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結構動力辛計算及控制之方法(二)
Symplectic computation and control for structural dynamics (2)

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一、摘要

對於線性二次最佳控制法則，應用於結構物受到地震力、風力等事先未能知其歷時變化大小之外力作用時，所遭遇的困難，本計畫提出解決之道，總結為縮辛群控制律。縮辛群即縮群與辛群的直積，其中的辛群保證了控制律的最佳化，縮群保證了控制律的穩定性。

換句話說，本計畫在擴大的哈米爾頓架構下，處理多自由度線性結構之振動主動控制問題，已建立結構物受到外力歷時作用下之穩定最佳線性二次控制律。利用李群理論之保群法則，以縮辛群建構合適之計算方法及控制方法，即縮辛算法及縮辛控制法，使得長時間計算得以準確穩定不發散，而且穩定的最佳控制得以實現。

對於單自由度結構振動之穩定最佳線性二次控制，本計畫已得到運動方程式及控制力等之解析正解。

值得注意的是，本研究得到的線性二次最佳控制運動方程式，是初始值問題列式，而非兩點邊界值問題列式。因此所發展的縮辛控制法，可實現為即時實時控制，雖然將外力對結構的累積效應納入控制律中，但所需要的外力只到現在為止，並不需要用到尚未發生的將來外力。

本計畫研究辛特徵值、辛模態、辛正交、辛相似、辛群，發現線性二次最佳控制法則的諸多內在關係，可以辛正交及辛相似關係式總結之。

對於單自由度結構的振動阻尼問題，本計畫參考黏彈性組成律、動態系統理論、分數微積分的作法，發展積分型、微分型、分數微分型、鬆弛譜、遲緩譜等模式，以及其間的關係。對於多自由度結構的振動阻尼，有三種作法：一為參考多自由度動態系統理論的作法，一為採用一般的模態展開加入單自由度阻尼後再合成，一為採辛模態展開加入單自由度阻尼後再合成。

關鍵詞：多自由度結構、振動、阻尼、穩定化、線性二次控制、最佳控制、哈米爾頓架構、辛群、縮群、保群法則。

Abstract

The project examined the difficulties encountered by the linear quadratic (LQ) optimal control theories such as the so called LQR and LQG when they are applied to the vibration control of engineering structures against external loads, e.g., earthquake and wind forces, whose histories are unknown a priori. We established a control law based on a contractive symplectic group, which is the direct product of a contractive group and a symplectic group, the symplectic group ensuring optimality of the control law while the contractive group stabilizing the control algorithm.

In other words, under the extended Hamiltonian formalism, the project has achieved a stabilized optimal linear quadratic law for controlling the vibrations of structures of multiple degrees of freedom (MDOF) subjected to external disturbances of general nature. On the basis of Lie group of transformations with time being the parameter of the group, we developed group preserving schemes and algorithms, which preserve characteristic quantities of the contractive symplectic group as time elapses. Thus the schemes have long time computational stability and the LQ optimal control algorithms can be stabilized.

For oscillators of single degree of freedom, we have obtained exact solutions to all relevant quantities including solutions to the equations of motion and *control force*.

Note that obtained in the project was an initial value problem (IVP) formulation rather than a two-point boundary value problem (TPBVP) formulation. The significance of the IVP is that the developed control law can be used in real time because although taking into consideration the accumulating effect of external loads, it needs the external loads only up to the current time and does not need the not yet known future loads as conventional LQ algorithms being TPBVP do.

We studied symplectic eigenvalues, symplectic modes, symplectic orthogonality, symplectic similarity and symplectic group, and found that the expressions of symplectic orthogonality and symplectic similarity can summarize almost all important identities and properties in LQ optimal control theories.

For the problem of modeling vibration damping for oscillators of single degree of freedom, referring to the theory of viscoelasticity, dynamical system theory and fractional calculus, we wrote damping models of various types, including integral, differential, fractional derivative, relaxation spectrum, retardation spectrum, and clarified their relationship. For MDOF

damping models, we found three approaches; one may refer to MDOF dynamical systems, or apply modal analysis and add damping and synthesize the modes, or apply symplectic modal analysis and add damping and synthesize the symplectic modes.

