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The Riemann complex boundary element method for the solutions of two-dimensional Elliptic equations

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Abstract

In this paper, a new boundary integral equation model Riemann complex boundary element method (RCBEM), is proposed based on the boundary element method (BEM) and the theory of Vekua and its modification as well as complex Riemann function as the fundamental solution. The RCBEM method is used to solve the linear, second order, elliptical partial differential equations in the fluid flow problems. In comparison to the generally used BEM, for RCBEM, there are two distinct differences. First one is that, RCBEM applies complex Riemann function as the fundamental solution of the adjoint operator while in direct BEM, on the other hand Green function is used. The second one is that the governing equations should be transformed into complex domain because there exist two characteristics in complex plane for elliptic systems, while in the direct BEM is not, since the Green function is adopted instead. The singular problem occurring in direct BEM can be avoided in RCBEM, especially for regular domain problems. The efficiency and accuracy of the RCBEM depends on the complex variable integration. To verify the feasibility and accuracy of RCBEM, the model is applied to different case studies of potential flows, Helmholtz equation problem and advection–diffusion problem and results are compared with analytical solutions and other numerical models. The results are satisfactory and prove the applicability of RCBEM for various two-dimensional elliptic equation problems.

Keywords: Riemann complex functions; Boundary element method; Fluid flow problems

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1. Introduction

Boundary element method (BEM) has been established as a powerful numerical tool in the solution of various fluid flow problems [1]. In BEM, the computational domain becomes the enclosing boundary and the effective dimensions of the problem considered will be reduced by one. Hence it is much easier in discretization and data preparation for the problem considered. These advantages make BEM more suitable for the solution of various fluids flow problems.

BEM are usually derived from the Green's theorem with an appropriate free-space Green's function [2] using real variables. In recent times, a boundary element approach using complex variables for boundary integration, known as complex variable boundary element method (CVBEM) has been introduced [3]. The CVBEM is a generalization of the Cauchy integral formula into a boundary integral equation method, followed by formulation into a workable computer algorithm for effective mathematical simulation. This generalization allows an immediate and valuable transfer of the modeling technique and makes the process simpler and more efficient than using the real variables. But this limits its application to two-dimensional harmonic (Laplace) problems [3].

Generally an integral equation is solved by a numerical model that assumes the boundary of the problem domain is discretized into piecewise-polynomial curves, and the known and the unknown boundary values are approximated as piecewise-continuous functions along the boundary. Unlike the Green's function formulations, the complex variable method does not depend on the shape of the contour between nodes [3,4]. Most of the complex variable boundary element methods adopt piecewise-linear representations of the complex functions that result in a second order accurate integration and normally give second-order accurate solutions for the boundary element solutions as well [4]. In complex BEM, the simplicity and elegance of complex analysis carries over to the computations as well. While the Green's function formulation and the complex method are not directly compared here, studies by Dold and Peregrine [5] and Hromadka and Lai [3] indicate that the latter method is clearly superior to others in many fluid flow problems.

In the present study, a new complex boundary element method called Riemann Complex Boundary Element Method (RCBEM) is proposed. In RCBEM, the solutions of two-dimensional fluid flow problems are developed based on the theory of Vekua [6] and it's modification and, thereby Riemann function [7] is taken as the fundamental solution to solve the linear second order elliptic equations.

In comparison with the generally used direct BEM, there are two major differences in RCBEM. In direct BEM, Greens functions are employed as solutions of adjoint operator while in RCBEM complex Riemann functions are used as the fundamental solutions of the adjoint operators. Secondly in RCBEM, the governing equations must be transformed into a complex domain because there exist two characteristics in complex plane for elliptic systems (it should be noted that hyperbolic systems remain in real plane), while the direct BEM is solved in real planes, since the Green function is adopted instead. The main difference of RCBEM with the CVBEM [3] is, while in RCBEM, the Riemann functions are used as the fundamental solutions, in CVBEM, Cauchy's functions are used as the fundamental solutions which will restrict the applications of more engineering problems beyond the harmonic functions.

In RCBEM, since the generic meaning of Riemann function is a characteristic boundary value problem, and the characteristic curves are closely associated with the propagation of certain types of singularities, we can infer that the Riemann function used is a regular solution, meaning that the singularities are far behind infinite. Some of the important features offered by RCBEM are: (1) the Riemann functions used are regular solutions and hence they satisfy the governing equations throughout the region enclosed by the problem boundary, the approximation is made only at the boundary; (2) the integration of the boundary integrals along each boundary element are carried out exactly; (3) mathematical means can be devised to evaluate approximation errors; and (4) substantial modeling simplification are possible resulting from the complex variable application and the boundary element approach. However, it should be noted that the present theory of RCBEM is for two-dimensional problems only.

In this paper, the RCBEM is applied for the solutions of various fluid flow problems governed by Laplace equation, Helmholtz equation and steady state advection–diffusion equations. The feasibility and accuracy of the RCBEM has been shown by solving a variety of problems governed by the above mentioned equations. RCBEM solutions are verified by comparing with available exact solutions and direct BEM solutions. Good agreements are observed in all the cases.

2. Complex Riemann boundary integral equations

2.1. Riemann function in complex variables

Initially, let us take account of some results of the theory of solutions of linear second order elliptic differential equations essentially based on the complex Riemann function and the Volterra type integral equation described by Vekua [6]. Consider a second order linear elliptic differential equation in two independent variables x and y,

$$L[u] = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = f(x,y) \tag{2.1}$$

where the coefficients a, b, c and f are functions of the variables x and y, are analytic and continuous in a domain D. Let the coordinate transformation of the complex variables be represented as $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$ and the differential operators as,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)$$

Now using the new complex variables, Eq. (2.1) can be transformed into,

$$M[u] = \frac{\partial^2 u}{\partial z \partial \bar{z}} + A(z,\bar{z})\frac{\partial u}{\partial z} + B(z,\bar{z})\frac{\partial u}{\partial \bar{z}} + C(z,\bar{z})u = F(z,\bar{z}) \tag{2.2}$$

which is a complex form of hyperbolic partial differential equation. A, B, C and F are defined as,

$$A(z,\bar{z}) = \frac{1}{4}\left[a\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2\mathrm{i}}\right) + \mathrm{i}b\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2\mathrm{i}}\right)\right], \quad C(z,\bar{z}) = \frac{1}{4}c\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2\mathrm{i}}\right)$$

$$B(z,\bar{z}) = \frac{1}{4}\left[a\left(\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2\mathrm{i}}\right) - \mathrm{i}b\left(\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2\mathrm{i}}\right)\right], \quad F(z,\bar{z}) = \frac{1}{4}f\left(\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2\mathrm{i}}\right)$$

Eq. (2.2) is the complex form of (2.1), and the coefficients A, B, C and F are holomorphic with respect to variables z and $\bar{z}$ in a domain $z \in D$ and $\bar{z} \in D_1$ where D and D_1 are simply connected domains.

