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Lorentz group $SO_o(5, 1)$ for perfect elastoplasticity with large deformation and a consistency numerical scheme

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Abstract

How to effectively deal with non-linearity and accurately fulfill the consistency condition is essential for modeling and computing in plasticity. Utilizing the concepts of two phases and homogeneous coordinates, we obtain a linear representation of a constitutive model of perfect elastoplasticity with large deformation, and, furthermore, a linear irreducible representation, which contains a five-order spin tensor. The underlying vector space is found to be the projective realization of a composite space resulting from a surgery on Minkowski spacetime $\mathbb{M}^{5+1}$. The irreducible representation in the vector space admits of the projective proper orthochronous Lorentz group $PSO_o(5, 1)$ in the on (or elastoplastic) phase and the special Euclidean group $SE(5)$ in the off (or elastic) phase. The input path dictates symmetry switching between the two groups. Based on such symmetry a numerical scheme is devised which preserves the consistency condition for every time step. The *consistency scheme* is shown to be stabler, more accurate, and more efficient than the current numerical schemes developed directly based upon the model itself, because the new scheme preserves the internal symmetry $SO_o(5, 1)$ of the model in the on phase so as to locate the stress point automatically on the yield surface at each time step without iterations at all.

Keywords: Elastoplasticity; Large deformation; Irreducible representation; Lorentz group; Consistency scheme

1. Introduction

A symmetry of a constitutive model is a statement that when one makes changes in the states of the model, a particular expression for certain constitutive phenomena he formulates does not change. The changes or transformations he makes to the constitutive model which leave the form of the expression unchanged are naturally linked with the invariance of a conserved quantity. A numerical scheme which preserves symmetry and utilizes the invariance property from one time stop to the next one or few stops will be more capable of capturing key features during elastoplastic deformation and have long-term stability and much improved

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efficiency and accuracy. Therefore the issue of internal symmetries in constitutive laws of plasticity is not only important in its own right, but will also find applications to computational plasticity.

Hong and Liu recently found symmetry groups in a constitutive model of perfect elastoplasticity [1, 2] and in a constitutive model of bilinear elastoplasticity [1, 3]. The models considered were all restricted to small deformation. It is interesting to observe that the symmetry groups the models admit possess richer structures than what the models imply. For example, the symmetry group for the perfect elastoplasticity in the on (i.e. elastoplastic) phase is $SO_o(k, 1)$ in an augmented stress space. It is known that an element $\mathbf{A}$ of the real Lie algebra $so(k, 1)$ of the Lorentz group $SO_o(k, 1)$ has the general form:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_s^s & \mathbf{A}_0^s \\ \mathbf{A}_s^0 & 0 \end{bmatrix}$$

with the properties

$$(\mathbf{A}_s^s)^{\mathrm{t}} = -\mathbf{A}_s^s, \quad \mathbf{A}_s^0 = (\mathbf{A}_0^s)^{\mathrm{t}},$$

where the superscript t indicates the transpose. But the form of $\mathbf{A}$ previously found for the small deformation model of perfect elastoplasticity has zero $\mathbf{A}_s^s$ [2] and is, therefore, less general than it may. One may generalize the model by including a non-vanishing, skew-symmetric tensor $\mathbf{A}_s^s$ if he no longer assumes negligibly small spinning but instead considers large deformation and rotation.

As we know the numerical schemes developed up to now for the integration of the constitutive equations of elastoplasticity are executed in the stress space, for example, the tangent stiffness-radial return method [4, 5], the radial return method [4], the elastic predictor-radial corrector method [5], the generalized midpoint rule [6], the closest-point-projection algorithm [7] and also the recently developed plastic predictor-elastic corrector method [8]. In order to enforce the consistency condition at every time step the abovementioned algorithms require some iterative calculations to force the stress point at the end of each time step to converge to the yield surface [7], which is known as a main source of numerical errors and of consumption of computational time.

In this paper we consider a large deformation constitutive model of perfect elastoplasticity (in Sections 2 and 3) and manage to put it in a more appropriate setting (in Sections 4 and 5) such that the internal spacetime structure (see Sections 6, 7 and 10) and the internal symmetries (see Sections 8 and 9) of the model are brought out. Finally, using internal symmetry inherent in the constitutive model we develop a *consistency scheme* (in Section 11). One direct benefit of the scheme is that the stress point is automatically updated on the yield surface without iterative calculations for every time step. This is what the conventional constitutive numerical schemes desired and failed to achieve directly in the stress space. Another, not less important, benefit is that the formulation of the model becomes exactly linear in the augmented stress space.

2. The constitutive model considered

The constitutive law of elastoplasticity of solid materials proposed by Prandtl [9] and Reuss [10] can be re-postulated (cf. [11]) and enlarged to take account of large deformation as in the following system of axioms:[1]

$$\mathbf{D} = \mathbf{D}^{\mathrm{e}} + \mathbf{D}^{\mathrm{p}}, \tag{1}$$

$$\mathring{\mathbf{s}} = 2G\mathbf{D}^{\mathrm{e}}, \tag{2}$$

[1] The volumetric part of the Prandtl–Reuss law is linearly elastic and was thus excluded from the present study in order to focus on the more interesting elastic–plastic behavior of the deviatoric part.

$$\mathbf{s}\dot{\bar{\gamma}}^{\mathrm{p}} = 2\tau_y \mathbf{D}^{\mathrm{p}}, \tag{3}$$

$$\|\mathbf{s}\| \leqslant \sqrt{2}\tau_y, \tag{4}$$

$$\dot{\bar{\gamma}}^{\mathrm{p}} \geqslant 0, \tag{5}$$

$$\|\mathbf{s}\|\dot{\bar{\gamma}}^{\mathrm{p}} = \sqrt{2}\tau_y \dot{\bar{\gamma}}^{\mathrm{p}}, \tag{6}$$

in which the two material constants, namely the shear modulus G and the shear yield stress (or strength) τ_y, are determined experimentally and both are assumed to be positive. The bold-faced symbols $\mathbf{D}$, $\mathbf{D}^{\mathrm{e}}$, $\mathbf{D}^{\mathrm{p}}$ and $\mathbf{s}$ stand for the *deviatoric* parts of the deformation rate, elastic deformation rate, plastic deformation rate, and Cauchy's stress, respectively, all being symmetric and traceless tensors, whereas $\bar{\gamma}^{\mathrm{p}}$ is a scalar, called the equivalent shear plastic (engineering) strain. All the $\mathbf{D}$, $\mathbf{D}^{\mathrm{e}}$, $\mathbf{D}^{\mathrm{p}}$, $\mathbf{s}$ and $\bar{\gamma}^{\mathrm{p}}$ are functions of one and the same independent variable, which in most cases is taken either as the ordinary time or as the arc length of the controlled strain path; however, for convenience, the independent variable no matter what it is will be simply called (the external) time and given the symbol t.

A superimposed dot denotes (material) differentiation with respect to time, that is d/dt, and a surmounted circle "$\circ$" on $\mathbf{s}$ represents a Lie derivative of $\mathbf{s}$ with respect to $\mathbf{W}$, that is the Jaumann rate

$$\mathring{\mathbf{s}} := \dot{\mathbf{s}} - \mathbf{W}\mathbf{s} + \mathbf{s}\mathbf{W}. \tag{7}$$

Here $\mathbf{W}$ is the spin tensor, defined as the skew-symmetric part of the velocity gradient. A dot is placed between two tensors to denote their Euclidean inner product, and as usual the Euclidean norm of a tensor, $\mathbf{s}$ say, is represented by $\|\mathbf{s}\| := \sqrt{\mathbf{s}\cdot\mathbf{s}}$.

