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Solution of Stokes Flow Using an Iterative DRBEM Based on Compactly-Supported, Positive-Definite Radial Basis Function

D. L. YOUNG*, C. C. TSAI AND T. I. ELDHO
Department of Civil Engineering & Hydrotech Research Institute
National Taiwan University, Taipei, Taiwan 10617, R.O.C.
dlyoung@hy.ntu.edu.tw

A. H.-D. CHENG
Department of Civil and Environmental Engineering
University of Delaware, Newark, DE 19716, U.S.A.

Abstract—This paper describes an iterative dual reciprocity boundary element method (DRBEM) based on the compactly-supported, positive definite radial basis function for the solution of Stokes flow problems. The method involves the solution of Laplace equations for vorticity, and Poisson equations for velocity with the solenoidal components of vorticity as right-hand sides. These equations are solved using iterative BEM and DRBEM. For the DRBEM, the compactly supported, positive definite radial basis function is used to approximate the body force term to convert it to boundary integrals. A relaxation method is devised such that the resulting algebraic equations are solved without the need of assembling a matrix. Hence, large systems of equations such as the fine discretization needed for Stokes flow problems can be handled easily. The results of Stokes flow in a square cavity and a circular cavity are presented and compared with other numerical model results. The iterative DRBEM solution is found to be satisfactory. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Velocity-vorticity formulation, Stokes flow, Iterative solution, Dual reciprocity boundary element method, Radial basis function.

1. INTRODUCTION

The incompressible viscous flows in slow motion, known as Stokes flows, have many important industrial applications. Stokes flows can be considered as a subset of Navier-Stokes flows, whose nonlinear convective terms are negligibly small. For the solution of Stokes flows, three formulations are well known: they are in terms of primitive variables of pressure and velocity, stream function and vorticity, and velocity and vorticity. For the solution of steady Stokes flows using the velocity-vorticity formulation, the governing equations have been written as a system of Laplace and Poisson-type equations for the components of vorticity and velocity fields, respectively. The main advantage of this formulation lies in the numerical separation of the kinematic and kinetic aspects of fluid flow from the pressure computation, which is determined afterwards from the known velocity and vorticity fields.

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Early boundary solutions of Stokes flows were based on the primitive variable formulation by distributing stokeslets on the solution boundary [1]. This type of formulation, which leads to Fredholm integral equations of the first kind, is mathematically ill posed. As a remedy, formulation based on double layer potential, which leads to Fredholm integral equations of the second kind, has been proposed [2–4]. Another well-posed formulation using the primitive variables is based on the Somigliana integral [5,6]. A boundary element method (BEM) has also been developed for the stream function and vorticity formulation [7,8]. The velocity-vorticity formulation, however, is rarely used in conjunction with BEM due to the inhomogeneous right-hand side, which requires a domain type discretization.

The dual reciprocity boundary element method (DRBEM) has been established as a powerful numerical tool in the solution of various fluid flow problems [9,10]. DRBEM extracts an operator that is easy to handle to the left-hand side. The right-hand side is then approximated by a radial basis function interpolation [9]. By finding the particular solution corresponding to the radial basis functions, the need for domain integration is eliminated; hence, the advantages of the boundary method are preserved. The DRBEM can be used in the solution of incompressible viscous flow problems [10]. In this paper, we shall use the DRBEM to solve Stokes flow problems using the velocity-vorticity formulation.

In solving large-scale incompressible viscous flow problems, a large number of discrete unknowns are required to accurately represent the geometry and the solution variation. The same as BEM schemes, DRBEM produces fully populated solution matrices. When the matrix is assembled, its size can be so large that its storage and solution by direct elimination can cause problems for an ordinary computer. Consequently, the matrix size becomes the limiting factor that large problems can be solved with a particular computer. With this type of problem in mind, a few researchers recently developed iterative BEM schemes for various engineering problems [11–13]. In those implementations, matrices were assembled, and iterative techniques were used to solve the matrices.

To further remove the constraint in the storage of an assembled matrix, an iterative procedure that does not assemble the matrix, and thus, resembles the relaxation method of the finite difference method, has been devised. The first attempt in this direction is attributed to Cahan *et al.* [14] who used the direct BEM in the solution of Laplace equation. Later, an iterative BEM based on the indirect formulation was devised for solving stochastic boundary value problems [15]. Recently, Cheng *et al.* [16] formulated an iterative DRBEM for the solution of Poisson equations using compactly-supported positive-definite radial basis functions.

In this paper, we devise a scheme for solving the Stokes flow problems using the velocity-vorticity formulation. The compactly-supported positive-definite radial basis functions are used to interpolate the right-hand side of the Poisson equation. The solution by DRBEM produces the vorticity boundary condition needed in the solution of the Laplace equation. For the whole solution procedure, matrix assemblage is avoided and an iterative scheme is used to solve the linear system. To demonstrate the feasibility and accuracy of the proposed iterative DRBEM procedure, we solve the well-known benchmark problems of flow in a driven square cavity and a circular cavity.