Keywords：MDOF structures, vibration, damping, stabilization, linear quadratic control, optimal control, Hamiltonian formalism, symplectic group, contractive group, group-preserving scheme.

二、緣由與目的

結構動力學除了牛頓體系外，尚有拉格朗日體系及哈米爾頓體系。這在學界是盡人皆知之事，但在土木結構工程界顯然很少拿來活用。多自由度結構的運動方程式如下：

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{C}\dot{\mathbf{q}}(t)+\mathbf{K}\mathbf{q}(t)=\mathbf{w}(t) \quad (1)$$

此式如何計算？方法很多，直接時間積分或者以振態解耦，一個振態一個振態時間積分，再疊加，這兩個方式如果改在哈米爾頓架構下怎麼做？好不好？這是本計畫有興趣要做的。接著，若結構(1)加上控制力，要求最佳控制，這個問題如何在哈米爾頓體系下來做，這是本計畫的核心工作目標。我們的目的，主要在於探討長時間準確穩定的算法，以及具有追蹤外力 $\mathbf{w}(t)$ 能力的真正最佳且穩定的控制。

對於第一個問題，重點在於算法的長時間穩定性，以及如何擴充哈米爾頓架構，以便處理有阻尼的結構振動，不要只限制在沒有阻尼的結構振動。對於第二個問題，重點在於如何擴充哈米爾頓架構，以便有穩定的控制。

線性二次(LQ)控制是目前比較完備的一種理論，應用十分廣泛，是現代控制理論中最重要的成果之一。在土建工程的結構控制，受控對象不可避免地有外力 $\mathbf{w}(t)$ 作用，譬如地震力或風力。但應用線性二次最佳控制理論於受外力作用的結構時，對外力有非常嚴格的要求，必須事先知道整個歷時，或至少在統計意義上知道。具體而言，需事先知道外力的歷時，或假設外力小到可忽略，或假設外力是白噪音，或假設外力是線性濾波器在高斯白噪音為輸入下的輸出。因此大部份文獻都是採用忽略外力作用的線性二次調節器 (LQR) 控制理論，或採用假設外力是統計上已知的，例如是高斯分布的線性二次高斯 (LQG) 控制理論。基於上述的困難，在各個方面皆考慮外力的線性二次最佳控制理論常被認為實際上不可行的，因為實時控制計算需要將來尚未發生的外力（載重歷時）的數據。

我們的研究[1]顯示，外力作用是完全不能忽略的，控制法則必須考慮內狀態變數，只要引進包括外狀態與內狀態的全狀態觀念，就能將外力對結構物的作用及累積效應納入控制律中，得到線性二次最佳控制。這是一個具有追蹤外力能力的真正線性二次最佳控制理論。全狀態向量構成辛空間，因此這個線性二次最佳控制理論，很自然地，是在哈米爾頓架構下列式的，形成一種哈米爾頓動態系統，不同時間的全狀態向量彼此之間是以辛變換連繫的。

但是根據我們的實算經驗，只有少數的哈米爾頓動態系統，可以維持長時間的積分穩定。理論上來講，$2m$ 維的哈米爾頓動態系統可以分解為兩個 m 維的哈米爾頓動態系統，其中一個穩定，一個不穩定。所以實算經驗顯示大多數的哈米爾頓動態系統並沒有辦法維持長時間的積分穩定實在是良有以也。所以，穩定、即時、最佳這幾個重點，是本計畫面對的極具挑戰性要解決的課題。

三、研究內容—單自由度解析正解

Introduction

Recent decades witnessed many successful applications of linear quadratic (LQ) optimal control theory to active control of plants in various fields. The theory provides the active control of plants with a two-point boundary value problem (TPBVP) formulation over a time interval $[t_0, t_f]$. Hence, to search for an optimal control, one has to solve the TPBVP. However, when the plants are engineering structures designed to withstand external disturbances (for example, buildings against earthquakes), solving the TPBVP encounters formidable obstacles since it has to be solved backward from the terminal time t_f, so that external disturbances which do not yet occur must be known in advance. Unless external disturbances are entirely absent or they are white noise stochastic processes or modeled as the output of a prescribed linear filter subjected to a Gaussian white noise input, the control laws currently being used do not achieve the goal of optimalizing the performance index that is set up. Unfortunately, most of external disturbances acting on engineering structures do not meet those restrictions.