Now the Riemann function can be expressed in two alternative forms [6],

$$R(t,\bar{z};t,\tau) = \exp\left[\int_{\tau}^{\bar{z}} A(t,\eta)\,\mathrm{d}\eta\right] \quad \text{on } z = t \tag{2.3}$$

$$R(z,\tau;t,\tau) = \exp\left[\int_{t}^{z} B(\xi,\tau)\,\mathrm{d}\xi\right] \quad \text{on } \bar{z} = \tau \tag{2.4}$$

where t and τ are fixed parameters. The Riemann function R satisfies the following normalized condition,

$$R(z,\tau;t,\tau) = 1 \quad \text{on } z = t,\ \ \bar{z} = \tau \tag{2.5}$$

By taking the adjoint of Eq. (2.2), using the Riemann functions and integrating with respect z, $\bar{z}$, we have the Volterra type integral equation of second kind [6],

$$\begin{aligned} R(z,\bar{z};t,\tau) &- \int_{\tau}^{\bar{z}} A(z,\eta)R(z,\eta;t,\tau)\,\mathrm{d}\eta - \int_{t}^{z} B(\xi,\bar{z})R(\xi,\bar{z};t,\tau)\,\mathrm{d}\xi \\ &+ \int_{t}^{z}\int_{\tau}^{\bar{z}} C(\xi,\eta)R(\xi,\eta;t,\tau)\,\mathrm{d}\xi\,\mathrm{d}\eta = 1 \end{aligned} \tag{2.6}$$

As described in Vekua [6], using the adjoint property of Eq. (2.2) and using the Riemann functions, exchanging the pairs of $(z,\bar{z})$ and (t,τ) and integrating with respect to t in the interval (z_0,z), τ in $(\bar{z}_0,\bar{z})$, one can obtain

$$\begin{aligned} u(z,\bar{z}) = u(z_0,\bar{z}_0)R(z,\bar{z};z_0,\bar{z}_0) &+ \int_{z_0}^{z} \Phi_1(t)R(z,\bar{z};t,\bar{z}_0)\,\mathrm{d}t + \int_{\bar{z}_0}^{\bar{z}} \Phi_2(\tau)R(z,\bar{z};z_0,\tau)\,\mathrm{d}\tau \\ &+ \int_{z_0}^{z}\int_{\bar{z}_0}^{\bar{z}} F(t,\tau)R(z,\bar{z};t,\tau)\,\mathrm{d}\tau\,\mathrm{d}t \end{aligned} \tag{2.7}$$

where $\Phi_1(z) = \dfrac{\partial u(z,\bar{z}_0)}{\partial z} + B(z,\bar{z}_0)u(z,\bar{z}_0)$; $\Phi_2(z) = \dfrac{\partial u(z_0,\bar{z})}{\partial \bar{z}} + A(z_0,\bar{z})u(z_0,\bar{z})$.

It should be noted that the coefficients of Eq. (2.1) are real functions and u is also a real function and hence after taking integration by parts, the solution of (2.1) can be written as,

$$u(x,y) = \mathrm{Re}\left[H_0(z)\phi(z) + \int_{z_0}^{z} H(z,t)\phi(t)\,\mathrm{d}t + \int_{z_0}^{z}\int_{\bar{z}_0}^{\bar{z}} R(z,\bar{z};t,\tau)F(t,\tau)\mathrm{d}\tau\,\mathrm{d}t\right] \tag{2.8}$$

where Re means real part and

$$H_0(z) = R(z,\bar{z};z_0,\bar{z}_0) \tag{2.9}$$

$$H(z,t) = -\frac{\partial}{\partial t}R(z,\bar{z};t,z_0) + B(z,\bar{z}_0)R(z,\bar{z};t,z_0) \tag{2.10}$$

$$\phi(z) = 2u(z,\bar{z}) - u(z_0,\bar{z}_0)R(z,\bar{z}_0;z_0,\bar{z}_0) \tag{2.11}$$

$\phi(z)$ is an arbitrary holomorphic function in D determined by the boundary condition. Assuming without loss of generality that $z_0 = 0$ lying inside the region D and also $A(0,\bar{z}) = B(z,0) = 0$, we get,

$$H(z,t) = -\frac{\partial}{\partial t}R(z,\bar{z};t,0) \tag{2.12}$$

$$\phi(z) = 2u(z,0) - u(0,0)R(z,0;0,0) \tag{2.13}$$

The detailed procedure of the derivation of Eq. (2.8) is given in Vekua [6], which is not repeated here.

2.2. Riemann functions for the elliptic differential equations

Before the application of boundary element procedure for the concerned elliptic differential equations using RCBEM, we have to find the fundamental solutions (Riemann functions in RCBEM). The fundamental solution is derived from the general Eq. (2.6). In this section, we consider some important elliptic equations in different forms, such as Laplace equation, modified Helmholtz equation and Helmholtz equation. Equations like steady-state advection–diffusion equation are solved after converting it into the modified Helmholtz equation.