The current work sets the stage for solving unsteady and three-dimensional Navier-Stokes equations. The time dependent vorticity transport equation can be solved by finite differencing in the time domain. The nonlinear convection terms need to be taken as sources, and approximated by the radial basis function interpolation. Other approaches, such as the Eulerian-Lagrangian BEM method [17], can also be adopted.

2. GOVERNING EQUATIONS

The governing equations of Stokes flow for the velocity-vorticity formulation in steady-state can be derived from the Navier-Stokes equations [18] and written as

$$\nabla^2 \bar{\omega} = 0, \tag{1}$$

$$\nabla^2 \bar{u} = -\nabla \times \bar{\omega}, \tag{2}$$

where $\bar{u}$ is the velocity vector, and $\bar{\omega}$ the vorticity vector. The vorticity vector $\bar{\omega}$ can be expressed as

$$\bar{\omega} = \nabla \times \bar{u}. \tag{3}$$

In two dimensions, if (u, v) are the velocity vectors and ω is the associated vorticity vector, then the vorticity transport equation (1) reduces to the scalar form

$$\nabla^2 \omega = 0, \tag{4}$$

and equation (2) can be written as

$$\nabla^2 u = -\frac{\partial \omega}{\partial y}, \tag{5}$$

$$\nabla^2 v = \frac{\partial \omega}{\partial x}. \tag{6}$$

In steady-state conditions, the solution of equation (4), in conjunction with the Poisson equations (5) and (6), gives the velocity and vorticity distribution all over the domain.

3. NUMERICAL FORMULATION

3.1. DRBEM Formulation

Consider a Poisson equation

$$\nabla^2 u = b(\bar{x}), \tag{7}$$

where $b(\bar{x}) = -\frac{\partial \omega}{\partial y}$ corresponds to the right-hand side of (5), with the boundary conditions

$$u = u_\Gamma, \quad \text{on } \Gamma_1, \qquad \frac{\partial u}{\partial n} = q_r, \quad \text{on } \Gamma_2, \tag{8}$$

where n is the unit outward normal vector. In the current iterative algorithm, the right-hand side of equation (7) is known from the previous step by solving the vorticity transport equation (4).

In the standard DRBEM formulation, the right-hand side of equation (7) is approximated by a series of basis functions [9,19]

$$b(\bar{x}) = \sum_{i=1}^{n_p} \beta_i p_i(\bar{x}) + \sum_{i=1}^{n_r} \alpha_i \phi(r_i), \tag{9}$$

where $\phi(r)$ is a radial basis function, $r_i = \|\bar{x}_i - \bar{x}\|$ is the Euclidean distance between a field point $\bar{x}$ and the i^{th} collocation point $\bar{x}_i$, $p_i(\bar{x})$ is a multivariate monomial, n_r is the number of collocation nodes, n_p is the number of monomial terms, and α_i and β_i are the unknown coefficients to be determined by collocation and constraint equations prior to the BEM implementation. The collocation nodes are distributed on the boundary as well as in the interior of the domain.

Now using the DRBEM principle [9] and a radial basis function interpolation, a boundary only representation of equation (7) can be written as

$$C_i u_i + \int_r q^* u \, d\Gamma - \int_\Gamma u^* q \, d\Gamma = \sum_{j=1}^{N+L} \alpha_j \left(C_i \Phi_{ij} + \int_\Gamma q^* \Phi_j \, d\Gamma - \int_\Gamma u^* \hat{q}_j \, d\Gamma \right), \tag{10}$$

where Γ is the solution boundary, u^* is the free-space Green's function which in two dimensions is $u^* = -\ln(r)/2\pi$, and C_i is the Green's constant. The term $\hat{q}_j$ is defined as $\frac{\partial \Phi_j}{\partial n}$, q as $\frac{\partial u}{\partial n}$, and q^* as $\frac{\partial u^*}{\partial n}$. The function Φ represents the particular solution of a Laplacian operator given by

$$\nabla^2 \Phi = \phi(r). \tag{11}$$

Note that equation (10) does not contain a domain integral. The source term b in equation (7) has been substituted by equivalent boundary integrals.

Equation (10) is in a natural form for an iteration procedure. As in the case of relaxation methods, a set of initial trial values can be assigned to all boundary nodes. Then using equation (10), by placing the base point on a boundary node, where u is unknown, and performing integration, an updated value of u at the concerned node can be found. We can then move on to the next node and repeat the procedure until all unknowns are found. In the same way, the iterative DRBEM procedure can be applied to the other velocity Poisson equation (6).

For the solution of the Laplace equation (4), the regular BEM [20] can be used. The boundary integral equation is written as

$$C_i \omega_i + \int_\Gamma q^* \omega \, d\Gamma - \int_\Gamma u^* q \, d\Gamma = 0, \tag{12}$$

where q is defined as $\frac{\partial \omega}{\partial n}$.

3.2. Radial Basis Functions

As indicated earlier, the set of collocation equation (9), together with a set of constraint equations, form a linear system that can be solved for the coefficients α_i and β_i. To avoid the assemblage of the matrix, we practice an iterative procedure that instantly generates the matrix elements and then discards them without storage. Due to their simple form, the computational effort involved in the repeated generation of matrix elements is minimal.