Tracing back to the origin of the difficulties encountered---the TPBVP with the presence of external disturbances and the stability of the controlled structures, Hong [2,3] was able to obtain a stable initial value problem (IVP) formulation for the considered issue. In the current report we restrict the controlled engineering structure to be of single degree of freedom and examine further its behavior and properties in details and in exact form.

An Initial Value Problem Formulation

Consider a linear oscillator of single degree of freedom subjected to an external disturbance $w(t)$ and a control force $u(t)$. The equation of motion of the oscillator may be written as

$$M\ddot{q}(t)+C\dot{q}(t)+K(t)=w(t)+u(t) \quad (1)$$

in its lifetime $[t_b, t_d] \subset$ R along with the values of $q(t_0)$ and $\dot{q}(t_0)$ prescribed at a certain time instant t_0. Here t is (the current) time, t_b and t_d are the birth and death times, respectively, of the oscillator, and t_0 is the initial time of a certain time interval $[t_0, t_f]$ in which we are interested. A superposed dot indicates time differentiation. q, $\dot{q}$ and $\ddot{q}$ are the displacement relative to the base, the velocity and the acceleration, respectively. M, C, K are respectively the mass, damping, and stiffness of the oscillator. To be an engineering structure M and K are positive and C is nonnegative.

This dynamical problem (an equation of motion together with the initial conditions on the displacement and velocity) can be transformed to the following state space description:

$$\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}u(t)+\mathbf{E}w(t) \quad \forall t\in[t_b,t_d], \quad \mathbf{x}(t_0)=\mathbf{x}_0 \quad (2)$$

where

$$\mathbf{x}(t)=\begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}, \quad \mathbf{x}_0=\begin{bmatrix} q(t_0) \\ \dot{q}(t_0) \end{bmatrix}, \quad (3)$$

and

$$\mathbf{A}=\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\omega^2 & -2\xi\omega \end{bmatrix}, \quad \mathbf{B}=\begin{bmatrix} \mathbf{0} \\ 1/M \end{bmatrix}, \quad \mathbf{E}=\begin{bmatrix} \mathbf{0} \\ 1/M \end{bmatrix} \quad (4)$$

in which $\omega=\sqrt{K/M}$ and $\xi=C/(2\sqrt{KM})$ are the natural frequency and damping ratio, respectively, of the oscillator. In the state IVP (2), the state $\mathbf{x}\in \mathrm{R}^2$, the control $u\in \mathrm{R}$, and the disturbance $w\in \mathrm{R}$ are all functions of time; the initial state $\mathrm{x}_0 \in \mathrm{R}^2$ is a constant column matrix; and $\mathbf{A}\in \mathrm{R}^{2\times 2}$, $\mathbf{B}\in \mathrm{R}^{2\times 1}$, $\mathbf{E}\in \mathrm{R}^{2\times 1}$ are constant matrices.

For a crucial consideration which will be made clear later let us introduce

$$\underline{\mathbf{x}}:=e^{at}\mathbf{x}, \ \underline{u}:=e^{at}u, \ \underline{w}:=e^{at}w, \ \underline{\mathbf{x}}_0:=e^{at_0}\mathbf{x}_0 \quad (5)$$

so that (2) becomes

$$\dot{\underline{\mathbf{x}}}(t)=(\mathbf{A}+a\mathbf{I})\underline{\mathbf{x}}(t)+\mathbf{B}\underline{u}(t)+\mathbf{E}\underline{w}(t) \quad \forall t\in[t_b,t_d], \quad \underline{\mathbf{x}}(t_0)=\underline{\mathbf{x}}_0. \quad (6)$$

The a is a positive real number to be determined later on. As usual **I** and **0** denote the identity and zero matrices, respectively, of appropriate orders as implied in the context.