3. A non-linear representation

Let us first analyze the constitutive model (1)–(6). Substituting Eqs. (2), (3) and (7) into Eq. (1), we obtain

$$\frac{1}{2G}[\dot{\mathbf{s}} - \mathbf{W}\mathbf{s} + \mathbf{s}\mathbf{W}] + \frac{1}{2\tau_y}\dot{\bar{\gamma}}^{\mathrm{p}}\mathbf{s} = \mathbf{D}. \tag{8}$$

Define (the internal time)

$$X^0 := \exp\frac{\bar{\gamma}^{\mathrm{p}}}{\gamma_y}, \tag{9}$$

where

$$\gamma_y := \frac{\tau_y}{G} \tag{10}$$

is the shear yield (engineering) strain. Then Eq. (8) becomes

$$\frac{1}{2G}\left[\frac{\mathrm{d}}{\mathrm{d}t}(X^0\mathbf{s}) - X^0\mathbf{W}\mathbf{s} + X^0\mathbf{s}\mathbf{W}\right] = X^0\mathbf{D}. \tag{11}$$

The inner product of $\mathbf{s}$ with Eq. (8) is

$$\frac{1}{2G}\mathbf{s}\cdot\dot{\mathbf{s}} + \frac{1}{2\tau_y}\dot{\bar{\gamma}}^{\mathrm{p}}\mathbf{s}\cdot\mathbf{s} = \mathbf{s}\cdot\mathbf{D}. \tag{12}$$

Hence

$$\| \mathbf{s} \| = \sqrt{2}\tau_y \Rightarrow \tau_y \dot{\gamma}^{\mathrm{p}} = \mathbf{s} \cdot \mathbf{D}. \tag{13}$$

Recalling $\tau_y > 0$, we have

$$\| \mathbf{s} \| = \sqrt{2}\tau_y \Rightarrow \{\mathbf{s} \cdot \mathbf{D} > 0 \Leftrightarrow \dot{\gamma}^{\mathrm{p}} > 0\}, \tag{14}$$

asserting

$$\{ \| \mathbf{s} \| = \sqrt{2}\tau_y \quad \text{and} \quad \mathbf{s} \cdot \mathbf{D} > 0\} \Rightarrow \dot{\gamma}^{\mathrm{p}} > 0. \tag{15}$$

Conversely, if $\dot{\gamma}^{\mathrm{p}} > 0$, Eq. (6) assures $\| \mathbf{s} \| = \sqrt{2}\tau_y$, which together with Eq. (14) asserts that

$$\dot{\gamma}^{\mathrm{p}} > 0 \Rightarrow \{ \| \mathbf{s} \| = \sqrt{2}\tau_y \quad \text{and} \quad \mathbf{s} \cdot \mathbf{D} > 0\}. \tag{16}$$

Statements (15) and (16) tell us that the yield condition $\| \mathbf{s} \| = \sqrt{2}\tau_y$ and the straining condition $\mathbf{s} \cdot \mathbf{D} > 0$ are sufficient and necessary for plastic irreversibility $\dot{\gamma}^{\mathrm{p}} > 0$. In view of Eqs. (4), (5) and (13), the two statements are logically equivalent to the following criteria:

$$\dot{\gamma}^{\mathrm{p}} = \frac{1}{\tau_y} \mathbf{s} \cdot \mathbf{D} > 0 \quad \text{if } \| \mathbf{s} \| = \sqrt{2}\tau_y \quad \text{and} \quad \mathbf{s} \cdot \mathbf{D} > 0, \tag{17a}$$

$$\dot{\gamma}^{\mathrm{p}} = 0 \quad \text{if } \| \mathbf{s} \| < \sqrt{2}\tau_y \quad \text{or} \quad \mathbf{s} \cdot \mathbf{D} \leqslant 0. \tag{17b}$$

From Eqs. (8), (17a), (17b) and (10) follows a two-phase non-linear system of differential equations:

$$\dot{\mathbf{s}} - \mathbf{W}\mathbf{s} + \mathbf{s}\mathbf{W} = -\frac{\mathbf{s} \cdot \mathbf{D}}{\tau_y \gamma_y} \mathbf{s} + 2G\mathbf{D} \quad \text{if } \| \mathbf{s} \| = \sqrt{2}\tau_y \quad \text{and} \quad \mathbf{s} \cdot \mathbf{D} > 0, \tag{18a}$$

$$\dot{\mathbf{s}} - \mathbf{W}\mathbf{s} + \mathbf{s}\mathbf{W} = 2G\mathbf{D} \quad \text{if } \| \mathbf{s} \| < \sqrt{2}\tau_y \quad \text{or} \quad \mathbf{s} \cdot \mathbf{D} \leqslant 0. \tag{18b}$$

According to criteria (17a) and (17b) and the complementary trios (4)–(6) and further to the two-phase system (18a), (18b), the model of elastoplasticity has precisely two phases: the on phase in which $\dot{\gamma}^{\mathrm{p}} > 0$ and $\| \mathbf{s} \| = \sqrt{2}\tau_y$ and the off phase in which $\dot{\gamma}^{\mathrm{p}} = 0$ and $\| \mathbf{s} \| \leqslant \sqrt{2}\tau_y$. In the on phase the plasticity mechanism is on so that the model exhibits elastoplastic behavior, which is irreversible, while in the off phase the plasticity mechanism is off so that the model responds elastically and reversibly. Thus, Eqs. (17a) and (17b) are called the on–off switching criteria for the mechanism of plasticity, and Eqs. (18a) and (18b) is the *non-linear* differential representation of the constitutive model.

4. A linear representation

In this section and the next one we will manage to put the constitutive model in such a form as to reveal internal symmetry. For these purposes let us introduce

$$\tilde{\mathbf{X}} = \begin{bmatrix} \tilde{\mathbf{X}}^s \\ X^0 \end{bmatrix} := \frac{X^0}{\tau_y} \begin{bmatrix} s^{11}/\sqrt{2} \\ s^{22}/\sqrt{2} \\ s^{33}/\sqrt{2} \\ s^{23} \\ s^{13} \\ s^{12} \\ \tau_y \end{bmatrix}. \tag{19}$$

The components of $\tilde{\mathbf{X}}$ are the homogeneous coordinates of the vector space of $(s^{11}/\sqrt{2}, s^{22}/\sqrt{2}, s^{33}/\sqrt{2}, s^{23}, s^{13}, s^{12})/\tau_y$. Here and henceforth the index s refers to the totality of the "internal space" coordinates, while the index 0 refers to the "internal time" coordinate; a bold-faced symbol even with the index(es) denotes a tensor (and sometimes the matrix of the components).

Thus Eq. (11) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{X}}^s = \tilde{\mathbf{A}}^s_s\tilde{\mathbf{X}}^s + \tilde{\mathbf{A}}^s_0 X^0, \tag{20}$$

in which

$$\tilde{\mathbf{A}}^s_s := \begin{bmatrix} 0 & 0 & 0 & 0 & \sqrt{2}W_{13} & \sqrt{2}W_{12} \\ 0 & 0 & 0 & \sqrt{2}W_{23} & 0 & -\sqrt{2}W_{12} \\ 0 & 0 & 0 & -\sqrt{2}W_{23} & -\sqrt{2}W_{13} & 0 \\ 0 & -\sqrt{2}W_{23} & \sqrt{2}W_{23} & 0 & -W_{12} & -W_{13} \\ -\sqrt{2}W_{13} & 0 & \sqrt{2}W_{13} & W_{12} & 0 & -W_{23} \\ -\sqrt{2}W_{12} & \sqrt{2}W_{12} & 0 & W_{13} & W_{23} & 0 \end{bmatrix}. \tag{21}$$

Mehrabadi et al. [12] had shown that the $\tilde{\mathbf{A}}^s_s$ in the form of Eq. (21) is a spin *tensor* in a six-dimensional space.

In view of Eq. (12) and the positivity of G, τ_y, γ_y and X^0, the on–off switching criteria (17a) and (17b) become

$$\dot{X}^0 = \tilde{\mathbf{A}}^0_s\tilde{\mathbf{X}}^s > 0 \quad \text{if } \tilde{\mathbf{X}}^{\mathrm{t}}\tilde{\mathbf{g}}\mathbf{X} = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}[(\tilde{\mathbf{X}}^s)^{\mathrm{t}}\tilde{\mathbf{g}}_{ss}\tilde{\mathbf{X}}^s] > 0, \tag{22a}$$

$$\dot{X}^0 = 0 \quad \text{if } \tilde{\mathbf{X}}^{\mathrm{t}}\tilde{\mathbf{g}}\tilde{\mathbf{X}} < 0 \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}t}[(\tilde{\mathbf{X}}^s)^{\mathrm{t}}\tilde{\mathbf{g}}_{ss}\tilde{\mathbf{X}}^s] \leqslant 0, \tag{22b}$$

where

$$\tilde{\mathbf{g}} = \begin{bmatrix} \tilde{\mathbf{g}}_{ss} & \tilde{\mathbf{g}}_{s0} \\ \tilde{\mathbf{g}}_{0s} & \tilde{g}_{00} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_6 & \mathbf{0}_{6\times 1} \\ \mathbf{0}_{1\times 6} & -1 \end{bmatrix}, \tag{23}$$

in which $\mathbf{I}_i$ is the identity of order i, i being a natural number, and

$$\tilde{\mathbf{A}}^0_s = (\tilde{\mathbf{A}}^s_0)^{\mathrm{t}}, \tag{24}$$

in which

$$\tilde{\mathbf{A}}^s_0 := \frac{2}{\gamma_y}\begin{bmatrix} D_{11}/\sqrt{2} \\ D_{22}/\sqrt{2} \\ D_{33}/\sqrt{2} \\ D_{23} \\ D_{13} \\ D_{12} \end{bmatrix}. \tag{25}$$