As reported in [16], in the iterative scheme using the Gauss-Seidel or the conjugate gradient method, the linear system created with standard radial basis functions (RBFs), such as spline or conical functions, fails to convergence. As commented in [21], the spline or conical type RBFs are globally defined. The resulting interpolation matrix is dense and can be ill conditioned. Indeed, in our tests the iterative solution always fails to converge. Cheng *et al.* [16] have demonstrated that the difficulty in convergence can be overcome by the use of compactly supported, positive definite RBFs (CS-PD-RBF) presented by Wendland [22]. This kind of radial basis function results in a positive definite matrix; its inverse is guaranteed. We find that the iterative scheme is stable with these matrices.

Wendland [22] showed that a positive definite radial basis function in the form of a univariable polynomial of minimal degree always exists for a given dimension d and smoothness C^{2k}, and is unique within a constant factor. For the present study, the following two CS-PD-RBFs are used [22]: for $d = 3$ and $k = 0$,

$$\begin{aligned} \phi(r) &= \left(1 - \frac{r}{a}\right)^2, && \text{for } 0 \le r \le a, \\ \phi(r) &= 0, && \text{for } r > a, \end{aligned} \tag{13}$$

and for $d = 3$ and $k = 1$,

$$\begin{aligned} \phi(r) &= \left(1 - \frac{r}{a}\right)^4 \left(1 + \frac{4r}{a}\right), && \text{for } 0 \le r \le a, \\ \phi(r) &= 0, && \text{for } r > a, \end{aligned} \tag{14}$$

where a is the influence radius beyond which the function is truncated to zero. In the above equations, although $d = 3$ is used, the corresponding CS-PD-RBFs are valid for any lower dimensions, and hence, can be used for the current two-dimensional problems. The influence radius controls the density of the matrix. If a is larger than the largest span of the domain, the matrix is fully populated; if a is smaller than the smallest distance between two collocation nodes, the matrix becomes diagonal. In practice, there exists a trade-off: a smaller a will result in a sparser matrix, which is more efficient in an iterative scheme, but the interpolation is less accurate. For optimal efficiency and accuracy, a multilevel scheme should be practiced [23]. This is, however, not attempted in this paper.

The interpolation equation for CS-PD-RBF is given by

$$b(\bar{x}) = \sum_{i=1}^{n_r} \alpha_i \phi(r_i). \tag{15}$$

The coefficients α_i are determined from the collocation equations

$$\sum_{i=1}^{n_i} \alpha_i \phi(r_{ij}) = b(\bar{x}_j), \qquad j = 1, 2, \dots, n_r. \tag{16}$$

The particular solutions corresponding to (13) satisfying (11) for the DRBEM implementation can be written as [16,21]

$$\begin{aligned} \Phi &= \frac{r^2}{4} - \frac{2r^3}{9a} + \frac{r^4}{16a^2}, && \text{for } 0 \le r \le a, \\ \Phi &= \frac{13a^2}{144} + \frac{a^2}{12} \ln\left(\frac{r}{a}\right), && \text{for } r > a. \end{aligned} \tag{17}$$

For (14), the corresponding particular solution is

$$\begin{aligned} \Phi &= \frac{r^2}{4} - \frac{5r^4}{8a^2} + \frac{4r^5}{5a^3} - \frac{5r^6}{12a^4} + \frac{4r^7}{49a^5}, && \text{for } 0 \le r \le a, \\ \Phi &= \frac{529a^2}{5880} + \frac{a^2}{14} \ln\left(\frac{r}{a}\right), && \text{for } r > a. \end{aligned} \tag{18}$$

3.3. Iterative Scheme

There are two parts of iteration involved. The first part is the BEM part resulting from the discretization of equations (10) and (12), and the second part is the RBF part shown in equation (16). Here, only the iteration involving the BEM equations will be discussed. The solution of (16) by conjugate gradient method has been discussed in [16].

Equations (10) and (12) can be written in discretized form using N nodes, as

$$\sum_{k=1}^{N} a_{jk} u^k - \sum_{k=1}^{N} b_{jk} u_n^k = c_j, \qquad j = 1, \dots, N, \tag{19}$$

where u^k is the discrete value of u at node k, and u_n^k is the discrete value of $\frac{\partial u}{\partial n}$ at node k, and coefficients a_{jk}, b_{jk}, and c_j are constants obtained by integration over the elements. After substituting the boundary conditions, the unknown variables $\frac{\partial u}{\partial n}$ and u, respectively, can be found from the following formulae:

$$u_n^j = \frac{1}{b_{jj}} \left[-c_j + \sum_{k=1}^{N} a_{jk} u^k - \sum_{\substack{k=1 \\ k \ne j}}^{N} b_{jk} u_n^k \right], \qquad \text{if } u^j \text{ is given}, \tag{20}$$

$$u^j = \frac{1}{a_{jj}} \left[c_j - \sum_{\substack{k=1 \\ k \ne j}}^{N} a_{jk} u^k + \sum_{k=1}^{N} b_{jk} u_n^k \right], \qquad \text{if } u_n^j \text{ is given}. \tag{21}$$

The above equations form the basis for the present iterative scheme. As in the case of the Gauss-Seidel scheme, here an updated datum can be used in the evaluation of the next datum.