According to LQ optimal control theory, (6) is to be controlled during the fixed, finite time interval$[t_0, t_f] \subset [t_b, t_d] \subset \mathrm{R}$ such that the quadratic functional, called the performance index,

$$J=\frac{1}{2}\int_{t_0}^{t_f} [\underline{\mathbf{x}}^t(t)\mathbf{Q}\underline{\mathbf{x}}(t)+R\underline{u}^2(t)+2\underline{\mathbf{x}}^t(t)\mathbf{T}\underline{u}(t)]\,dt \quad (7)$$

is minimized, where the symmetric weighting matrix

$$\begin{bmatrix} \mathbf{Q} & \mathbf{T} \\ \mathbf{T}^t & R \end{bmatrix}\in \mathrm{R}^{3\times 3}$$

is positive semidefinite and the weight R is a positive real number. The superscript t stands for the transpose.

To minimize the performance index J subjected to the constraint (6), we adjoin the weighted residue of the IVP (6) to J,

$$L=J+\int_{t_b}^{t_d}\underline{\lambda}^t(t)[(\mathbf{A}+a\mathbf{I})\underline{\mathbf{x}}(t)+\mathbf{B}\underline{u}(t)+\mathbf{E}\underline{w}(t)-\dot{\underline{\mathbf{x}}}(t)]dt$$

$$+\underline{\lambda}^t(t_0)[\underline{\mathbf{x}}(t_0)-\underline{\mathbf{x}}_0], \quad (8)$$

and then minimize L. The column matrix $\underline{\lambda}\in \mathrm{R}^2$ contains two Lagrange multipliers.

Note that since the governing equation $(6)_1$ of the IVP (6) is valid throughout the lifetime$[t_b,t_d]$ of the oscillator the constraint$(6)_1$ should be imposed for all the time t, $t_b\le t\le t_d$, as we have done in the above; this has led us to use the different time intervals $[t_0,t_f]$ and $[t_b,t_d]$ in the two integrals in (8) and this concept of nested intervals is crucial in obtaining an IVP formulation. Among all admissible varied functions $\underline{\mathrm{x}}(t)+\delta\underline{\mathrm{x}}(t)$, $\underline{u}(t)+\delta\underline{u}(t)$, $\underline{\lambda}(t)+\delta\underline{\lambda}(t)$ $\forall t\in[t_b,t_d]$, which are otherwise arbitrary except that $\delta\underline{\mathrm{x}}(t)$ vanishes at the two ends t_b and t_d and that $\underline{u}(t)$ and hence $\delta\underline{u}(t)$ are zero outside the control interval $[t_0,t_f]$, the necessary conditions for the optimal functions $\underline{\mathrm{x}}(t)$, $\underline{u}(t)$, $\underline{\lambda}(t)$ that extremize L are found to be

$$\dot{\underline{\lambda}}(t)=-(\mathbf{A}+a\mathbf{I})^t\underline{\lambda}(t) \quad \forall t\in[t_b,t_d], \quad (9)$$

$$\dot{\underline{\lambda}}(t)=-(\mathbf{A}+a\mathbf{I})^t\underline{\lambda}(t)-\mathbf{Q}\underline{\mathbf{x}}(t)-\mathbf{T}\underline{u}(t) \quad \forall t\in[t_0,t_f], \ \underline{\lambda}(t_0)=0, \quad (10)$$

$$\dot{\underline{\lambda}}(t)=-(\mathbf{A}+a\mathbf{I})^t\underline{\lambda}(t) \quad \forall t\in[t_f,t_d], \quad (11)$$

and

$$\underline{u}(t)=-\mathbf{T}^t\underline{\mathbf{x}}(t)/R-\mathbf{B}^t\underline{\lambda}(t)/R \quad \forall t\in[t_0,t_f], \quad (12)$$

as well as the IVP (6). All of these have resulted from extremizing the J of equation (8).