Organizing Eqs. (20), (22a) and (22b) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{X}} = \tilde{\mathbf{A}}\tilde{\mathbf{X}}, \tag{26}$$

where

$$\tilde{\mathbf{A}} = \begin{bmatrix} \tilde{\mathbf{A}}_s^s & \tilde{\mathbf{A}}_0^s \\ \tilde{\mathbf{A}}_s^0 & 0 \end{bmatrix} \quad \text{if } \tilde{\mathbf{X}}^{\mathrm{t}}\tilde{\mathbf{g}}\tilde{\mathbf{X}} = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}[(\tilde{\mathbf{X}}^s)^{\mathrm{t}}\tilde{\mathbf{g}}_{ss}\tilde{\mathbf{X}}^s] > 0, \tag{27a}$$

$$\tilde{\mathbf{A}} = \begin{bmatrix} \tilde{\mathbf{A}}_s^s & \tilde{\mathbf{A}}_0^s \\ \mathbf{0}_{1\times 6} & 0 \end{bmatrix} \quad \text{if } \tilde{\mathbf{X}}^{\mathrm{t}}\tilde{\mathbf{g}}\tilde{\mathbf{X}} < 0 \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}t}[(\tilde{\mathbf{X}}^s)^{\mathrm{t}}\tilde{\mathbf{g}}_{ss}\tilde{\mathbf{X}}^s] \leqslant 0. \tag{27b}$$

This is a linear, (6 + 1)-dimensional representation of the constitutive model (1)–(6).

5. A linear irreducible representation

Due to the vanishing traces of the deviatoric tensors **s** and **D**, i.e.,

$$s^{33} = -s^{11} - s^{22}, \quad D_{33} = -D_{11} - D_{22}, \tag{28}$$

the third component of the vector Eq. (26) can be obtained from the first two components. To delete this redundancy, let us introduce[2]

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}^s \\ X^0 \end{bmatrix} = \begin{bmatrix} X^1 \\ X^2 \\ X^3 \\ X^4 \\ X^5 \\ X^0 \end{bmatrix} := \frac{X^0}{\tau_y} \begin{bmatrix} a_1 s^{11} + a_2 s^{22} \\ a_3 s^{11} + a_4 s^{22} \\ s^{23} \\ s^{13} \\ s^{12} \\ \tau_y \end{bmatrix}, \tag{29}$$

where

$$a_1 := \sin\left(\theta + \frac{\pi}{3}\right), \quad a_2 := \sin\theta, \quad a_3 := \cos\left(\theta + \frac{\pi}{3}\right), \quad a_4 := \cos\theta, \tag{30}$$

with θ being any real number.[3] The on–off switching criteria turn out to be

$$\dot{X}^0 = \mathbf{A}_s^0\mathbf{X}^s > 0 \quad \text{if } \mathbf{X}^{\mathrm{t}}\mathbf{g}\mathbf{X} = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}[(\mathbf{X}^s)^{\mathrm{t}}\mathbf{g}_{ss}\mathbf{X}^s] > 0, \tag{31a}$$

$$\dot{X}^0 = 0 \quad \text{if } \mathbf{X}^{\mathrm{t}}\mathbf{g}\mathbf{X} < 0 \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}t}[(\mathbf{X}^s)^{\mathrm{t}}\mathbf{g}_{ss}\mathbf{X}^s] \leqslant 0, \tag{31b}$$

where

$$\mathbf{g} = \begin{bmatrix} \mathbf{g}_{ss} & \mathbf{g}_{s0} \\ \mathbf{g}_{0s} & g_{00} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_5 & \mathbf{0}_{5\times 1} \\ \mathbf{0}_{1\times 5} & -1 \end{bmatrix}, \tag{32}$$

$$\mathbf{A}_s^0 = (\mathbf{A}_0^s)^{\mathrm{t}}, \tag{33}$$

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[2] A special case of Eq. (29) in which $\theta = 0$ can be viewed as the homogeneous coordinates (non-dimensionalized with respect to the tensile yield strength $\sqrt{3}\tau_y$) of the Il'yushin stress space $(\frac{3}{2}s^{11}, \sqrt{\frac{3}{2}}s^{11} + \sqrt{3}s^{22}, \sqrt{3}s^{23}, \sqrt{3}s^{13}, \sqrt{3}s^{12})$.

[3] For Eq. (30) see (and compare) Il'yushin [13], Ohashi et al. [14], and Hong and Liu [11], for example.

in which

$$\mathbf{A}_0^s = \begin{bmatrix} A_0^1 \\ A_0^2 \\ A_0^3 \\ A_0^4 \\ A_0^5 \end{bmatrix} := \frac{2}{\gamma_y} \begin{bmatrix} a_1 D_{11} + a_2 D_{22} \\ a_3 D_{11} + a_4 D_{22} \\ D_{23} \\ D_{13} \\ D_{12} \end{bmatrix}. \tag{34}$$

Then Eq. (26) is reduced to

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}, \tag{35}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_s^s & \mathbf{A}_0^s \\ \mathbf{A}_s^0 & 0 \end{bmatrix} \quad \text{if } \mathbf{X}^{\mathrm{t}}\mathbf{g}\mathbf{X} = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}[(\mathbf{X}^s)^{\mathrm{t}}\mathbf{g}_{ss}\mathbf{X}^s] > 0, \tag{36a}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_s^s & \mathbf{A}_0^s \\ \mathbf{0}_{1\times 6} & 0 \end{bmatrix} \quad \text{if } \mathbf{X}^{\mathrm{t}}\mathbf{g}\mathbf{X} < 0 \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}t}[(\mathbf{X}^s)^{\mathrm{t}}\mathbf{g}_{ss}\mathbf{X}^s] \leqslant 0. \tag{36b}$$

in which

$$\mathbf{A}_s^s := \begin{bmatrix} 0 & 0 & 2a_2 W_{23} & 2a_1 W_{13} & 2(a_1 - a_2) W_{12} \\ 0 & 0 & 2a_4 W_{23} & 2a_3 W_{13} & 2(a_3 - a_4) W_{12} \\ \frac{2}{\sqrt{3}}(2a_3 - a_4) W_{23} & \frac{2}{\sqrt{3}}(a_2 - 2a_1) W_{23} & 0 & -W_{12} & -W_{13} \\ \frac{2}{\sqrt{3}}(a_3 - 2a_4) W_{13} & \frac{2}{\sqrt{3}}(2a_2 - a_1) W_{13} & W_{12} & 0 & -W_{23} \\ -\frac{2}{\sqrt{3}}(a_3 + a_4) W_{12} & \frac{2}{\sqrt{3}}(a_1 + a_2) W_{12} & W_{13} & W_{23} & 0 \end{bmatrix}. \tag{37}$$

However, we may go further to prove that $\mathbf{A}_s^s$ as in Eq. (37) is skew-symmetric, i.e.

$$a_2 = \frac{-1}{\sqrt{3}}(2a_3 - a_4), \quad a_1 = \frac{-1}{\sqrt{3}}(a_3 - 2a_4), \quad a_1 - a_2 = \frac{-1}{\sqrt{3}}(a_3 + a_4),$$

$$a_4 = \frac{-1}{\sqrt{3}}(a_2 - 2a_1), \quad a_3 = \frac{-1}{\sqrt{3}}(2a_2 - a_1), \quad a_3 - a_4 = \frac{-1}{\sqrt{3}}(a_1 + a_2). \tag{38}$$

It is obvious that the first two equations imply the last four equations, so only the first two equations need to be checked. To this end substitution of the first two of Eq. (30) into the first two of Eq. (38) gives

$$\sin\theta = \frac{-1}{\sqrt{3}}\left[2\cos\left(\theta + \frac{\pi}{3}\right) - \cos\theta\right], \quad \sin\left(\theta + \frac{\pi}{3}\right) = \frac{-1}{\sqrt{3}}\left[\cos\left(\theta + \frac{\pi}{3}\right) - 2\cos\theta\right],$$

which obviously hold for any real number of θ, thereby completing the proofs. That is

$$(\mathbf{A}_s^s)^{\mathrm{t}} = -\mathbf{A}_s^s. \tag{39}$$

Indeed, by extending the reasoning for the six-dimensional case by Mehrabadi et al. [12], the $\mathbf{A}^s_s$ can be shown to be a spin tensor occurring in the five-dimensional internal space, representing the spin tensor $\mathbf{W}$ occurring in the three-dimensional external space.