Comparing to a typical finite difference relaxation scheme, in which the nodal value is updated from only the four neighboring nodes, solution of the current scheme is dependent on all nodes. We expect the correction of solution to propagate faster and less attenuated. In the evaluation of a nodal solution, half of the data are already known from the boundary conditions; hence, they have a stabilizing effect. In most cases, even an all-zero initial trial value assignment can bring the solution to a reasonable convergence within a few iterations. In comparison, an interior node of the finite difference scheme can access the boundary data only in an indirect and attenuated way through propagation. In conclusion, the iterative solution based on the boundary integral equation is expected to be stable and efficient.

In a noniterative solution of the BEM, the main matrix representing (19) is assembled, which is of the size $N \times N$. In the present scheme, the main matrix is not assembled, and hence, only arrays of the size N exist. To accomplish this, elements of the main matrix are calculated on the fly and immediately discarded. The unchanged integral quantities on the right-hand side of equation (10) are evaluated and stored in an array of size N. The shortcoming of this scheme, however, is the repeated evaluation of the matrix elements. To alleviate this additional computational cost, low order elements, such as constant or linear elements for which exact integration is available, are used in the present study.

4. SOLUTION PROCEDURE

The iterative BEM procedure is applied to the velocity-vorticity formulation of Stokes flow. In most incompressible viscous flow problems, the natural boundary condition is the prescribed velocity. The vorticity boundary conditions are seldom known and should be determined from the computation. In the present model, the velocity Poisson equations are first solved to obtain the vorticity boundary conditions that are used in the solution of the vorticity Laplace equation. The solution procedure is described in the following steps.

1. Solve the velocity Poisson equations (5) and (6) using the iterative DRBEM.
2. Calculate the unknown boundary velocity or velocity flux values.
3. Calculate the velocity distribution and velocity derivatives at all nodal points.
4. Determine new vorticity boundary values using the definition of vorticity in equation (3).
5. Solve the vorticity Laplace equation (4) using iterative BEM.
6. Calculate the unknown vorticity values throughout the domain.
7. Calculate the derivatives of the vorticity vectors to be used in the velocity Poisson equations.
8. Check for the convergence of the velocity and vorticity components in the present iteration, for example,
$$\frac{\left|\omega_n^{k+1} - \omega_n^k\right|}{\left|\omega_n^k\right|} \leq \text{tolerance}. \tag{22}$$
9. If the convergence criterion is satisfied, then stop; otherwise go to Step 1.

5. NUMERICAL EXAMPLES

The present iterative DRBEM scheme has been applied on two test problems to verify the feasibility and accuracy of the method. The test problems are the classical driven flow in a square cavity and flow in a circular cavity problems for which analytical and numerical results are available in the literature.

The iterative solution of the RBF collocation equations requires an initialization of the solution. It is noticed that the diagonal terms of the CS-PD-RBF collocation matrix are all equal to unity; hence, the initial trial values are assigned as
$$\alpha_i = b(\bar{x}_i), \qquad i = 1, 2, \ldots, n_r. \tag{23}$$

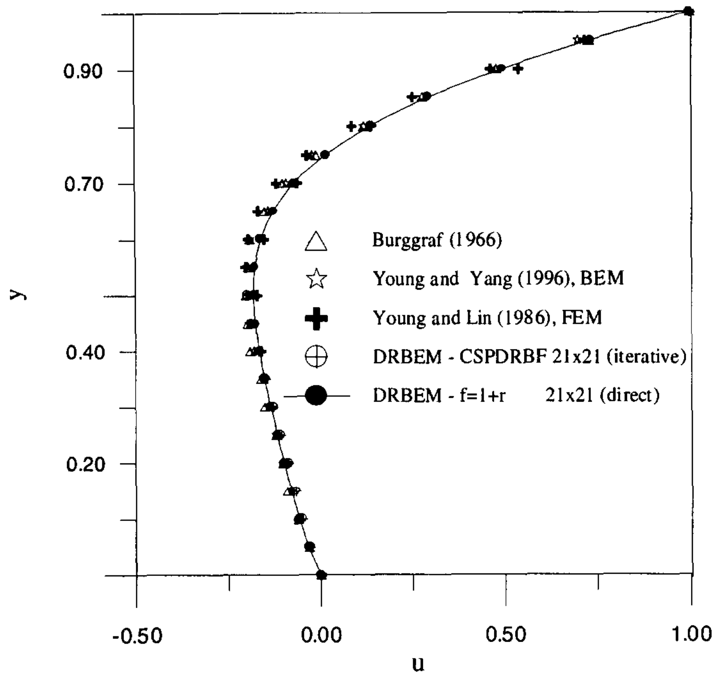

Figure 1. Velocity profile (u) on the centerline at $x = 0.5$ of a square cavity.