In this way, as a result of the extremization, $(6)_1$ is valid $\forall t\in[t_b,t_d]$ as it should be. If in constructing L the constraint $(6)_1$ is imposed only for$[t_0,t_f]$as was conventional, then through the extremization process $(6)_1$ will turn out to hold only for $[t_0,t_f]$, which is obviously absurd if we recall that an active control of, say, a building against an earthquake may last as short as t_f-t_0 = 50 seconds, but the building itself may stand on the site for more than t_d-t_b = 50 years, say.

Notice that both equations (10) and (12) are valid only over the control interval$[t_0,t_f]$. Let us now concentrate on the control interval $[t_0,t_f]$. Combining (6) and (10) and using (12), we have

$$\frac{d}{dt}\begin{bmatrix} \underline{\mathbf{x}}(t) \\ \underline{\lambda}(t) \end{bmatrix}=\mathbf{H}\begin{bmatrix} \underline{\mathbf{x}}(t) \\ \underline{\lambda}(t) \end{bmatrix}+\begin{bmatrix} \mathbf{E}\underline{w}(t) \\ \mathbf{0} \end{bmatrix} \quad \forall t\in[t_0,t_f], \quad \begin{bmatrix} \underline{\mathbf{x}}(t_0) \\ \underline{\lambda}(t_0) \end{bmatrix}=\begin{bmatrix} \underline{\mathbf{x}}_0 \\ \mathbf{0} \end{bmatrix} \quad (13)$$

where the matrix **H** is defined as

$$\mathbf{H}:=\begin{bmatrix} \mathbf{A}_1 & \mathbf{N} \\ -\mathbf{P} & -\mathbf{A}_1^t \end{bmatrix}, \quad (14)$$

in which

$$\mathbf{A}_1:=\mathbf{A}+a\mathbf{I}-\mathbf{B}\mathbf{T}^t/R, \ \mathbf{N}:=-\mathbf{B}\mathbf{B}^t/R, \ \mathbf{P}:=\mathbf{Q}-\mathbf{T}\mathbf{T}^t/R, \quad (15)$$

Therefore, when the control is regulated by the optimal law (12), the optimal $\underline{\mathrm{x}}(t)$ and $\underline{\lambda}(t)$ are governed by the IVP (13).

It is easy to check that N and P are symmetrical and thus **H** is a Hamiltonian matrix, which is, by definition, a square matrix satisfying

$$(\mathbf{J}\mathbf{H})^t=\mathbf{J}\mathbf{H}, \ \mathbf{J}:=\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}, \quad \mathbf{J}^{-1}=\mathbf{J}^t=-\mathbf{J}, \ \mathbf{J}^2=-\mathbf{I}, \quad (16)$$

It can be shown that **N** is negative semidefinite. As such, the IVP (13) is a constant coefficient linear Hamiltonian system defined in a symplectic space endowed with the canonical metric J defined in $(16)_2$.

Stabilization

Although we have been able to get an IVP formulation such as (13) rather than a TPBVP, the IVP (13) is still suffering from instability. To see this, let us examine the system matrix **H** of (13), which is a real Hamiltonian matrix. Recall that the eigenvalues of a real Hamiltonian 4x4 matrix are of four types: (i) a quadruple of truly complex eigenvalues $i = 1,2$, (ii) two pairs of real eigenvalues $\pm\alpha_1$, $\pm\alpha_2$, (iii) two pairs of purely imaginary eigenvalues $\pm\beta_{1i}$, $\pm\beta_{2i}$, and (iv) the eigenvalue 0 of multiplicity 4. Hence, for the more frequently occurring types (i) and (ii), two of four eigenvalues have positive real parts, leading to unstable solutions of (13).

To stabilize the solutions let us now define

$$z := \begin{bmatrix} \mathbf{x} \\ \underline{\lambda} \end{bmatrix}, \tag{17}$$

where, similar to (5),

$$\underline{\lambda} := e^{at}\lambda \tag{18}$$

The z may be called the complete state[1], consisting of the state **x** and the costate λ. From (13) together with (5) and (18), we find that the complete state z of the controlled oscillator is governed by the IVP

$$\dot{z}(t) = (\mathbf{H} - a\mathbf{I})z(t) + f(t) \;\; \forall t \in [t_0, t_f], \; z(t_0) = z_0 \tag{19}$$

where

$$f(t) := \begin{bmatrix} \mathbf{E}w(t) \\ \mathbf{0} \end{bmatrix}, \;\; z_0 := \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{0} \end{bmatrix},$$

the latter being the initial complete state.