2.2.1. Laplace equation

Considering the Laplace equation,

$$\nabla^2 u = 0 \tag{2.14}$$

Comparing with the Eq. (2.2), the coefficients $A(z,\eta) = B(\xi,\bar{z}) = C(\xi,\eta) = 0$. From Eq. (2.6), the fundamental solution (Riemann function) for the Laplace equation is obviously,

$$R(z,\bar{z};t,\tau) = 1 \tag{2.15a}$$

Hence the Eqs. (2.9) and (2.12) for potential flow problems can be written as,

$$H_0(z) = 1 \tag{2.15b}$$

$$H(z,t) = -\frac{\partial}{\partial t}R(z,\bar{z};t,o) = 0 \tag{2.15c}$$

2.2.2. Modified Helmholtz equation

Consider the modified Helmholtz equation,

$$\nabla^2 u - \lambda^2 u = 0 \tag{2.16}$$

where λ is assumed a constant. In comparison with Eq. (2.2), the coefficients $A(z,\eta) = B(\xi,\bar{z}) = 0$, and $C(\xi,\eta) = -\lambda^2/4$. Now Eq. (2.6) can be expressed as,

$$R(z,\bar{z};t,\tau)-\frac{1}{4}\lambda^2\int_t^z\int_\tau^{\bar{z}}R(\xi,\eta;t,\tau)\,d\xi\,d\eta=1 \tag{2.17}$$

By the methods of successive approximations [8], the solution is,

$$R(z,\bar{z};t,\tau)=1+\int_t^z\int_\tau^{\bar{z}}\Gamma_0(z,\bar{z};\xi,\eta;t,\tau)\,d\eta\,d\xi \tag{2.18}$$

where $\Gamma_0(z,\bar{z};\xi,\eta;t,\tau)=\sum_{v=1}^{\infty}N_0^{(v)}(z,\bar{z};\xi,\eta;t,\tau)$

$$N_0^{(v)}(z,\bar{z};\xi,\eta;t,\tau)=\int_\xi^z\int_\eta^{\bar{z}}N_0^{(v-1)}(s,\sigma;\xi,\eta;t,\tau)N_0\,d\sigma\,ds;\quad N_0^{(1)}=N_0=-C(\xi,\eta)$$

$$\text{since } N_0^{(1)}=-C(\xi,\eta)=\lambda^2/4,\ N_0^{(2)}=\int_\xi^z\int_\eta^{\bar{z}}\left(\frac{1}{4}\lambda^2\right)\left(\frac{1}{4}\lambda^2\right)d\sigma\,ds=\frac{\lambda^4}{16}(z-\xi)(\bar{z}-\eta)$$

$$N_0^{(3)}=\int_\xi^z\int_\eta^{\bar{z}}\left(\frac{1}{4}\lambda^2\right)\left[\frac{\lambda^4}{16}(z-\xi)(\bar{z}-\eta)\right]d\sigma\,ds=\frac{\lambda^6}{64}(z-\xi)^2(\bar{z}-\eta)^2\ \ldots$$

Now we have,

$$\begin{aligned}R(z,\bar{z};t,\tau)&=1+\int_t^z\int_\tau^{\bar{z}}\Gamma_0(z,\bar{z};\xi,\eta;t,\tau)\,d\eta\,d\xi=1+\int_t^z\int_\tau^{\bar{z}}\left[\sum_{v=1}^{\infty}N_0^{(v)}(z,\bar{z};\xi,\eta,t,\tau)\right]d\eta\,d\xi\\&=1+\int_t^z\int_\tau^{\bar{z}}\left[\frac{1}{4}\lambda^2+\frac{\lambda^4}{16}(z-\xi)(\bar{z}-\eta)+\frac{\lambda^6}{64}(z-\xi)^2(\bar{z}-\eta)^2+\cdots\right]d\eta\,d\xi\\&=\sum_{k=1}^{\infty}[(z-t)(\bar{z}-\tau)/2]^{2k}/k!\Gamma(k+1)=I_0\left(\lambda\sqrt{(z-t)(\bar{z}-\tau)}\right)\end{aligned} \tag{2.19a}$$

where I_0 is the modified Bessel function of the first kind of order zero which also shows the regular behavior in the defined domain.

Hence the Eqs. (2.9) and (2.12) can be written as,

$$H_0(z)=I_0(\lambda r),\quad r=\sqrt{x^2+y^2} \tag{2.19b}$$

$$H(z,t)=\frac{1}{2}\lambda\sqrt{\bar{z}}\frac{I_1\left(\lambda\sqrt{\bar{z}(z-t)}\right)}{\sqrt{z-t}} \tag{2.19c}$$

where I_j is the modified Bessel function of the first kind of order one which is a regular function for the defined domain.

2.2.3. Helmholtz equation

For the Helmholtz equation (λ is constant),

$$\nabla^2u+\lambda^2u=0 \tag{2.20}$$

In comparison with Eq. (2.2), the coefficients $A(z,\eta) = B(\xi,\bar{z}) = 0$, and $C(\xi,\eta) = \lambda^2/4$. Using the successive approximation as mentioned above, we obtain

$$R(z,\bar{z};t,\tau) = J_0\left(\lambda\sqrt{(z-t)(\bar{z}-\tau)}\right) \tag{2.21a}$$

where J_0 is the Bessel function of the first kind of order zero, which is also a regular function for the defined domain.

Hence the Eqs. (2.9) and (2.12) can be written as,

$$H_0(z) = J_0(\lambda r), \quad r = \sqrt{x^2+y^2} \tag{2.21b}$$

$$H(z,t) = -\frac{\partial}{\partial t}R(z,\bar{z};t,0) = -\frac{1}{2}\lambda\sqrt{\bar{z}}\frac{J_1\left(\lambda\sqrt{\bar{z}(z-t)}\right)}{(\sqrt{z-t})} \tag{2.21c}$$

where J_1 is the Bessel function of the first kind of order one, which shows the regular behavior in the defined domain.

2.3. Boundary conditions

In this section, the application of two types of boundary conditions, Dirichlet boundary conditions and mixed boundary conditions, are illustrated with reference to the RCBEM model development.

2.3.1. Dirichlet boundary conditions

Let D be a simply connected domain bounded by a contour Γ. Considering the function $u(x,y)$ satisfying the real boundary condition,

$$u(x,y) = g(x,y) = g(s) \quad s \in \Gamma \tag{2.22}$$

Assuming that $g(s)$ satisfies the Hölder condition on boundary Γ and the boundary value $\phi(t)$ of Eq. (2.13) also satisfies the Hölder condition which can be expressed in the form of double layer (or the Cauchy kernel integral),

$$\phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\phi(t)}{t-z}\,dt \tag{2.23}$$

for harmonic function [6] and

$$\phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\mu(t)}{t-z}\,dt \tag{2.24}$$

for the other functions [6] where $\mu(t)$ is a real density function of the dipole.