Note that Eq. (35) is a linear, irreducible, (5 + 1)-dimensional representation of the constitutive model (1)–(6), in which $\mathbf{X}$ and $\mathbf{A}$ are the augmented stress vector and the control tensor, respectively. The control tensor $\mathbf{A}$ organizes the input information of the deviatoric deformation rate tensor $\mathbf{D}$ (normalized with respect to the shear yield strain $\gamma_y/2$) and the spin tensor $\mathbf{W}$. If Eq. (35) is viewed as a matrix representation, the $(5+1)\times 1$ matrix $\mathbf{X}$ contains the contravariant components of the augmented stress vector $\mathbf{X}$ and the $(5+1)\times(5+1)$ matrix $\mathbf{A}$ contains the mixed components of the control tensor $\mathbf{A}$.

6. The cone and the discs

In this section we study the properties of the augmented stress space of $\mathbf{X}$ induced by the constitutive model (1)–(6). The model formulated in the deviatoric stress space of $\mathbf{s}$ may be converted into a model in the augmented stress space of $\mathbf{X}$; the first five equations of (35) and Eqs. (6), (4) and (5) thus become successively

$$\begin{bmatrix} \mathbf{I}_5 & \mathbf{0}_{5\times 1} \\ \mathbf{0}_{1\times 5} & \mathbf{X}^{\mathrm{t}}\mathbf{g}\mathbf{X} \end{bmatrix} \dot{\mathbf{X}} = \begin{bmatrix} \mathbf{A}^s_s & \mathbf{A}^s_0 \\ \mathbf{0}_{1\times 5} & 0 \end{bmatrix} \mathbf{X}, \tag{40}$$

$$\mathbf{X}^{\mathrm{t}}\mathbf{g}\mathbf{X} \leqslant 0, \tag{41}$$

$$\dot{X}^0 \geqslant 0, \tag{42}$$

in terms of the Minkowski metric $\mathbf{g}$ (in the space-like convention) of Eq. (32).[4] The vector space of augmented stresses $\mathbf{X}$ endowed with the Minkowski metric tensor $\mathbf{g}$ is referred to as *Minkowski spacetime* and designated as $\mathbb{M}^{5+1}$.

Thus a deviatoric stress point $\mathbf{s}$ on the yield hypersphere $\|\mathbf{s}\| = \sqrt{2}\tau_y$ in Euclidean space $\mathbb{E}^5$ corresponds to an augmented stress point $\mathbf{X}$ on the right circular cone $\{\mathbf{X} \,\|\, \mathbf{X}^{\mathrm{t}}\mathbf{g}\mathbf{X} = 0\}$ emanating from $\mathbf{X} = \mathbf{0}$ of Minkowski spacetime $\mathbb{M}^{5+1}$, henceforth referred to as *the* cone, while an $\mathbf{s}$ within the yield hypersphere corresponds to an $\mathbf{X}$ in the interior $\{\mathbf{X}|\mathbf{X}^{\mathrm{t}}\mathbf{g}\mathbf{X} < 0\}$ of the cone. The exterior $\{\mathbf{X}|\mathbf{X}^{\mathrm{t}}\mathbf{g}\mathbf{X} > 0\}$ of the cone is uninhabitable since $\|\mathbf{s}\| > \sqrt{2}\tau_y$ is forbidden according to Eq. (4). Even though admitting an infinite number of Riemannian metrics, the yield hypersphere $\mathbb{S}^{5-1}$ of $\mathbf{s}$ does not admit a Minkowskian metric, nor does the cylinder of $(\mathbf{s}, X^0)$. It is the cone in the $\mathbf{X}$-space which admits the Minkowski metric. When X^0 is frozen in the off phase as indicated by Eq. (31b), the augmented stress $\mathbf{X}$ stays in the closed 5-disc $\mathbb{D}^5$ (i.e. closed 5-ball $\mathbb{B}^5$) on the hyperplane $X^0 =$ constant in the space of $(X^1, X^2, X^3, X^4, X^5, X^0)$; the hyperplane is identified to be Euclidean 5-space $\mathbb{E}^5$, which is endowed with the Euclidean metric $\mathbf{I}_5$. In summary, the augmented stress $\mathbf{X}$ either evolves on the cone when in the on phase, or stays in the discs of simultaneity, which are stacked up one by one in the interior of the cone and are glued to the cone, when in the off phase. This inspires us to remould the spacetime by removing the interior of the cone and gluing (identifying) a continuously infinite number of stacking Euclidean closed 5-discs to the cone. This surgery results in a composite space endowed with the Minkowski metric (32) on the cone and the Euclidean metric $\mathbf{I}_5$ on the closed discs.

Spacetime of this sort underlying the plasticity theory may be called *internal spacetime*, because we can think of it as having to do with the intrinsic nature of the mechanical behavior of the solid materials which

[4] The reason for adopting the space-like convention is that the nature of elastoplasticity rejects time-like paths, as to be elaborated in Section 7, so that the space-like convention favorably results in non-negative squared lengths exclusively, as shown in Eq. (49). Otherwise the time-like convention would have resulted in embarrassing negative squared lengths.

the constitutive model describes, rather than their position or motion in ordinary (external) space and time. Thus the "temporal" component X^0 and the "spatial" components $\mathbf{X}^s = (X^1, X^2, X^3, X^4, X^5)$ may be thought of as the *internal time* and the *internal space*, respectively. Thus the *internal* time X^0 is distinguished from time t, which is the independent variable of the constitutive model and may therefore be deemed as the *external* time. The external time t is dictated by the external input whereas the internal time X^0 is coined for the age of the internal state of the model. Since Eqs. (1)–(6) is one of the so-called "local" constitutive laws, the *external space* does not appear explicitly in the model. In such a point of view X^0 and $\mathbf{X}^s$ are no more disparate and incompatible as the usual (5 + 1)-dimensional vector in Euclidean spacetime $\mathbb{E}^{5+1}$; they are now being organized to an integrated object in Minkowski spacetime $\mathbb{M}^{5+1}$, which is not a simple extension of ordinary Euclidean 5-space to 5 + 1 dimensions, with X^0 as just one more dimension. Because the corresponding entries in the metric (32) have different signs, -1 versus positive definiteness, the internal time coordinate X^0 is not on the same footing as the five internal space coordinates $(X^1, X^2, X^3, X^4, X^5)$, and the structure built on the spacetime consequently has *internal symmetries* (see Sections 8 and 9) quite unlike those on Euclidean space.

7. No time-like paths in internal spacetime

From Eqs. (3), (5) and (6) and $\tau_y > 0$, it is not difficult to prove that

$$\dot{\bar{\gamma}}^{\mathrm{p}} = \sqrt{2}\,\|\mathbf{D}^{\mathrm{p}}\|, \tag{43}$$

which indicates that $\bar{\gamma}^{\mathrm{p}}/\sqrt{2}$ is the arc length of a path in the plastic strain space. Criteria (17a) and (17b) ensure that

$$\dot{\bar{\gamma}}^{\mathrm{p}}\mathbf{s}\cdot\dot{\mathbf{s}} = 0 \tag{44}$$

no matter whether in the on or in the off phase. Substituting Eqs. (2), (3) and (7) into the above equation and considering the positivity of G and τ_y, we obtain

$$\mathbf{D}^{\mathrm{p}}\cdot\mathbf{D}^{\mathrm{e}} = 0. \tag{45}$$

This and Eqs. (1) and (43) give

$$\dot{\bar{\gamma}}^{\mathrm{p}} \leqslant \sqrt{2}\,\|\mathbf{D}\| \tag{46}$$

no matter whether in the on or off phase. This inequality tells us that the maximum value of the specific dissipation power $\dot{\Lambda} = \tau_y\dot{\bar{\gamma}}^{\mathrm{p}}$ an admissible path in the state space may discharge is $\sqrt{2}\tau_y\|\mathbf{D}\|$. (On the other hand, axiom (5) tells us that the minimum value of the specific dissipation power an admissible path may discharge is zero.)