Now it is very easy to choose a value of a such that all the eigenvalues of $\mathbf{H} - a\mathbf{I}$ have negative real parts so as to stabilize the controlled oscillator.

Symplectic Eigenvalues and Eigenvectors

To solve the IVP (19) and to explore its properties, we will now in this section find the eigenvalues and eigenvectors of the system matrix $\mathbf{H} - a\mathbf{I}$ of (19) and then in the next section perform a "modal" analysis of (19). It is recalled that **H** is a real Hamiltonian matrix and contains most of remarkable properties of the controlled oscillator; it is thus desirable that our eigenanalysis of **H** and the "modal" analysis of (19) are able to preserve those properties (see (21) below).

Accordingly, we first attempt to make a symplectic eigenanalysis of **H**; that is, given **H**, we find Σ and Ψ, where

$$\Sigma = diag[\sigma_1, \sigma_2, \sigma_3, \sigma_4]; \;\; \Psi = [\psi_1, \psi_2, \psi_3, \psi_4];$$

such that both

$$\mathbf{H}\Psi = \Psi\Sigma \tag{20}$$

and

$$\Psi^t \mathbf{J} \Psi = \mathbf{J}. \tag{21}$$

Equation (20) often written $\mathbf{H}\psi_i = \sigma_i\psi_i$ is typical of eigenvalue problems and equation (21) requires that the eigenvectors ψ_i be symplectically orthonormal, namely

$$\psi_i^t \mathbf{J} \psi_j = \begin{cases} 1 & \text{for } j = i+2, \\ 0 & \text{for } i \le j \ne i+2, \end{cases}$$

which taken together are equivalent to equation (21). Similar to (yet to be contrasted with) the relation of orthogonality $\Psi^t \mathbf{I} \Psi = \mathbf{I}$ in Euclidean space, equation (21) is the relation of symplectic orthogonality in a symplectic space. It is indeed the definition for a square matrix Ψ to be symplectic. It can be seen in (21) that **J**, the canonical metric of the symplectic space, replaces **I**, the canonical metric of Euclidean space.

For definiteness the weighting matrix is explicitly written

$$\begin{bmatrix} \mathbf{Q} & \mathbf{T} \\ \mathbf{T} & R \end{bmatrix} = \begin{bmatrix} Q_d & 0 & 0 \\ 0 & Q_v & 0 \\ 0 & 0 & R \end{bmatrix},$$

in which $Q_d \ge 0, Q_v \ge 0, R > 0$ are the weights for displacement, velocity and control force, respectively, and all cross-term weights are here assumed to be zero for the sake of simplicity. Thus the matrix **H** becomes

$$\mathbf{H} = \begin{bmatrix} a & 1 & 0 & 0 \\ -\omega^2 & -2\xi\omega + a & 0 & -1/(M^2R) \\ -Q_d & 0 & -a & \omega^2 \\ 0 & -Q_v & -1 & 2\xi\omega - a \end{bmatrix},$$

With the above **H** substituted into (20) the eigenvalues are found to be

$$\sigma_1 = \sqrt{s + \Gamma},$$

$$\sigma_2 = -\sqrt{s - \Gamma},$$

$$\sigma_3 = -\sqrt{s + \Gamma} = -\sigma_1,$$

$$\sigma_4 = \sqrt{s - \Gamma} = -\sigma_2,$$

where

$$S := \frac{Q_v}{2M^2R} + \left[a^2 - 2a\omega\xi + \omega^2\left(2\xi^2 - 1\right)\right],$$

$$\Gamma := \left\{\left(\frac{Q_v}{2M^2R}\right)^2 + a\left[\omega\left(2\xi^2 - 1\right) - 2a\xi\right]\frac{Q_v}{M^2R} + \left[4\omega^2(a - \omega\xi)^2\left(\xi^2 - 1\right) - \frac{Q_d}{M^2R}\right]\right\}^{1/2},$$