Substituting Eq. (2.24) into Eq. (2.8), the Fredholm integral equation of the first kind can be obtained as,

$$u(x,y) = \int_\Gamma \mu(t)M(z,t)\,ds \tag{2.25}$$

where

$$M(z,t) = \mathrm{Re}\left[\frac{H_0(z)\mathrm{d}t/\mathrm{d}s}{2\pi\mathrm{i}(t-z)} + \frac{\mathrm{d}t/\mathrm{d}s}{2\pi\mathrm{i}}\int_0^z \frac{H(z,t_1)}{t-t_1}\,\mathrm{d}t_1\right] \tag{2.26}$$

Let an interior point z approaches to an arbitrary point t_0, and satisfies the boundary condition Eq. (2.22), the Fredholm type integral equation of second kind can be obtained,

$$\left(1-\frac{\bar{\alpha}}{2\pi}\right)\mu(t_0) + \int_\Gamma M(t_0,t)\mu(t)\,\mathrm{d}s = g(t_0) \tag{2.27}$$

for the unknown function μ, ($\bar{\alpha}$ is contour angle, $0 \leqslant \bar{\alpha} \leqslant 2\pi$) [6], where,

$$M(t_0,t) = \mathrm{Re}\left[\frac{H_0(t_0)\mathrm{d}t/\mathrm{d}s}{2\pi\mathrm{i}(t-t_0)} + \frac{\mathrm{d}t/\mathrm{d}s}{2\pi\mathrm{i}}\int_0^{t_0} \frac{H(t_0,t_1)}{t-t_1}\,\mathrm{d}t_1\right] \tag{2.28}$$

Eqs. (2.25)–(2.28) are convenient to evaluate in numerical methods since the integral equations are all regular.

2.3.2. Mixed boundary conditions

Considering the following more general mixed type boundary condition,

$$p(s)\frac{\partial u}{\partial x} + q(s)\frac{\partial u}{\partial y} + r(s)u = g(s) \quad s \in \Gamma \tag{2.29}$$

and the fact that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \qquad \frac{\partial}{\partial y} = \mathrm{i}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right),$$

Eq. (2.29) can be written as

$$A(s)\frac{\partial u}{\partial z} + \overline{A}(s)\frac{\partial u}{\partial \bar{z}} + r(s)u = g(s) \tag{2.30}$$

where $A(s) = p(s) + \mathrm{i}q(s)$, $\overline{A}(s) = p(s) - \mathrm{i}q(s)$.

Substituting Eq. (2.30) into Eq. (2.8) yields,

$$\mathrm{Re}\left[L(t_0)\phi'(t_0) + M(t_0)\phi(t_0) + \int_0^{t_0} N(t_0,t_1)\phi(t_1)\,\mathrm{d}t\right] = g(t_0) \tag{2.31}$$

where $L(t_0) = A(t_0)H_0(t_0)$ and

$$M(t_0) = A(t_0)\frac{\partial H_0(t_0)}{\partial t_0} + A(t_0)H(t_0,t) + \overline{A}(t_0)\frac{\partial H_0(t_0)}{\partial \bar{t}_0} + r(t_0)H_0(t_0) \tag{2.32}$$

$$N(t_0,t_1) = A(t_0)\frac{\partial H_0(t_0,t_1)}{\partial t_0} + \overline{A}(t_0)\frac{\partial H(t_0,t_1)}{\partial \bar{t}_0} + r(t_0)H(t_0,t_1) \tag{2.33}$$

Assuming that the boundary values $\phi(z)$ and $\phi'(z)$ satisfy the Hölder condition on boundary Γ, we can use an integral representation of the following,

$$\phi(z) = \frac{1}{2\pi i}\int_\Gamma \mu(t)\ln(t-z)\,dt \quad z \in D \tag{2.34}$$

With the help of Eqs. (2.34), (2.31) can be written as,

$$\mathrm{Re}\left\{-\left(1-\frac{\bar{\alpha}}{2\pi}\right)L(t_0)\mu(t_0) - \int_\Gamma\left[\frac{L(t_0)}{t_0-t} - E_0(t_0,t)\right]\mu(t)\,dt\right\} = g(t_0) \tag{2.35}$$

where,

$$E(t_0,t) = \frac{1}{2\pi i}\int_0^{t_0} N(t_0,t)\ln(t-t_1)\,dt_1 + M(t_0)\ln(t-t_0) \tag{2.36}$$

Eqs. (2.34)–(2.36) are convenient to evaluate in numerical methods since the integral equations are all regular. The above equations can be used in the evaluation of mixed type and Neuman type boundary conditions in the RCBEM procedure.

3. Boundary integral procedure

Using the results described in Section 2, RCBEM is developed to solve the singular integral equation. The first step in the RCBEM procedure is that the boundary of the domain of the problem considered is divided into piecewise smooth elements as shown in Fig. 1. On any one of the elements considered, the boundary conditions are of the same kind, say Dirichlet, Neumann or mixed etc. Assuming t_0 as a base point (as in Fig. 1), we have

$$t - t_0 = re^{i\alpha} \tag{3.1}$$

$$dt = e^{i\theta}\,ds \tag{3.2}$$

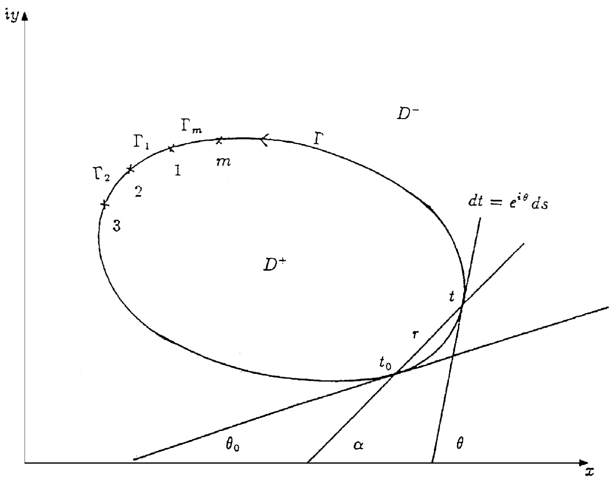

Fig. 1. Domain and boundary for complex plane.