What does this important observation imply for a path in the augmented stress space of Minkowski spacetime? From Eqs. (46), (5) and (9) it follows that

$$2(X^0)^2\mathbf{D}\cdot\mathbf{D} - \gamma_y^2(\dot{X}^0)^2 \geqslant 0,$$

via Eq. (34) which further reduces to

$$(X^0)^2\mathbf{A}^0_s\mathbf{A}^s_0 - (\dot{X}^0)^2 \geqslant 0.$$

Substituting Eq. (35) into the above equation, we obtain

$$(\mathring{\mathbf{X}}^s)^{\mathrm{t}}\mathbf{g}_{ss}\mathring{\mathbf{X}}^s - (\dot{X}^0)^2 \geqslant 0, \tag{47}$$

where

$$\overset{\circ}{\mathbf{X}}^s := \dot{\mathbf{X}}^s - \mathbf{A}^s_s \mathbf{X}^s. \tag{48}$$

(Compare definitions (7) and (48).) Thus

$$(\mathrm{d}X)^2 := \mathrm{d}\mathbf{X}^{\mathrm{t}}\mathbf{g}\,\mathrm{d}\mathbf{X} \geqslant 0. \tag{49}$$

This defines the Minkowskian length $\mathrm{d}X$ of a differential element $\mathrm{d}\mathbf{X}$ of a path. We should be cautious to distinguish $\mathrm{d}X$ from $\mathrm{d}X^0$, the latter of which is the "temporal" component of $\mathrm{d}\mathbf{X}$. Recalling that a path such that $\mathrm{d}\mathbf{X}^{\mathrm{t}}\mathbf{g}\,\mathrm{d}\mathbf{X} > 0$ (resp. $= 0$, < 0) is called a space-like (resp. null, time-like) path in $\mathbb{M}^{5+1}$, we thereby conclude that the curve $\{\mathbf{X}(t') \,|\, t_i < t' \leqslant t\}$ in the augmented stress space is a space-like or null path in the Minkowski spacetime $\mathbb{M}^{5+1}$ no matter whether in the on phase or in the off phase. Here t_i denotes an initial time and t the current time. Indeed, Eq. (49) conveys an important message that *no time-like paths may exist in the internal spacetime of elastoplasticity*.

Moreover, from Eqs. (1), (2), (43) and (45) it follows that

$$\mathrm{d}s := \|\mathrm{d}\mathbf{s}\| = 2G\sqrt{\|\mathbf{D}\|^2 - (\mathrm{d}\bar{\gamma}^{\mathrm{p}})^2/2},$$

which, with the aid of Eqs. (11), (10), (29), (32) and (49), gives

$$\frac{\mathrm{d}X}{X^0} = \frac{\mathrm{d}s}{\sqrt{2}\tau_y} \tag{50}$$

no matter whether in the on phase or in the off phase. In a word, the Minkowskian length $\mathrm{d}X$ of a differential element of a path in the augmented stress space $\mathbb{M}^{5+1}$ of $(X^1, X^2, X^3, X^4, X^5, X^0)$ is $X^0/(\sqrt{2}\tau_y)$ times the Euclidean length $\mathrm{d}s$ of the corresponding differential element of the corresponding path in the deviatoric stress space $\mathbb{E}^5$ of $\mathbf{s}$.

The vector $\mathbf{X}(t) - \mathbf{X}(t_i)$ and the path $\{\mathbf{X}(t') \,|\, t_i < t' \leqslant t\}$ are said to be future-pointing if $X^0(t) > X^0(t_i)$ strictly. Therefore, the solution to Eq. (35) with the on-phase $\mathbf{A}$ of Eq. (36a) can be viewed as a future-pointing space-like or null path on the cone $\{\mathbf{X} \,|\, \mathbf{X}^{\mathrm{t}}\mathbf{g}\mathbf{X} = 0\}$, while the solution to Eq. (35) with the off-phase $\mathbf{A}$ of Eq. (36b) is a space-like path on a closed disc of simultaneity $\{\mathbf{X} \,|\, \mathbf{X}^{\mathrm{t}}\mathbf{g}\mathbf{X} \leqslant 0$ and $\dot{X}^0 = 0\}$. It is worth stressing that the interior of the cone is sliced into stacking discs of simultaneity tagged with different values of X^0; therefore, the admissible augmented stress space can be reached either along the paths in the discs of simultaneity when in the off phase or along the future-pointing space-like or null paths on the cone when in the on phase.

8. $PSO_o(5, 1)$ symmetry in the on phase

The solution of Eq. (35) may be expressed in the following transition formula from the augmented stress $\mathbf{X}(t_1)$ at time t_1 to the augmented stress $\mathbf{X}(t)$ at time t:

$$\mathbf{X}(t) = [\mathbf{G}(t)\mathbf{G}^{-1}(t_1)]\mathbf{X}(t_1), \tag{51}$$

in which $\mathbf{G}(t)$ is the fundamental solution of Eq. (35), that is a transformation tensor (represented by a square matrix of order $(5 + 1)$ containing the mixed components) satisfying

$$\dot{\mathbf{G}}(t) = \mathbf{A}(t)\mathbf{G}(t), \tag{52}$$

$$\mathbf{G}(0) = \mathbf{I}_6. \tag{53}$$

From Eqs. (5), (9) and (29) it follows that

$$X^0(t) \geqslant X^0(t') \geqslant X^0(t_i), \quad \forall t \geqslant t' \geqslant t_i, \tag{54}$$

applicable to both the on and off phases.

In the remainder of this section we concentrate on the on phase to bring out internal symmetry inherent in the model in the on phase. Denote by I_{on} an open, maximal, continuous time interval during which the mechanism of plasticity is on exclusively. From Eqs. (36a) and (32) it is easy to verify that the control tensor **A** in the on phase satisfies

$$\mathbf{A}^{\mathrm{t}}\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}. \tag{55}$$

Hence, the corresponding transformation **G** satisfies (see, e.g. [3])

$$\mathbf{G}^{\mathrm{t}}\mathbf{g}\mathbf{G} = \mathbf{g}, \tag{56}$$

$$\det \mathbf{G} = 1, \tag{57}$$

$$G^0_0 \geqslant 1. \tag{58}$$

Thereby the on-phase control tensor **A** is an element of the real Lie algebra $so(5,1)$ and generates the on-phase transformation **G**, which is thus an element of the proper orthochronous Lorentz group $SO_o(5,1)$. See, for example, Cornwell [15]. So the function $\mathbf{G}(t)$ of time $t \in I_{\mathrm{on}}$ may be viewed as a connected path of the Lorentz group and the algebraic and topological properties of the proper orthochronous Lorentz group are shared by the constitutive model in the on phase.

From Eq. (31a), $\dot{X}^0 > 0$ strictly when the mechanism of plasticity is on; hence,

$$X^0(t) > X^0(t_1), \quad \forall t > t_1, \quad t, t_1 \in I_{\mathrm{on}}, \tag{59}$$

which means that in the sense of irreversibility there exists future-pointing time-orientation from the augmented stress $\mathbf{X}(t_1)$ to $\mathbf{X}(t)$. (Compare Eq. (59) with Eq. (54).) Moreover, such time-orientation is a causal one, because the augmented stress transition formula (51) and inequality (59) establish a *causality relation* between the two augmented stresses $\mathbf{X}(t_1)$ and $\mathbf{X}(t)$ in the sense that the preceding augmented stress $\mathbf{X}(t_1)$ influences the following augmented stress $\mathbf{X}(t)$ according to formula (51). Accordingly, the augmented stress $\mathbf{X}(t_1)$ chronologically and causally precedes the augmented stress $\mathbf{X}(t)$. This is indeed a common property for all models with inherent symmetry of the proper orthochronous Lorentz group. By this symmetry a core connection among irreversibility, the time arrow of evolution, and causality has thus been established for plasticity in the on phase.

We solve Eq. (56) for the inverse

$$\mathbf{G}^{-1} = \mathbf{g}\mathbf{G}^{\mathrm{t}}\mathbf{g} \tag{60}$$

and partition **G** as

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}^s_s & \mathbf{G}^s_0 \\ \mathbf{G}^0_s & G^0_0 \end{bmatrix}, \tag{61}$$

where $\mathbf{G}^s_s$, $\mathbf{G}^s_0$ and $\mathbf{G}^0_s$ are of order 5×5, 5×1 and 1×5, respectively. Thus, Eq. (51) is partitioned into the following matrix equation:

$$\begin{bmatrix} \mathbf{X}^s(t) \\ X^0(t) \end{bmatrix} = \begin{bmatrix} \mathbf{G}^s_s(t)(\mathbf{G}^s_s)^{\mathrm{t}}(t_1) - \mathbf{G}^s_0(t)(\mathbf{G}^s_0)^{\mathrm{t}}(t_1) & \mathbf{G}^s_0(t)G^0_0(t_1) - \mathbf{G}^s_s(t)\mathbf{G}^0_s(t_1) \\ \mathbf{G}^0_s(t)(\mathbf{G}^s_s)^{\mathrm{t}}(t_1) - G^0_0(t)(\mathbf{G}^s_0)^{\mathrm{t}}(t_1) & G^0_0(t)G^0_0(t_1) - \mathbf{G}^0_s(t)\mathbf{G}^0_s(t_1) \end{bmatrix} \begin{bmatrix} \mathbf{X}^s(t_1) \\ X^0(t_1) \end{bmatrix}, \tag{62}$$

which is valid for the on phase.