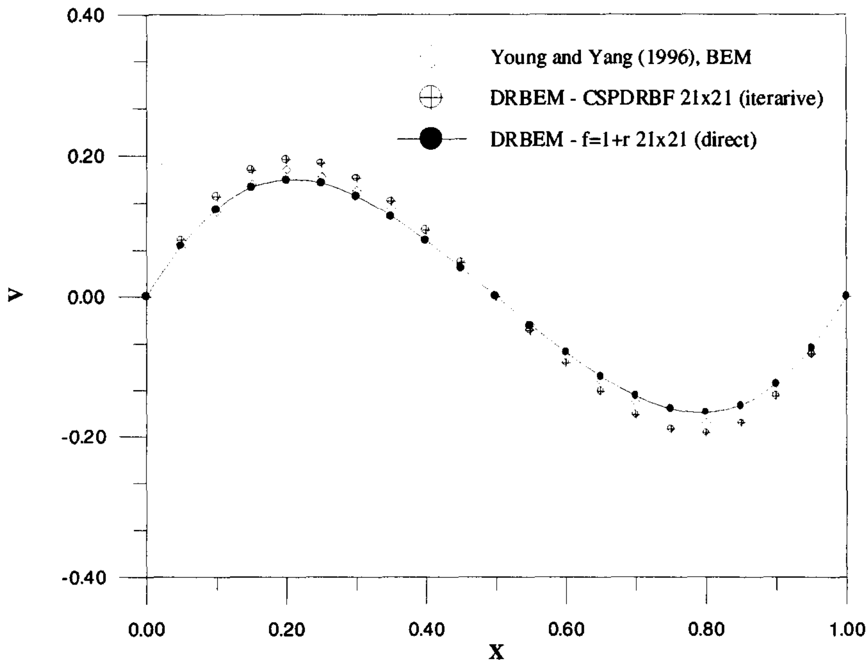

Figure 2. Velocity profile (v) on the centerline at $y = 0.5$ of a square cavity.

For the present two-dimensional problem, the size of the linear system is not intolerably large; hence, the influence radius a is chosen as 2, covering the entire solution domain. (See [16] for the effect of influence radius on the rate of convergence.) The first-order CS-PD-RBF defined in equation (13) is used in the present analysis. Constant elements are used for the boundary element discretization.

The solution converges to a relative tolerance of 10^{-3} within 10 iterations, and hence, the scheme is highly efficient. Further numerical investigations with finer meshes with a tolerance

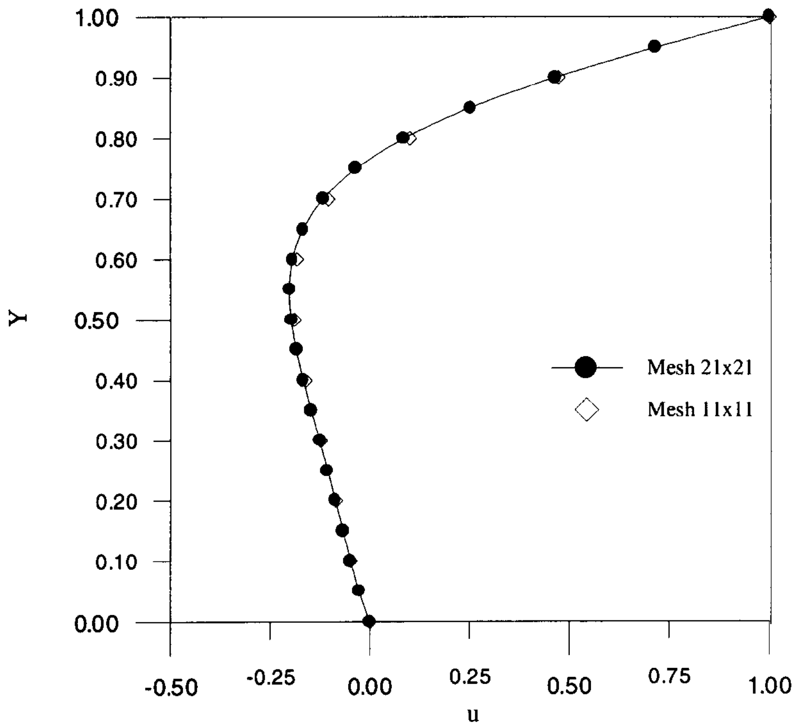

Figure 3. Velocity profile (u) on the centerline at $x = 0.5$ of a square cavity; comparison of mesh convergence.