Then the kernel of Eq. (2.27) or (2.28) can be expressed as,

$$\begin{aligned} M(t_0,t) &= \frac{1}{2\pi}\mathrm{Im}\left[\frac{H_0(t_0)\mathrm{e}^{\mathrm{i}(\theta-\alpha)}}{r} + \mathrm{e}^{\mathrm{i}\theta}\int_0^{t_0}\frac{H(t_0,t_1)}{t-t_1}\,\mathrm{d}t_1\right] \\ &= \frac{1}{2\pi}\frac{H_0(t_0)\sin(\theta-\alpha)}{r} + \frac{1}{2\pi}\mathrm{Im}\left[\mathrm{e}^{\mathrm{i}\theta}\int_0^{t_0}\frac{H(t_0,t_1)}{t-t_1}\,\mathrm{d}t_1\right] \end{aligned} \tag{3.3}$$

The first term of Eq. (3.3) is regular as t approaches to t_0 [8].

Our aim is to finally devise a method that will convert Eq. (2.27) into a series of algebraic equations. For convenience, considering a linear element, the unknowns of Eq. (2.27) can be represented as follows [9],

$$\mu(t) = \sum_{i=1}^{2} N_i\mu_i = \frac{(\mu_{j+1}-\mu_j)\xi + \xi_{j+1}\mu_j - \xi_j\mu_{j+1}}{(\xi_{j+1}-\xi_j)} \qquad \xi_j \leqslant \xi \leqslant \xi_{j+1} \tag{3.4}$$

in which ξ is the distance along the element (see Fig. 2), N_i is the shape function of the linear element and,

$$r_i = (\xi^2 + \eta_i^2)^{1/2} \tag{3.5}$$

$$\alpha_i = \tan^{-1}\left(\frac{\xi\sin\theta + \eta_i\cos\theta}{\xi\cos\theta - \eta_i\sin\theta}\right) \tag{3.6}$$

are used in the integration.

The last term of Eq. (3.3) is the imaginary part of complex variable integration, which can be evaluated using the simple trapezoidal rule, Simpson 3/8 rule or by Gaussian quadrature. Here the implementation using Gaussian quadrature is explained briefly.

In the Gaussian quadrature method, the complex variable integration is done by separating the real and imaginary parts, and integrating each separately. For example, consider the Helmholtz equation,

$$H(t_0,t_1) = -\frac{\lambda}{2}\sqrt{t_0}\,\frac{J_1\left(\lambda\sqrt{t_0(t_0-t_1)}\right)}{\sqrt{t_0-t_1}} \tag{3.7}$$

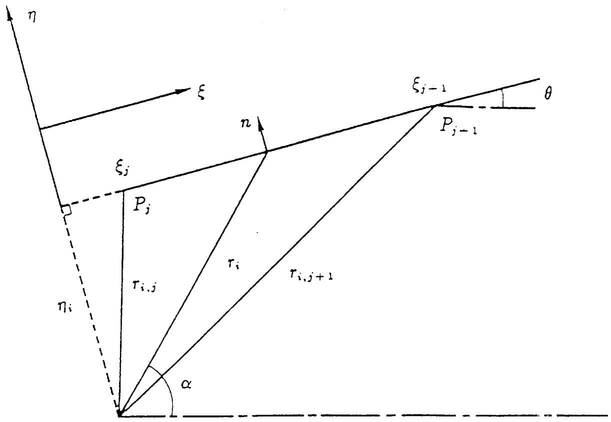

Fig. 2. Local ξ–η coordinate system for RCBEM.

Substituting Eq. (3.7) into Eq. (2.28), the complex integration can be expressed as,

$$\int_0^{t_0} \frac{J_1(\lambda\sqrt{t_0(t_0-t_1)})}{(t_j-t_1)\sqrt{t_0-t_1}}\,\mathrm{d}t_1 \quad 0 \leqslant t_1 \leqslant t_0, \quad t_0 = x_0 + \mathrm{i}y_0 \tag{3.8}$$

Since Eq. (3.8) is independent of path, a straight-line integration can be chosen between 0 and t_0.

Let

$$t_1(s) \equiv x(s) + \mathrm{i}y(s) = s + \mathrm{i}\frac{y_0}{x_0}s \quad 0 \leqslant s \leqslant x_0 \tag{3.9}$$

$$\mathrm{d}t_1 = \left(1 + \mathrm{i}\frac{y_0}{x_0}\right)\mathrm{d}s \tag{3.10}$$

Substituting Eqs. (3.9) and (3.10) into Eq. (3.8), the integral can be separated into real part and imaginary part and the Gaussian quadrature can be applied for numerical integration.

The last term of Eq. (3.3) can be written as follows after the numerical discretization for each μ_i

$$\frac{1}{2\pi}\mathrm{Im}\left[\mathrm{e}^{\mathrm{i}\theta}\int_0^{t_0}\frac{H(t_0,t_1)N_j}{t_i-t_1}\,\mathrm{d}t_1\right]\mu_j = B_{ij}\mu_j, \quad i = 1,2,\ldots N \tag{3.11}$$

The integral on the first term of Eq. (3.3) can be integrated over the element between p_j and p_{j+1} with respect to the base point p_i, by omitting the constant of $H_0(t_0)$.

$$I^{\mathrm{e}} = \int_{\xi_j}^{\xi_{j+1}} \frac{\sin(\theta_j - \alpha_i)}{r_i}\mu(\xi)\,\mathrm{d}\xi = [K^{\mathrm{e}}]\begin{pmatrix}\mu_j\\ \mu_{j+i}\end{pmatrix} \tag{3.12}$$

where

$$|K^{\mathrm{e}}| = |(K^{\mathrm{e}})_{i,j}, (K^{\mathrm{e}})_{i,j+1}| = |\xi_{j+1}I_{11} - I_{12}, I_{12} - \xi_j I_{11}| \tag{3.13}$$

in which

$$I_{11} = (\sin\theta_i I_1 - \cos\theta_j I_2)\frac{1}{2\pi}; \quad I_{12} = (\sin\theta_j I_3 - \cos\theta_j I_4)\frac{1}{2\pi} \tag{3.14}$$

$$\begin{aligned} I_1 = \frac{1}{(\xi_{j+1}-\xi_j)}\int_{\xi_j}^{\xi_{j+1}}\frac{\cos\alpha_i}{r_i}\,\mathrm{d}\xi &= \frac{1}{(\xi_{j+1}-\xi_j)}\left[\frac{1}{2}\cos\theta_j \ln\left(\frac{\xi_{j+1}^2+\eta_i^2}{\xi_j^2+\eta_i^2}\right)\right. \\ &\left. - \sin\theta_j\left(\tan^{-1}\frac{\xi_{j+1}}{\eta_i} - \tan^{-1}\frac{\xi_j}{\eta_i}\right)\right] \end{aligned} \tag{3.15}$$