How can one determine the deviatoric stress tensor $\mathbf{s}(t)$ once he has the augmented stress vector $\mathbf{X}(t)$? From Eq. (29) it follows that

$$\begin{bmatrix} s^{11} \\ s^{22} \\ s^{23} \\ s^{13} \\ s^{12} \end{bmatrix} = \begin{bmatrix} \begin{matrix} a_4 & -a_2 \\ -a_3 & a_1 \end{matrix} & \mathbf{0}_{2\times 3} \\ \mathbf{0}_{3\times 2} & \frac{\sqrt{3}}{2}\mathbf{I}_3 \end{bmatrix} \frac{2\tau_y}{\sqrt{3}X^0}\mathbf{X}^s. \tag{63}$$

This is indeed a projective realization of the response. By this and the on-phase transition (62) one can map $\mathbf{s}(t_1)$ to the current response $\mathbf{s}(t)$.

9. *SE*(5) symmetry in the off phase

Contrary to Eq. (62) of the on phase, the transition for the off phase is very simple. To find it, recall that Eqs. (51)–(53) are still applicable but with

$$\mathbf{G}(t) = \begin{bmatrix} \mathbf{R}(t) & \mathbf{T}(t) \\ \mathbf{0}_{1\times 5} & 1 \end{bmatrix}, \tag{64}$$

where $\mathbf{R}$ and $\mathbf{T}$ are respectively of order 5×5 and 5×1 and governed by

$$\dot{\mathbf{R}} = \mathbf{A}^s_s\mathbf{R}, \quad \dot{\mathbf{T}} = \mathbf{A}^s_s\mathbf{T} + \mathbf{A}^s_0,$$

in which $\mathbf{A}^s_s$ is given in Eq. (37) and $\mathbf{A}^s_0$ in Eq. (34). Thus it is easy to show that $\mathbf{R}\in SO(5)$, $\mathbf{T}\in T(5)$, and $\mathbf{G}\in SE(5)$; therefore, the constitutive law in the off phase has an internal symmetry characterized by the special Euclidean (or proper motion) group $SE(5)$, which is the semi-direct product of the translation group $T(5)$ with the proper rotation group $SO(5)$. Even such an off-phase transformation (64) exists and is invertible, it is no longer an element of the Lorentz group because such $\mathbf{G}$ does not satisfy Eq. (56) although $\det\mathbf{G} = 1$ remains to hold and $G^0_0 = 1$.

The inverse of Eq. (64) is given by

$$\mathbf{G}^{-1} = \begin{bmatrix} \mathbf{R}^{\mathrm{t}} & -\mathbf{R}^{\mathrm{t}}\mathbf{T} \\ \mathbf{0}_{1\times 5} & 1 \end{bmatrix}. \tag{65}$$

Thus according to Eq. (51) we obtain

$$\begin{bmatrix} \mathbf{X}^s(t) \\ X^0(t) \end{bmatrix} = \begin{bmatrix} \mathbf{R}(t)\mathbf{R}^{\mathrm{t}}(t_1) & -\mathbf{R}(t)\mathbf{R}^{\mathrm{t}}(t_1)\mathbf{T}(t_1) + \mathbf{T}(t) \\ \mathbf{0}_{1\times 5} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{X}^s(t_1) \\ X^0(t_1) \end{bmatrix}, \tag{66}$$

which is valid for the off phase. In a similar way the stress response in the off phase can be realized by invoking Eq. (63) and the off-phase transition formula (66).

In summary the stress response is the projective realization (63) of the on-phase transition formula (62) or the off-phase transition formula (66); switching between the two depends upon the control tensor $\mathbf{A}$ and obeys the on–off switching criteria (31a) and (31b). The switching from a transformation of the Lorentz group in the on phase to a non-Lorentzian transformation in the off phase indicates that internal symmetry switches from one kind to another, and vice versa. As a result the constitutive model in the deviatoric stress space of $\mathbf{s}$ has symmetry switching between the special Euclidean group $SE(5)$ acting on the closed 5-ball of admissible states and the projective proper orthochronous Lorentz group $PSO_o(5,1)$ acting on the yield hypersphere.

10. Basis responses to basis controls

In this section we restrict ourselves to the on phase. First let us consider the stress response to a control tensor of single shearing:[5]

$$\mathbf{A}_0^i = \begin{bmatrix} & \vdots & & \vdots \\ \cdots & . & \cdots & 1 \\ & \vdots & & \vdots \\ \cdots & 1 & \cdots & . \end{bmatrix} \tag{67}$$

for $1 \leqslant i \leqslant 5$; namely $\mathbf{A}_0^i$ is equal to $\mathbf{0}_{6\times 6}$ except that the i0th and 0ith entries are both $+1$. To such an elementary control the fundamental solution can be readily found to be

$$\mathbf{G}_0^i = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \cosh t & \cdots & \sinh t \\ & & \vdots & & \vdots \\ & & \sinh t & \cdots & \cosh t \end{bmatrix} \tag{68}$$

for $1 \leqslant i \leqslant 5$, where $t \in \mathbb{R}$; namely $\mathbf{G}_0^i$ is equal to $\mathbf{I}_6$ except that the iith, 00th, i0th and 0ith entries are $\cosh t$, $\cosh t$, $\sinh t$ and $\sinh t$, respectively. Then the stress response is calculated via Eqs. (62) and (63), in which $\mathbf{X}(t_1)$ plays the role of the prescribed initial condition (at time t_1).

For example, if the material specimen is sheared homogeneously and uniformly in the x_1x_2 plane of the external space (x_1, x_2, x_3) at the constant rate of $\gamma_y/2$ per unit time, then $\mathbf{A}$ of Eqs. (36a) and (36b) is equal to $\mathbf{A}_0^5$ of Eq. (67) with $i = 5$. As a result the on-phase transformation $\mathbf{G}$ of Eq. (62) is equal to $\mathbf{G}_0^5$ of Eq. (68).

If the specimen is subjected to a rigid-body rotation at the constant rate:

$$\mathbf{A}_j^i = \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & . & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & -1 & \cdots & . & \cdots \\ & \vdots & & \vdots & \end{bmatrix} \tag{69}$$

for $1 \leqslant i < j \leqslant 5$; namely $\mathbf{A}_j^i$ is equal to $\mathbf{0}_{6\times 6}$ except that the ijth and jith entries are $+1$ and -1, respectively, then the fundamental solution is

[5] For notation do not mix up the basis control tensor $\mathbf{A}_j^i$ with the mixed component A_j^i of the control tensor $\mathbf{A}$, neither the basis response tensor $\mathbf{G}_j^i$ with the mixed component G_j^i of the transformation tensor $\mathbf{G}$.

$$\mathbf{G}^i_j = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & \cos t & \cdots & \sin t & & \\ & & \vdots & & \vdots & & \\ & & -\sin t & \cdots & \cos t & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \tag{70}$$

for $1 \leqslant i < j \leqslant n$, where $t \in \mathbb{R}$; namely $\mathbf{G}^i_j$ is equal to $\mathbf{I}_6$ except that the iith, jjth, ijth and jith entries are $\cos t$, $\cos t$, $\sin t$ and $-\sin t$, respectively. Similarly the stress response is calculated via Eqs. (62) and (63). In fact a rigid-body rotation induces no deformation and stress; therefore, Eq. (70) just reflects this fact and confirms that the constitutive model is frame-indifferent.

The proper orthochronous Lorentz group $SO_o(5, 1)$ acting on the Minkowski spacetime $\mathbb{M}^{5+1}$ endowed with the canonical metric (32) contains five subgroups of the type (68) and fifteen subgroups of the type (70). The basis elements of the real Lie algebra $so(5,1)$ are defined by

$$\mathbf{A}^i_j = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{G}^i_j(t)\bigg|_{t=0}. \tag{71}$$

Hence the real Lie algebra $so(5, 1)$ possesses a convenient basis composed of five basis elements of the type (67) and fifteen basis elements of the type (69).

Any control tensor $\mathbf{A} \in so(5, 1)$ is thus a linear combination of the basis elements (67) and (69):

$$\mathbf{A} = \sum_{1 \leqslant i < j \leqslant n} A^i_j \mathbf{A}^i_j + \sum_{1 \leqslant i \leqslant n} A^i_0 \mathbf{A}^i_0 = [A^i_j], \tag{72}$$

with the mixed components of $\mathbf{A}$ being $A^i_j = -A^j_i$ for $1 \leqslant i \leqslant j \leqslant 5$, $A^i_0 = A^0_i$ for $1 \leqslant i \leqslant 5$ and $A^0_0 = 0$. This is consistent with Eq. (36a) together with Eq. (39) and (33).