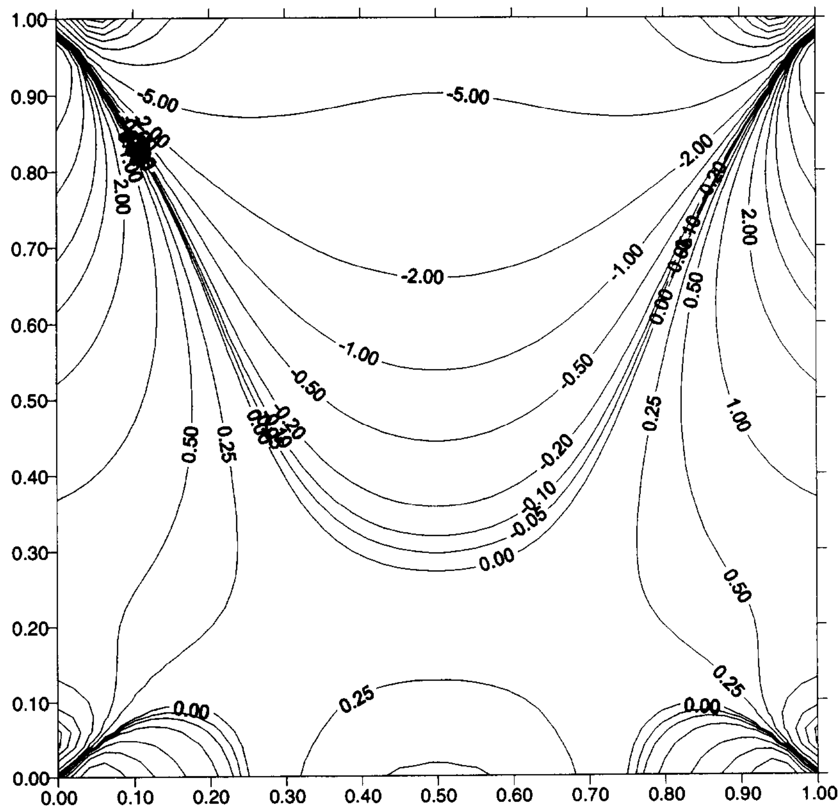

Figure 4. Vorticity distribution for Stokes flow in a square cavity.

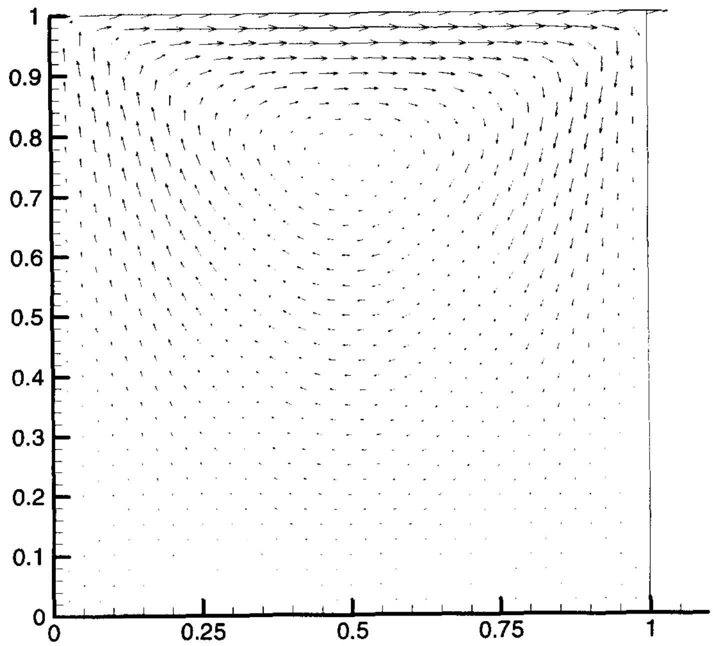

Figure 5. Velocity vectors for Stokes flow in a square cavity.

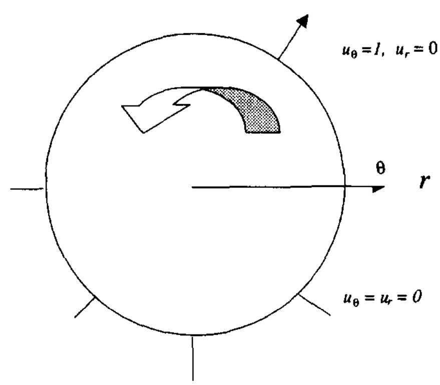

Figure 6. Circular cavity flow problem with boundary conditions.

of 10^{-3} show that the number of iterations needed for convergence is approximately related to the number of nodes N as follows:

$$\text{iteration number} \sim N^{1/2}. \tag{24}$$

This convergence rate is the same as the successive-over-relaxation (SOR) method in the finite difference method [24]. Detailed investigation regarding the convergence, influence radius, and other behaviors of the CS-PD-RBF can be found in [16].

EXAMPLE 1. The first model problem consists of a square cavity filled with an incompressible viscous fluid with a top lid moving at constant velocity. The Reynolds number is low such that the induced motion can be described as a Stokes flow. No slip and impervious conditions are imposed on all walls, with the velocity at the upper wall set to unity. Most results presented below are based on a computational mesh of 21×21 for RBF collocation, and 40 boundary nodes for BEM mesh.