$$\begin{aligned} I_2 = \frac{1}{(\xi_{j+1}-\xi_j)}\int_{\xi_j}^{\xi_{j+1}}\frac{\sin\alpha_i}{r_i}\,\mathrm{d}\xi &= \frac{1}{(\xi_{j+1}-\xi_j)}\left[\frac{1}{2}\sin\theta_j \ln\left(\frac{\xi_{j+1}^2+\eta_i^2}{\xi_j^2+\eta_i^2}\right)\right. \\ &\left. - \cos\theta_j\left(\tan^{-1}\frac{\xi_{j+1}}{\eta_i} - \tan^{-1}\frac{\xi_j}{\eta_i}\right)\right] \end{aligned} \tag{3.16}$$

$$I_3 = \frac{1}{(\xi_{j+1}-\xi_j)}\int_{\xi_j}^{\xi_{j+1}} \frac{\cos\alpha_i}{r_i}\xi\,\mathrm{d}\xi$$
$$= \frac{1}{(\xi_{j+1}-\xi_j)}\left\{\cos\theta_j\left[(\xi_{j+1}-\xi_j)-\eta_i\left(\tan^{-1}\frac{\xi_{j+1}}{\eta_i}-\tan^{-1}\frac{\xi_j}{\eta_i}\right)\right]-\frac{\eta_j}{2}\sin\theta_j\ln\left(\frac{\xi_{j+1}^2+\eta_i^2}{\xi_j^2+\eta_i^2}\right)\right\} \tag{3.17}$$

$$I_4 = \frac{1}{(\xi_{j+1}-\xi_j)}\int_{\xi_j}^{\xi_{j+1}} \frac{\sin\alpha_i}{r_i}\xi\,\mathrm{d}\xi$$
$$= \frac{1}{(\xi_{j+1}-\xi_j)}\left\{\sin\theta_j\left[(\xi_{j+1}-\xi_j)-\eta_i\left(\tan^{-1}\frac{\xi_{j+1}}{\eta_i}-\tan^{-1}\frac{\xi_j}{\eta_i}\right)\right]+\frac{\eta_i}{2}\cos\theta_j\ln\left(\frac{\xi_{j+1}^2+\eta_i^2}{\xi_j^2+\eta_i^2}\right)\right\} \tag{3.18}$$

Finally, the algebraic Eq. (2.27) can be expressed as,

$$\sum_{j=1}^{N}(A_{ij}+B_{ij})\mu_j = g_i \quad i = 1,2,\ldots N \tag{3.19}$$

where

$$A_{ij} = \left|(K^{\mathrm{e}})_{ij} + \left(1-\frac{\bar{\alpha}}{2\pi}\right)\delta_{ij}\right| \qquad \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} \tag{3.20}$$

Eq. (3.20) is used only in the case of interior problems. For the exterior problems, Eq. (3.20) is written as,

$$A_{ij} = \left|(K^{\mathrm{e}})_{ij} - \frac{\bar{\alpha}}{2\pi}\delta_{ij}\right| \qquad \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} \tag{3.21}$$

It should be noted that for exterior problem, the normal vector is in opposite direction.

4. Numerical applications

To test the feasibility and accuracy of the RCBEM model, here three numerical applications are presented. Initially, we consider the potential flow problems in which interior and exterior problems are considered. Secondly, a fluid flow problem governed by the Helmholtz equation is considered and finally a steady-state advection–diffusion problem is solved by transforming the governing equation into the modified Helmholtz equation. The RCBEM results in all the cases are compared with analytical and other numerical solutions.

4.1. Potential flow problems

The governing equation for the potential flow problems is the Laplace equation. Here the potential flows are considered with interior problems and exterior problems. In the case of po-

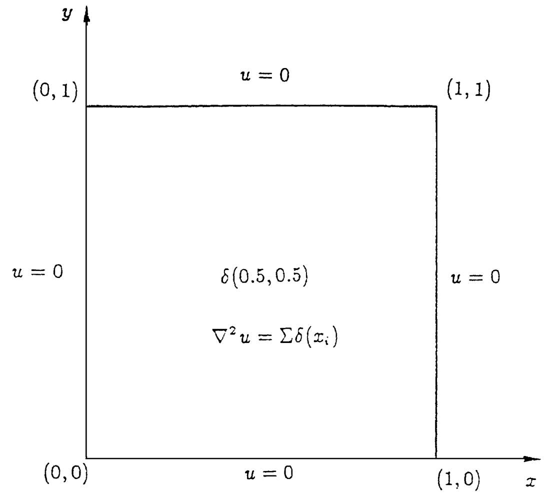

Fig. 3. Study domain and boundary conditions for the groundwater flow problem.

tential flow problems, we consider the solution of two problems using the RCBEM. Initially an interior problem of groundwater flow is considered and then an exterior problem of source outside a circular cylinder is considered.

4.1.1. Groundwater flow problem

Here we consider a two-dimensional groundwater flow problem in a homogeneous, isotropic porous media with a pumping well. The problem is a bounded square region, $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$ with zero potential on all the sides as shown in Fig. 3. There is a pumping well at the middle of the domain with unit discharge. In the RCBEM model, the boundary of the domain is discretized with 40 linear elements. Fig. 4 shows the potential variation with respect to the x-axis at $y = 0.5$ from the well position. The results are compared with exact solution and a BEM model [2]. Good agreement is observed between the results. A sensitivity study with different mesh discretization showed that RCBEM yields comparable results with other models even with 8 linear boundary elements. This case study shows the feasibility of the RCBEM model for the interior type problems.

4.1.2. Exterior flow problem

Here we consider the two-dimensional flow field past a circular cylinder of radius $r = 1$ due to a source at $x = 2$ with strength $m = 2\pi$. The aim is to find the stream function distribution over the circular cylinder due to the source effects. The problem domain with discretization is shown in Fig. 5. The boundary conditions of the stream function along the cylinder are assumed to be zero. Since the problem is exterior in nature, a sensitivity study of the mesh showed that a finer discretization is necessary [2]. In the RCBEM model, the total boundary of the domain is discretized with 160 linear elements. The problem can be simplified by considering the symmetry and discretizing half of the domain.

