eq. (15). Fourth, the speed-up, in terms of solving the effective system, can be estimated by $b/12m_{av}$ (see Eq. 25) before a nonlinear dynamic analysis is performed, where recall that b is the half-band width of the system and m_{av} is the average of the required number of basis vectors. Example three has shown that this estimation is quite good. In general a conservative estimation can be obtained if the value of m_{av} is taken to be four because, as mentioned before, m ranges from two to five mostly during the simulation. Fifth, the present method is more efficient than the Cholesky method when the half-band width b is greater than about 50; this can be obtained by letting the speed-up $b/12m_{av} = 1$ and assuming $m_{av} = 4$.

References

ABAQUS (1998) Version 5.7, Hibbitt, Karlsson & Sorenson, Inc., Providence, RI

Akin JE, Mofid M (1989) Numerical solution for response of beams with moving mass. ASCE J. Struct. Eng. 115: 120–131

Chang C, Engblom JJ (1991) Nonlinear dynamical response of impulsively loaded structure: A reduced basis approach. AIAA J. 29: 613–618

Clough RW, Wilson EL (1979) Dynamic analysis of large structural systems with local nonlinearities. Comput. Meth. Appl. Mech. Eng. 17/18: 107–129

DRAIN-2D (1973) A general purpose computer program for dynamic analysis of inelastic plane structures. NISEE/Computer Applications, CA

Gillies AG, Shepherd R (1983) Prediction of post-elastic seismic response of structures by a mode superposition technique. Bulletin of the New Zealand Nat. Soc. for Earthquake Eng. 16: 222–233

Idelsohn SR, Cardona A (1985a) A reduction method for nonlinear structural dynamic analysis. Comput. Meth. Appl. Mech. Eng. 49: 253–279

Idelsohn SR, Cardona A (1985b) A load-dependent basis for reduced nonlinear structural dynamics. Comput. and Struct. 20: 203–210

Kapania RK, Byun C (1993) Reduction methods based on eigenvectors and Ritz vectors for nonlinear transient analysis. Comput. Mech. 11: 65–82

Kirsch U (1991) Reduced basis approximations of structural displacements for optimal design. AIAA J. 29: 1751–1758

Léger P (1993) Mode superposition methods. In: Papadrakakis M (ed.). Solving Large-scale Problems in Mechanics. John Wiley & Sons, New York, pp 225–257

Léger P, Wilson EL (1988) Modal summation methods for structural dynamic computations. Earthquake Eng. and Struct. Dyn. 16: 23–27

Leu LJ (1999) A reduction method for boundary element reanalysis. Comput. Meth. Appl. Mech. Eng. 178: 125–139

Leu LJ, Huang CW (1997) Application of reduced-basis methods in structural static reanalysis. (in Chinese). Bulletin of the College of Eng., National Taiwan University 71: 73–89

Leu LJ, Huang CW (1998) A reduced-basis method for geometric nonlinear analysis of structures. IASS J. of Int. Assoc. Shell and Spatial Struct. 39: 71–76

Leu LJ, Huang CW (2000) Reanalysis-based optimal design of trusses. Int. J. Num. Meth. Eng. (in press)

Mohraz B, Elghadamsi FE, Chang CJ (1991) An incremental mode superposition for nonlinear dynamic analysis. Earthquake Eng. and Struct. Dyn. 20: 471–481

Muscolino G (1990) Mode-superposition methods for elastoplastic systems. ASCE J. Eng. Mech. 115: 2199–2215

Newmark NM (1959) A method of computation for structural dynamics. ASCE J. Eng. Mech. Div. 85: 67–94

Nickell RE (1976) Nonlinear dynamics by mode superposition. Comput. Meth. Appl. Mech. Eng. 7: 107–129

Noor AK (1981) Recent advances in reduction methods for nonlinear problems. Comput. and Struct. 13: 31–44

Nour-Omid B, Clough RW (1984) Dynamic analysis of structures using the Lanczos coordinates. Earthquake Eng. and Struct. Dyn. 12: 565–577

Olsson M (1985) Finite element, modal co-ordinate analysis of structures subjected to moving loads. J. Sound and Vib. 99: 1–12

Schwarz HR (1989) Numerical analysis: a compressive introduction. pp 35–37, John Wiley & Sons, Singapore

Shah VN, Bohm GJ, Nahavandi AN (1979) Modal superposition method for computationally economical nonlinear structural analysis. ASME J. Pressure Vessel Tech. 101: 134–141

Vukazich SM, Mish KD, Romstad KM (1996) Nonlinear dynamic response of frames using Lanczos modal analysis. ASCE J. Struct. Eng. 122: 1418–1426

Wilson EL, Yuan MW, Dickens JM (1982) Dynamic analysis by direct superposition of Ritz vectors. Earthquake Eng. and Struct. Dyn. 10: 813–821

Yang YB, Kuo SR (1994) Theory and analysis of nonlinear framed structures. Prentice-Hall, Singapore

Yau JD (1996) Dynamic response of bridges travelled by trains – analytical and numerical approaches. PhD Thesis, Department of Civil Engineering, National Taiwan University, Taipei, Taiwan