The basis element $\mathbf{A}^i_j$ may generate a one-parameter subgroup $\mathbf{G}^i_j(t)$, which corresponds to a rotation in the (X^i, X^j) plane of the Minkowski spacetime $(X^1, X^2, X^3, X^4, X^5, X^0)$ with metric $\mathbf{g}$, while the basis element $\mathbf{A}^i_0$ may generate a one-parameter subgroup $\mathbf{G}^i_0(t)$, which corresponds to a hyperbolic rotation in the (X^i, X^0) plane of the Minkowski spacetime (or a boost in the X^i-direction). The problems encountered in applications are often posed as follows: Given a control tensor $\mathbf{A}$, which contains the information of the normalized deviatoric deformation rate $2\mathbf{D}/\gamma_y$ and the spin $\mathbf{W}$, find the deviatoric stress response $\mathbf{s}$. Thus, we may deem $\mathbf{A}^i_j$, $i, j = 1, 2, 3, 4, 5, 0$, as (constant-valued) basis controls so that $\mathbf{G}^i_j(t)$, $i, j = 1, 2, 3, 4, 5, 0$, are the resulting basis responses. Since the basis controls $\mathbf{A}^i_j$, $i, j = 1, 2, 3, 4, 5, 0$, span the whole control space of $\mathbf{A}$, the resulting basis responses $\mathbf{G}^i_j(t)$, $i, j = 1, 2, 3, 4, 5, 0$, suffice to constitutively characterize the mechanical properties of the material.

11. Consistency scheme

The simplest scheme for Eq. (35) is a time-centered Euler scheme:

$$\mathbf{X}_{n+1} = \mathbf{X}_n + \tau\mathbf{A}(\mathbf{X}_{n+1} + \mathbf{X}_n), \tag{73}$$

where $\mathbf{X}_n$ denotes the value of $\mathbf{X}$ at the discrete time stop t_n and τ is one half of the time increment, that is $\tau := \Delta t/2 = (t_{n+1} - t_n)/2$. Using the Cayley transform we have

$$\mathbf{X}_{n+1} = \mathrm{Cay}(\tau\mathbf{A})\mathbf{X}_n := (\mathbf{I}_6 - \tau\mathbf{A})^{-1}(\mathbf{I}_6 + \tau\mathbf{A})\mathbf{X}_n = [\mathbf{I}_6 + 2\tau(\mathbf{I}_6 - \tau\mathbf{A})^{-1}\mathbf{A}]\mathbf{X}_n. \tag{74}$$

It is easy to check that this transform preserves the properties (56)–(58) of the Lorentz group, i.e., $\mathrm{Cay}(\tau\mathbf{A}) \in SO_o(5, 1)$ if τ is small enough.

A numerical algorithm is called a *group preserving scheme* if for every time increment the map from $\mathbf{X}_n$ to $\mathbf{X}_{n+1}$ preserves the group properties (56)–(58). Now let us investigate what $\mathrm{Cay}(\tau\mathbf{A}) \in SO_o(5, 1)$ implies as a numerical scheme for the constitutive law of plasticity. From Eqs. (29), (56) and (74) it follows that

$$\mathbf{X}^{\mathrm{t}}_{n+1}\mathbf{g}\mathbf{X}_{n+1} = \mathbf{X}^{\mathrm{t}}_n\mathbf{g}\mathbf{X}_n = (X^0_{n+1})^2\left(\frac{\|\mathbf{s}_{n+1}\|^2}{2\tau_y^2} - 1\right) = (X^0_n)^2\left(\frac{\|\mathbf{s}_n\|^2}{2\tau_y^2} - 1\right) = 0. \tag{75}$$

Because $X^0_{n+1} \geqslant X^0_n > 0$, the equalities (75) says nothing but for every time increment the points $\mathbf{s}_n$ and $\mathbf{s}_{n+1}$ are located on the yield hypersphere, i.e., $\|\mathbf{s}_{n+1}\| = \|\mathbf{s}_n\| = \sqrt{2}\tau_y$. In other words, the consistency condition is fulfilled exactly for every time stop in the on phase. This is what the conventional schemes of computational plasticity desired and failed to achieve directly in the stress space. Therefore, the new numerical scheme may be specifically called a *consistency scheme*.

Using Eq. (36a) and the following formula

$$\begin{bmatrix} \beta & \alpha \\ \alpha^{\mathrm{t}} & \beta_{00} \end{bmatrix}^{-1} = \begin{bmatrix} \beta^{-1} + \dfrac{1}{\beta_{00} - \alpha^{\mathrm{t}}\beta^{-1}\alpha}\beta^{-1}\alpha\alpha^{\mathrm{t}}\beta^{-1} & -\dfrac{1}{\beta_{00} - \alpha^{\mathrm{t}}\beta^{-1}\alpha}\beta^{-1}\alpha \\ -\dfrac{1}{\beta_{00} - \alpha^{\mathrm{t}}\beta^{-1}\alpha}\alpha^{\mathrm{t}}\beta^{-}1 & \dfrac{1}{\beta_{00} - \alpha^{\mathrm{t}}\beta^{-1}\alpha} \end{bmatrix}, \tag{76}$$

where β is a non-singular 5×5 matrix, α is a 5×1 matrix, β_{00} is a scalar such that $\beta_{00} \neq \alpha^{\mathrm{t}}\beta^{-1}\alpha$, we find the inverse of $\mathbf{I}_6 - \tau\mathbf{A}$ to be

$$(\mathbf{I}_6 - \tau\mathbf{A})^{-1} = \begin{bmatrix} \mathbf{B} + c\tau^2\mathbf{B}\mathbf{A}^s_0\mathbf{A}^0_s\mathbf{B} & c\tau\mathbf{B}\mathbf{A}^s_0 \\ c\tau\mathbf{A}^0_s\mathbf{B} & c \end{bmatrix}, \tag{77}$$

where

$$\mathbf{B} := (\mathbf{I}_5 - \tau\mathbf{A}^s_s)^{-1}, \quad c := \frac{1}{1 - \tau^2\mathbf{A}^0_s\mathbf{B}\mathbf{A}^s_0}. \tag{78}$$

Substituting Eqs. (77) and (36a) into Eq. (74) we obtain the discretized transition matrix in the on phase,

$$\mathrm{Cay}(\tau\mathbf{A}) = \begin{bmatrix} \mathbf{I}_5 + 2c\tau^3\mathbf{B}\mathbf{A}^s_0\mathbf{A}^0_s\mathbf{B}\mathbf{A}^s_s & 2c\tau^3\mathbf{B}\mathbf{A}^s_0\mathbf{A}^0_s\mathbf{B}\mathbf{A}^s_0 \\ \quad + 2\tau\mathbf{B}\mathbf{A}^s_s + 2c\tau^2\mathbf{B}\mathbf{A}^s_0\mathbf{A}^0_s & \quad + 2\tau\mathbf{B}\mathbf{A}^s_0 \\ 2c\tau^2\mathbf{A}^0_s\mathbf{B}\mathbf{A}^s_s + 2c\tau\mathbf{A}^0_s & 1 + 2c\tau^2\mathbf{A}^0_s\mathbf{B}\mathbf{A}^s_0 \end{bmatrix}. \tag{79}$$

The mapping in the off phase is obtained by Eqs. (74) and (36b) as in the following:

$$\mathbf{X}_{n+1} = \begin{bmatrix} \mathbf{I}_5 + 2\tau\mathbf{B}\mathbf{A}^s_s & 2\tau\mathbf{B}\mathbf{A}^s_0 \\ \mathbf{0}_{1\times 5} & 1 \end{bmatrix}\mathbf{X}_n, \tag{80}$$

from which it is obvious that $X^0_{n+1} = X^0_n$ in the off phase.

Note that $\mathbf{A}^s_s$ satisfies

$$(\mathbf{A}^s_s)^5 + 5w^2(\mathbf{A}^s_s)^3 + 4w^4\mathbf{A}^s_s = \mathbf{0}, \tag{81}$$