Figure 1 shows the x-component velocity profile (u) on the vertical centerline of the cavity, and Figure 2 shows the y-component velocity profile (v) on the horizontal centerline for steady-state Stokes flow on a 21×21 mesh. The results are compared with a series solution [25], an FEM

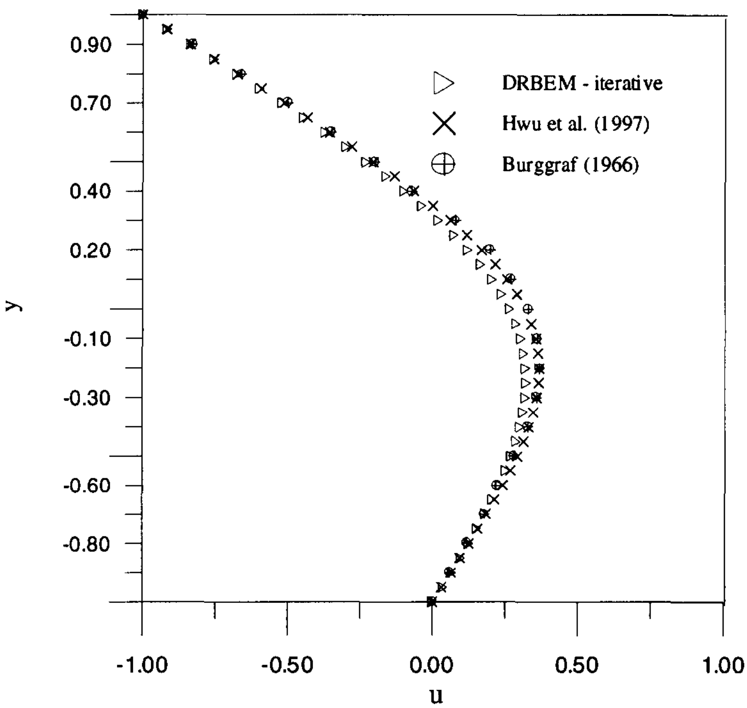

Figure 7. Comparison of u-velocity profile at $x = 0$ for a circular cavity.

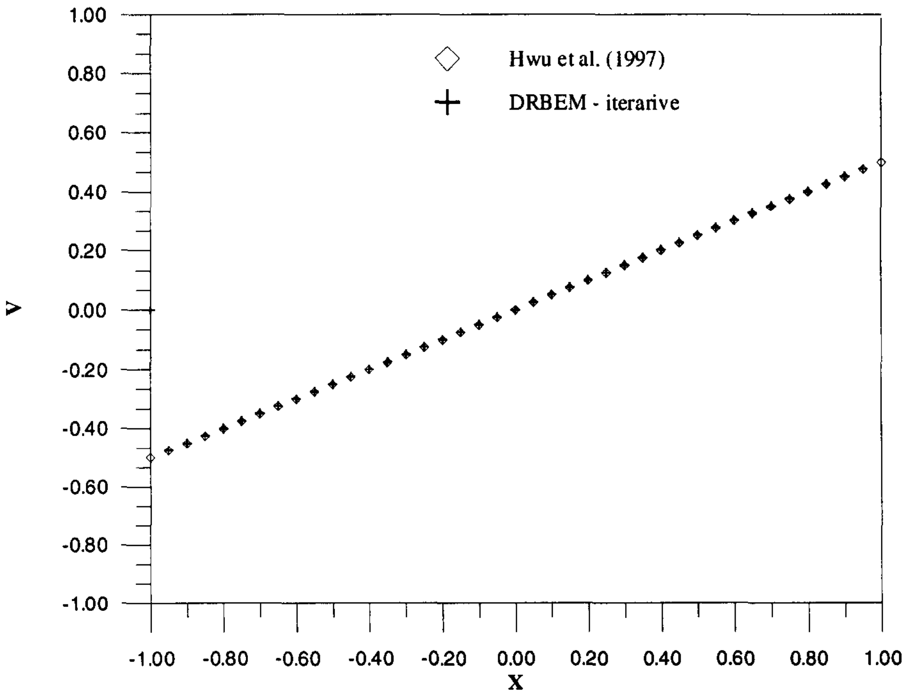

Figure 8. Comparison of v-velocity profile at $y = 0$ for a circular cavity.

solution [26], a BEM solution [27], and a DRBEM (direct method) solution [28] in Figure 1. As can be seen in Figure 1, the present iterative DRBEM scheme is more accurate than the regular DRBEM scheme [28] using the conical RBF when compared with the series solution [25].

As a convergence test for the grid, analysis has been performed for uniform meshes of 11×11 and 21×21. Figure 3 shows the x-component velocity profiles (u) on the vertical centerline with different meshes. The results differ only slightly. The distribution of vorticity ω is shown in Figure 4, and the velocity vector is shown in Figure 5, for the 21×21 mesh.