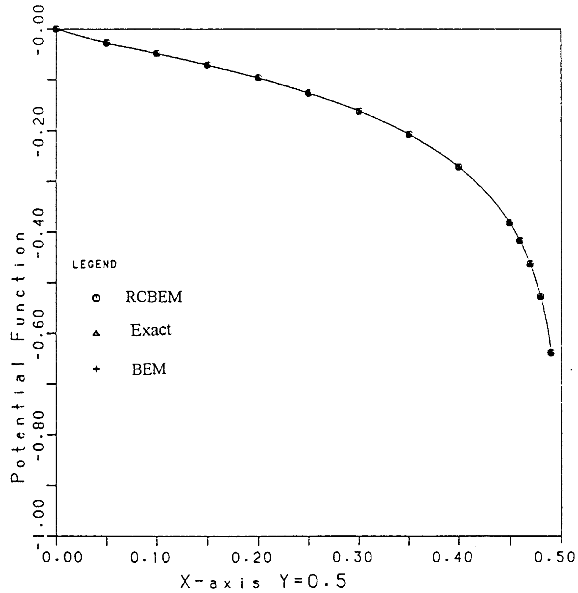

Fig. 4. Comparison of potential for groundwater flow problem.

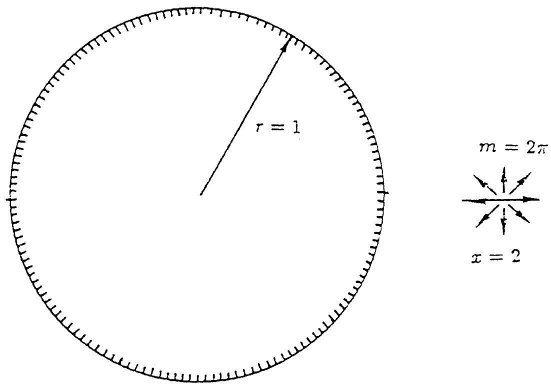

Fig. 5. Study domain for the exterior potential flow problem with source outside a circular cylinder.

In this problem, the influence of the point source can be directly taken into account or it can be separated into the perturbed part and the potential flow part and the solution can be combined [2]. In the present analysis, as the problem considered is linear in nature, it is convenient to separate the stream function (ψ) into two parts, $\psi = \psi_U + \psi_P$, where ψ_U defines the stream function due to potential flow and ψ_P is a perturbed stream function owing to the influence of the source. Fig. 6 shows the stream function variation around the circular cylinder. The results are compared with exact solution. Good agreement is observed between the results. This case study shows the feasibility of the RCBEM model for the exterior type problems.

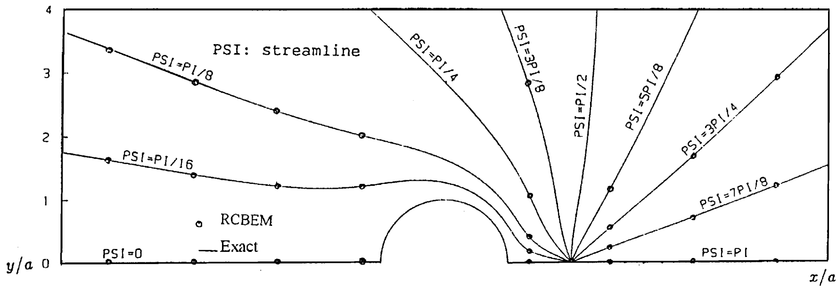

Fig. 6. Streamlines for the exterior potential flow problem: comparison of RCBEM and exact solution.

4.2. Helmholtz equation problem

The governing equation of the reduced wave problem is the Helmholtz equation. To illustrate the application of the Helmholtz equation problem, here we consider an interior problem in which the potential is propagated from a point source of unit strength. For the demonstration purpose, the value of λ is assumed as 1. The problem domain with discretization is shown in Fig. 7. The point source is located at the center of the domain. All over the boundary of the domain, a Dirichlet boundary condition $u(1,\theta) = 1$ is assumed. The boundary of the domain is discretized with 120 elements. The exact solution for the problem considered is given by,

$$u = \frac{J_0(\lambda r)}{J_0(\lambda a)} \tag{4.1}$$

Fig. 8 shows the propagation of potential for $a = 1$ along the axis using the RCBEM. The results are compared with the exact solution and a BEM model [2]. The results are almost identical. A sensitivity study with different mesh discretization showed that RCBEM yields

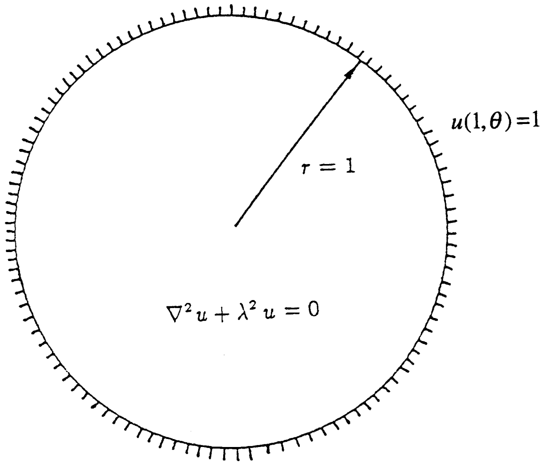

Fig. 7. Study domain and boundary conditions for the Helmholtz equation problem.

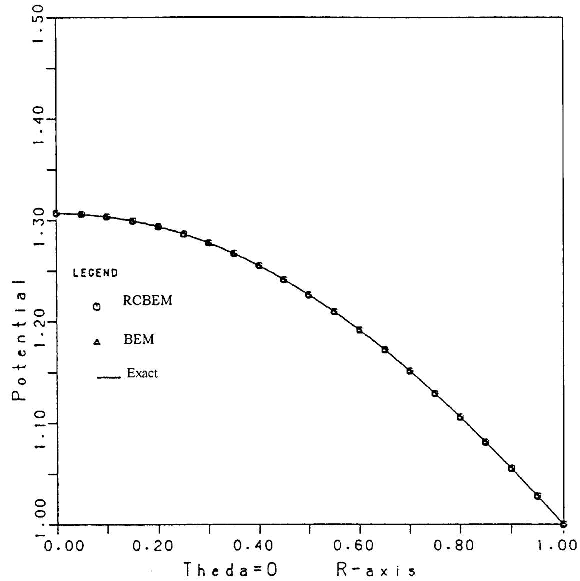

Fig. 8. Comparison of potential for the Helmholtz equation problem.

comparable results with other models even with 40 linear boundary elements. This case study shows the feasibility of the RCBEM model for the Helmholtz equation type problems.