After lengthy manupulations of (20) and (21) the corresponding (symplectically normalized) eigenvectors are found to be

$$\psi_i = \begin{bmatrix} \Psi_{1i} \\ \Psi_{2i} \\ \Psi_{3i} \\ \Psi_{4i} \end{bmatrix} = \frac{1}{2\sqrt{\Gamma}\sigma_i\left[Q_d + Q_v\left(a^2 - \sigma_i^2\right)\right]} \begin{bmatrix} \omega^2 + (\sigma_i + a)(\sigma_i + a - 2\omega\xi) \\ (\sigma_i - a)\left[\omega^2 + (\sigma_i + a)(\sigma_i + a - 2\omega\xi)\right] \\ a\left(Q_v\omega^2 - Q_d\right) - \sigma_i\left(Q_v\omega^2 + Q_d\right) + 2Q_d\omega\xi \\ Q_d + Q_v\left(a^2 - \sigma_i^2\right) \end{bmatrix} \tag{22}$$

for $i = 1,2,3,4$, where Ψ_{ji} is the ji-th entry of the 4×4 matrix Ψ and also the j-th entry of the column matrix ψ_i.

Once σ_i and ψ_i of **H** have been found, we then proceed to the eigenanalysis of $\mathbf{H} - a\mathbf{I}$. It is easy to show that

$$(\mathrm{H}-a\mathrm{I})\Psi = \Psi(\Sigma - a\mathrm{I}), \tag{23}$$

so that $\mathrm{H}-a\mathrm{I}$ has eigenvalue $\sigma_i - a$ and eigenvectors ψ_i. As stated in the last section, we must choose $a >$ max {Re σ_i } yieling all Re $\sigma_i - a < 0$ so as to stabilize the controlled oscillator.

Symplectic "Modal" Analysis and Superposition

By eigenvector expansions in the symplectic space we have

$$z(t) = \sum_{i=1}^{4} Z_i(t)\ \psi_i, \tag{24}$$

$$f(t) = \sum_{i=1}^{4} F_i(t)\ \psi_i, \tag{25}$$

where, due to the symplectic orthogonality relation (21),

$$Z_i(t) = -\psi_{2+i}^{t}\mathbf{J}\mathrm{z}(t), \quad Z_{2+i}(t) = \psi_i^{t}\mathbf{J}\mathrm{z}(t), \tag{26}$$

$$F_i(t) = -\psi_{2+i}^{t}\mathbf{J}f(t), \quad F_{2+i}(t) = \psi_i^{t}\mathbf{J}f(t), \tag{27}$$

for $i = 1,2,3,4$. Multiplying the IVP (19) on the left by $\Psi^t\mathbf{J}$, using equations (23)-(27), and noting $\sigma_{2+i} = -\sigma_i$, we obtain four uncoupled scalar IVPs for the "modal"responses:

$$\begin{aligned}
&\dot{Z}_i(t) = (\sigma_i - a)Z_i(t) + F_i(t), \\
&Z_i(t_0) = \Psi_{3,2+i}\, q(t_0) + \Psi_{4,2+i}\, \dot{q}(t_0), \\
&\dot{Z}_{2+i}(t) = (-\sigma_i - a)Z_{2+i}(t) + F_{2+i}(t), \\
&Z_{2+i}(t_0) = -\Psi_{3i}\, q(t_0) - \Psi_{4i}\, \dot{q}(t_0),
\end{aligned} \tag{28}$$

for $i = 1,2$, in which the "modal" excitations are computed by

$$F_i(t) = \Psi_{4,2+i} w(t)/M,\ F_{2+i}(t) = -\Psi_{4i} w(t)/M. \tag{29}$$

The "modal" IVPs (28) can be solved as follows:

$$\begin{aligned}
Z_i(t) &= \Psi_{3,2+i}\left\{e^{(\sigma_i - a)(t-t_0)} q(t_0)\right\} \\
&\quad + \Psi_{4,2+i}\left\{e^{(\sigma_i - a)(t-t_0)}\dot{q}(t_0) + \frac{1}{M}\int_{t_0}^{t} e^{(\sigma_i - a)(t-\tau)} w(\tau)d\tau\right\}, \\
Z_{2+i}(t) &= -\Psi_{3i}\left\{e^{(-\sigma_i - a)(t-t_0)} q(t_0)\right\} \\
&\quad - \Psi_{4i}\left\{e^{(-\sigma_i - a)(t-t_0)}\dot{q}(t_0) + \frac{1}{M}\int_{t_0}^{t} e^{(-\sigma_i - a)(t-\tau)} w(\tau)d\tau\right\},
\end{aligned} \tag{30}$$

for $i = 1,2$.

Once the "modal" responses (30) and eigenvectors (22) are obtained, "modal"superposition according to (24) will yield the complete state $z(t)$, which includes the state $\mathbf{x}(t)$ (containing the relative displacement $q(t)$ and velocity $\dot{q}(t)$ and the costate $\lambda(t)$ (containing the internal force and momentum). With $\mathbf{x}(t)$ and $\lambda(t)$ in hands the control $u(t)$ can be calculated via the optimal control law:

$$u(t) = -\mathbf{T}^t\mathbf{x}(t)/R - \mathbf{B}^t\lambda(t)/R \quad \forall t \in [t_0, t_f], \tag{31}$$

which is derived from equations (12), (5) and (18).

Controlled Responses to Earthquakes

It is noted that the integrals in equations (30) may be evaluated either in closed form for those external disturbances $w(t)$ that are described in some simple analytical forms, or with the aids of quadratures for those that are defined numerically or in complicated analytical forms. In the case of active control against earthquakes the earthquake excitations $w(t)$ are measured in real time at consecutive time instants and the variation in each time increment is usually assumed to be constant; as such, analytical forms of the complete state $z(t)$ and the control $u(t)$ can be found and then tailored to time stepping algorithms.

四、參考文獻

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[3] 洪宏基，結構動力辛計算及控制之方法，行政院國家科學委員會補助專題研究計畫成果報告，計畫編號：NSC 89-2211-E-002-033，執行期間：88年08月01日至89年07月31日，國立台灣大學土木工程學研究所。

五、計畫成果自評

本計劃為期兩年，在申請書中列有的工作項目，第一年八項：

(1) 多自由度結構運動方程式轉成哈米爾頓體系
(2) 多自由度結構運動方程式轉成哈米爾頓體系比較，尤其阻尼項
(3) 多自由度結構運動方程式轉成哈米爾頓體系物理意義探討
(4) 多自由度結構之線性二次最佳控制之初始問題列式
(5) 辛正交關係以及特徵值問題
(6) 辛群之應用研究
(7) 多自由度結構之辛算法
(8) 多自由度結構在外力作用下之最佳化控制

第二年十項：

(1) 單自由度結構振動阻尼
(2) 由縮辛群探討多自由度結構振動阻尼
(3) 由縮辛群再擴大來探討阻尼模式
(4) 哈米爾頓體系無阻尼振動之計算
(5) 擴大的哈米爾頓體系內多自由度結構振動之計算
(6) 單自由度結構振動之線性二次最佳控制之解析解
(7) 線性二次最佳控制
(8) 穩定的線性二次最佳控制
(9) 擴大的辛群的保群法則
(10) 擴大的哈米爾頓體系內相關的特徵值問題

均已大致完成；尤其核心目標已全數達成。現在不僅能做到最佳化，而且是穩定最佳化，可實現為實時控制，所需要的外力是只到現在為止，並不需要尚未發生的將來外力。這是一個重要的突破，可認為是有外力時的線性二次最佳控制理論的徹底解決。多自由度結構控制的理論的大貌已先發表在研討會[2]，也可參見第一年的精簡報告[3]。單自由度結構控制的解析正解理論的大貌則寫在本期第二年精簡報告的第三節研究內容裡。其他多項工作成果則扼要地敘述於本精簡報告的第一節摘要裡。