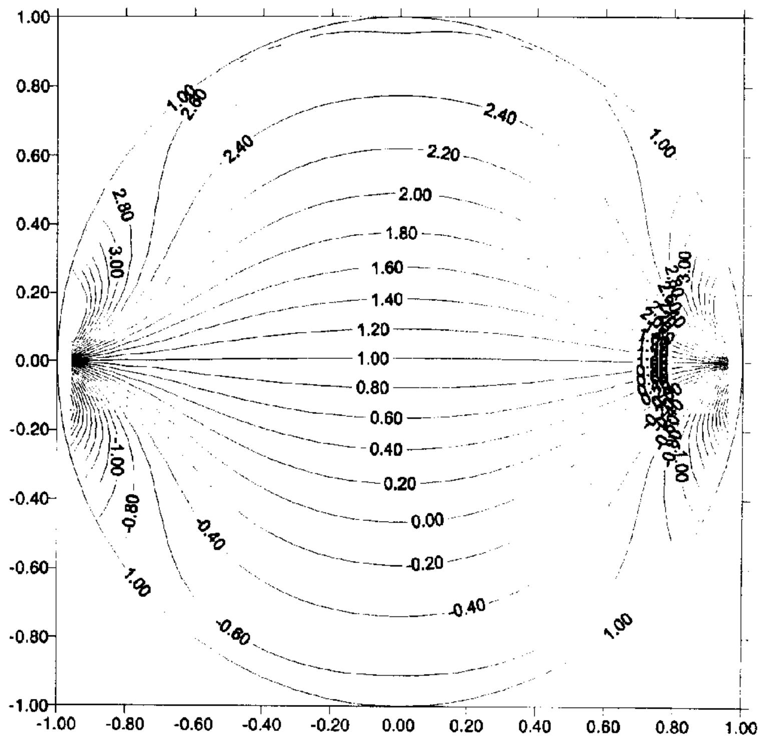

Figure 9. Vorticity distribution for Stokes flow in a circular cavity.

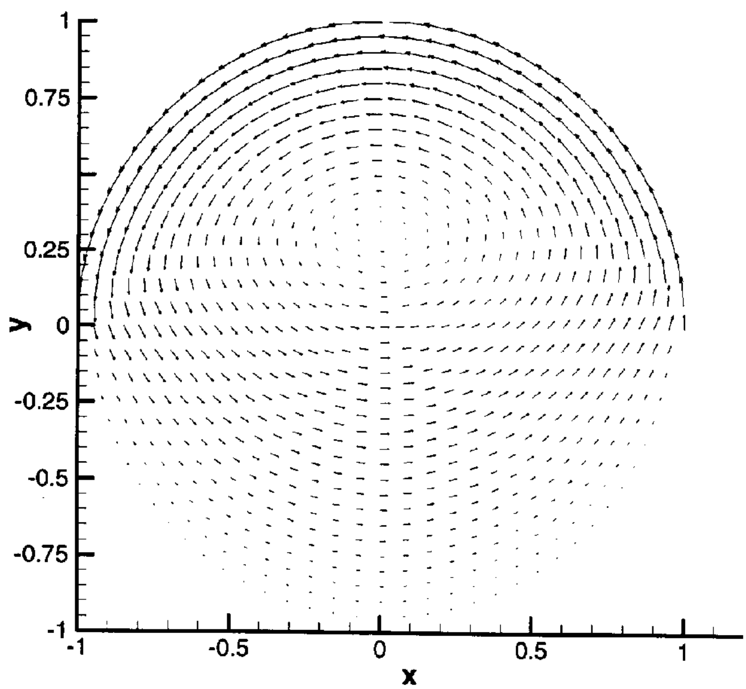

Figure 10. Velocity vectors for Stokes flow in a circular cavity.

The computational efficiency is investigated by comparing with an earlier work [28] using a noniterative DRBEM with the radial basis function in the form $1+r$. For the same problem solved, with a convergence to the tolerance of 10^{-3}, the present iterative scheme requires about 50% of the computational time of the noniterative case.

EXAMPLE 2. The second model problem consists of a recirculating flow in a circular cavity [25,29]. The radius of the circular cavity is assumed to be unity. The configuration and boundary conditions of the problem are shown in Figure 6. In the upper half of the boundary, the velocity $u_\theta = 1$ and $u_r = 0$ are prescribed, and in the lower half, $u_\theta = u_r = 0$. For the computation, 80 boundary nodes and 1600 internal nodes are used.

Figure 7 shows the x-component velocity profile (u) on the vertical centerline, and Figure 8 shows the y-component velocity profile (v) on the horizontal centerline. The results are compared with a series solution [25] and analytical solution [29], and show good agreement. The distribution of vorticity is shown in Figure 9, and the velocity vector is shown in Figure 10.

6. CONCLUSION

An iterative dual reciprocity boundary element method using compactly-supported, positive definite radial basis functions has been developed for the solution of the velocity-vorticity equations of Stokes flow problems. By iterating between the velocity Poisson equations and the vorticity Laplace equation, the vorticity boundary conditions are obtained from the DRBEM solution of the velocity equations. The use of the DRBEM enables a boundary only solution for the Poisson equations. The iterative solution of the DRBEM and BEM equations avoided the assemblage of matrix of the linear systems. A large system of equations can be solved without difficulty.

Low Reynolds number Stokes flows are investigated in a square and a circular cavity. Results are compared with analytical solutions and other numerical solutions. Even in a coarse mesh, the accuracy of the present solution compares quite well with other numerical methods using finer meshes. A comparison with a DRBEM solution using direct solution methods shows that the present scheme is more efficient. The advantage will be enhanced in a larger size problem.

The demonstrated good performance of the iterative DRBEM shows that it a useful tool for the numerical solution of incompressible viscous fluid flow in slow motion. Although only two-dimensional Stokes flow problems are solved, we expect the iterative DRBEM to outperform other methods in the solution of large-scale three-dimensional incompressible viscous flow problems governed by Stokes and Navier-Stokes equations, which is presently under investigation.

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