4.3. Advection–diffusion problem

Here the steady-state advection–diffusion problem is solved, by converting the governing equation into the modified Helmholtz equation. The governing equation of steady-state advection–diffusion problem is,

$$\nabla \cdot ([k]\nabla u) - (\overline{V} \cdot \nabla)u = f \tag{4.2}$$

where k is the diffusivity coefficient, $\overline{V}$ is the convective velocity and f is the source or sink term. The governing equation is made dimensionless using the following variables,

$$\bar{x}^* = \frac{\bar{x}}{L}, \quad \nabla^* = L\nabla, \quad [k^*] = \frac{[k]}{K}, \quad \overline{V}^* = \frac{\overline{V}}{U}, \quad u^* = \frac{u}{u_0} \tag{4.3}$$

where L is the characteristic length, K is the characteristic diffusivity and U is the characteristic velocity. Assuming $f = 0$, the non-dimensional form of Eq. (4.2) can be written as,

$$\nabla^* \cdot ([k^*]\nabla^* u^*) - Pe(\overline{V}^* \cdot \nabla^*)u^* = 0 \tag{4.4}$$

where $Pe = UL/K$ is the Peclet number. Using the following transformation [10],

$$u^{**} = \mathrm{e}^{-g}u^*; \quad \nabla^{**}g = \frac{1}{2}Pe[k^*]^{-1}\overline{V} \tag{4.5}$$

into Eq. (4.4), we can obtain the transformed modified Helmholtz equation as the following equation after omitting the superscript $**$ for brevity,

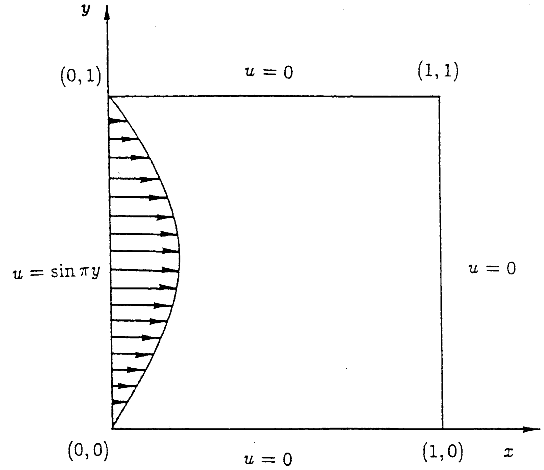

Fig. 9. Study domain and boundary conditions for the advection–diffusion problem.

$$\nabla \cdot ([k]\nabla u) - \lambda^2 u = 0 \tag{4.6}$$

where

$$\lambda^2 = \frac{1}{4} Pe^2 \overline{V}[k]^{-1}\overline{V} + \frac{1}{2} Pe \nabla \cdot \overline{V} \tag{4.7}$$

There are two limitations for this transformation viz. λ^2 have to be positive and the flow field should be irrotational. Here we consider an advection–diffusion problem in two-dimensional domain. The problem is a bounded square domain. A boundary condition with $\sin(\pi y)$ concentration profile is imposed on the boundary $(0, y)$ and zero concentration is assumed on all other boundaries. Fig. 9 depicts the studied domain with associated boundary conditions. The RCBEM simulates the flow field with constant $V_x = 1$, $V_y = 0$, $[k] = [I]$, and different Peclet numbers from diffusion (low Pe) to advection (high Pe) dominated flows. The boundary of the domain is discretized into 40 linear elements. The computations are carried out for Peclet numbers of 0, 1, 10, 40 and 80. For this problem, an exact solution can be derived as,

$$u = \frac{\sin \pi y}{e^a - e^b}\left[e^{a+bx} - e^{ax+b}\right] \tag{4.8}$$

where

$$a = \frac{Pe + \sqrt{4\pi^2 + Pe^2}}{2}, \qquad b = \frac{Pe - \sqrt{4\pi^2 + Pe^2}}{2} \tag{4.9}$$

Using RCBEM, for all the Peclet numbers, very stable and comparable results with analytical solutions are obtained. Fig. 10 shows the concentration along $(x, 0.5)$ compared with exact solution for various Peclet numbers. Good agreement is observed between the solutions and the results show that RCBEM is stable for higher Peclet number problems, which are more difficult and challengable to other numerical methods. This case study shows the effectiveness of the RCBEM model for the two-dimensional steady-state advection–diffusion problems.

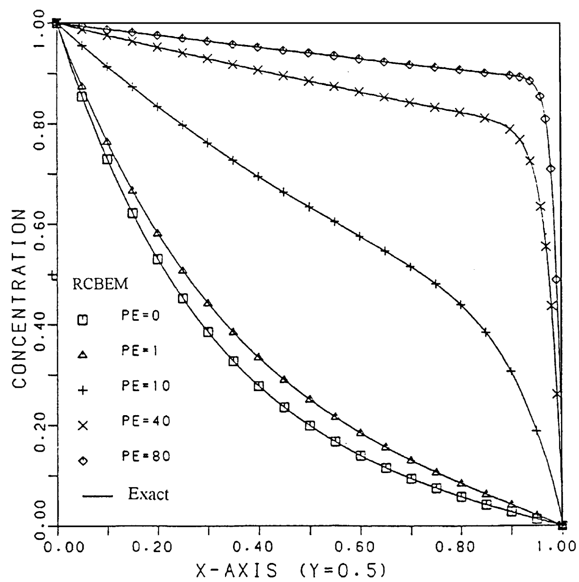

Fig. 10. Comparison of concentration for the advection–diffusion problem.

5. Concluding remarks

A new boundary integral equation model, RCBEM, is proposed based on the (BEM) and the theory of Vekua and its modification as well as complex Riemann function as the fundamental solution. The RCBEM method is used to solve the linear, second order, elliptical partial differential equations in the fluid flow problems. Compared to the generally used BEM, RCBEM applies complex Riemann function as the solution of the adjoint operator and the governing equations should be transformed into complex domain. The singular problem occurring in direct BEM can be avoided in RCBEM, especially for regular domain problems. The efficiency and accuracy of the RCBEM depends on the complex variable integration and the presented methodology applies only to two-dimensional problems. The simulations of various numerical examples presented demonstrated the accuracy and feasibility of RCBEM model.

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