where

$$w := \sqrt{W^2_{12} + W^2_{13} + W^2_{23}} \tag{82}$$

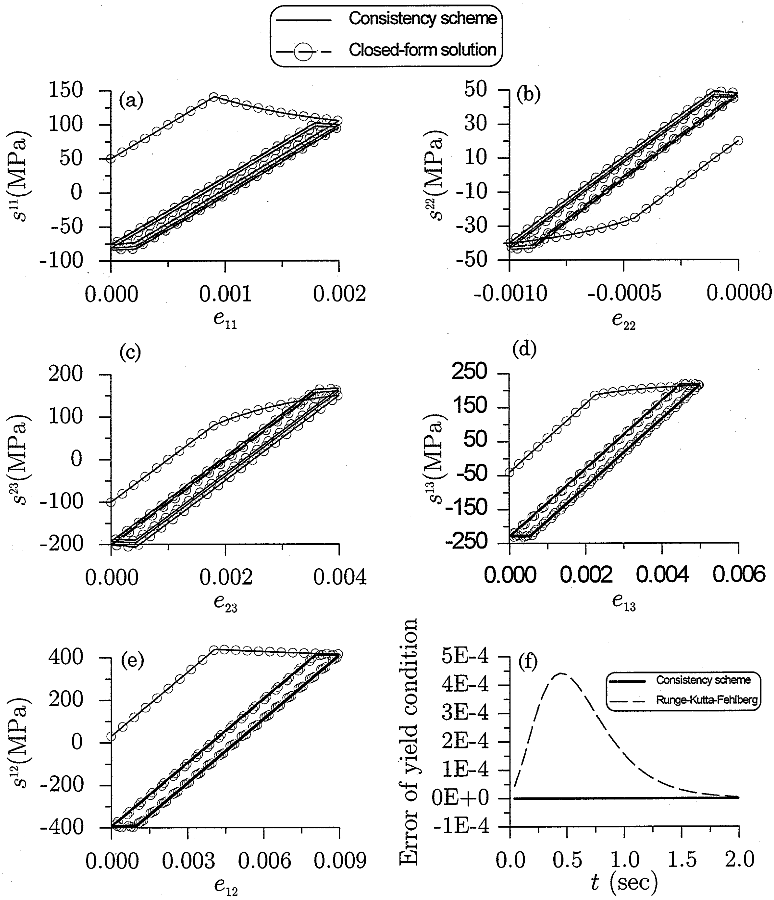

Fig. 1. In (a)–(e) the stress responses for the three straining cycles calculated by the consistency scheme are compared with the closed-form solutions. In (f) the errors of the fulfillment of the yield condition by using the consistency scheme and the Runge–Kutta–Fehlberg method are compared for the case where the initial and subsequent stresses are all located on the yield surface.

is the Euclidean norm of the axial vector of $\mathbf{W}$. By formula (81), $\mathbf{B}$ in terms of the powers of $\mathbf{A}_s^s$ can be expressed as follows:

$$\mathbf{B} = (\mathbf{I}_5 - \tau\mathbf{A}_s^s)^{-1} = \mathbf{I}_5 + d_1\mathbf{A}_s^s + d_2(\mathbf{A}_s^s)^2 + d_3(\mathbf{A}_s^s)^3 + d_4(\mathbf{A}_s^s)^4, \tag{83}$$

where

$$d_1 := \frac{\tau + 5w^2\tau^3}{1 + 5w^2\tau^2 + 4w^4\tau^4}, \quad d_2 := \frac{\tau^2 + 5w^2\tau^4}{1 + 5w^2\tau^2 + 4w^4\tau^4},$$
$$d_3 := \frac{\tau^3}{1 + 5w^2\tau^2 + 4w^4\tau^4}, \quad d_4 := \frac{\tau^4}{1 + 5w^2\tau^2 + 4w^4\tau^4}. \tag{84}$$

Once $\mathbf{X}_n$ is calculated at each time stop formula (63) gives the value of the response $\mathbf{s}_n$ at each time stop.

In order to gain the insight of the consistency scheme, and to understand what makes a difference in applying the new scheme from applying the Runge–Kutta–Fehlberg method [16] to solve Eq. (18a) directly, let us consider the case where both **D** and **W** are constant matrices. For such inputs the closed-form solution of the responses can be derived, so we can compare the results calculated by the two schemes with the closed-form solution. The material constants used in the calculations were $G = 50\,000$ MPa and $\tau_y = 500$ MPa. We applied three cycles, each cycle consisting of constant **D** and **W** for one second and then negative constant **D** and **W** for another one second, with

$$\mathbf{D} = \begin{bmatrix} 0.002 & 0.009 & 0.005 \\ 0.009 & -0.001 & 0.004 \\ 0.005 & 0.004 & -0.001 \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} 0 & 0.001 & 0.002 \\ -0.001 & 0 & -0.005 \\ -0.002 & 0.005 & 0 \end{bmatrix}.$$

Fig. 1a–Fig. 1e show that the consistency scheme gave very accurate responses in all respects, where the initial stresses were taken to be $s^{11} = 50$ MPa, $s^{22} = 20$ MPa, $s^{23} = -100$ MPa, $s^{13} = -40$ MPa, and $s^{12} = 30$ MPa. Furthermore, Fig. 1f shows that the consistency scheme supplied a more acurate result of the yield condition, $\|\mathbf{s}\| = \sqrt{2}\tau_y$, than the one calculated by the Runge–Kutta–Fehlberg scheme, where the initial stresses were taken to be located on the yield surface with $s^{11} = 300$ MPa, $s^{22} = 0$ MPa, $s^{23} = 400$ MPa, $s^{13} = 0$ MPa, and $s^{12} = 0$ MPa.

12. Conclusions

In this paper we have investigated internal symmetry inherent in a constitutive model of perfect elastoplasticity with large deformation. Even though the constitutive equations are non-linear in the deviatoric stress space of **s**, they can be converted to a linear system $\dot{\mathbf{X}} = \mathbf{AX}$ in the $(5 + 1)$-dimensional augmented stress space of **X**. In the augmented stress space not only the non-linearity of the model is unfolded, but also an internal spacetime structure of the Minkowskian type is brought out. The control tensor **A** for the on phase was proved to be an element of the real Lie algebra $so(5, 1)$ of the proper orthochronous Lorentz group $SO_o(5, 1)$, and the fundamental solution **G** of the system $\dot{\mathbf{X}} = \mathbf{AX}$ with the on-phase **A** was shown to be an element of the proper orthochronous Lorentz group, so that the causality relation of the augmented stresses was verified. To account for both the on and off phases we constructed a composite space endowed with a Minkowskian metric on the cone but with a Euclidean metric on each of the discs inside the cone. As a result we found that the perfect elastoplastic model with large deformation possesses two kinds of symmetry – $SE(5)$ in the off phase and $PSO_o(5, 1)$ in the on phase – and has symmetry switching between the two depending on the control input. The (basis) control tensors of single shearing and of rigid-body rotation were found to form a convenient basis of the real Lie algebra $so(5, 1)$, and the corresponding (basis) responses to be the corresponding subgroups of $SO_o(5, 1)$.

Based on the symmetry study, a numerical scheme which preserves the group properties for every time increment was developed. This group preserving scheme may be specifically called a consistency scheme, since it is capable, among other benefits derivable from the group properties, of updating the stress point to be automatically located on the yield surface at the end of each time increment in the on phase without any iterative calculations, that is, the consistency condition is fulfilled automatically and exactly. Moreover, another benefit derived from the group properties is that the new scheme is exactly linear in the augmented stress space, because $SO_o(5, 1)$ is a subgroup of the general linear group $GL(6, \mathbb{R})$. In this regard, the conventional numerical schemes typically do not share the group properties so that perform less accurate than the consistency scheme. Since the new scheme is easy to implement numerically and has high computational efficiency and accuracy, it is highly recommended especially for engineering applications which demand intensive calculations.

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References

[1] H.-K. Hong, C.-S. Liu, Analyses, experimentation and identification of dynamic system of elastoplasticity (2), Report prepared for the National Science Council Project No. NSC85-2211-E-002-001, Department of Civil Engineering, Taiwan University, Taipei, 1996.
[2] H.-K. Hong, C.-S. Liu, Internal symmetry in a constitutive model of perfect elastoplasticity, submitted for publication.
[3] H.-K. Hong, C.-S. Liu, Internal symmetry in bilinear elastoplasticity, Int. J. Non-Linear Mech. 34 (1999) 279–288.
[4] R.D. Krieg, D.B. Krieg, Accuracies of numerical solution methods for the elastic-perfectly plastic model, J. Press. Vess. Tech., ASME 99 (1977) 510–515.
[5] H.L. Schreyer, R.F. Kulak, J.M. Kramer, Accurate numerical solutions for elastic-plastic models, J. Press. Vess. Tech., ASME 101 (1979) 226–234.
[6] M. Ortiz, E.P. Popov, Accuracy and stability of integration algorithms for elastoplastic constitutive relations, Int. J. Numer. Meth. Engng 21 (1985) 1561–1567.
[7] J.C. Simo, R.L. Taylor, Consistent tangent operators for rate-independent elastoplasticity, Comput. Meth. Appl. Mech. Engng 48 (1985) 101–118.
[8] S. Nemat-Nasser, Rate-independent finite-deformation elastoplasticity: a new explicit constitutive algorithm, Mech. Mater. 11 (1991) 235–249.
[9] L. Prandtl, Spannungsverteilung in plastischen kœrpern, in: Proc. 1st Int. Congress on Applied Mechanics, Delft, 1924 pp. 43–54.
[10] E. Reuss, Beruecksichtigung der elastischen formaenderungen in der plastizitaetstheorie, Zeits. angew. Math. Mech. (ZAMM) 10 (1930) 266–274.
[11] H.-K. Hong, C.-S. Liu, Prandtl–Reuss elastoplasticity: on–off switch and superposition formulae, Int. J. Solids Struct. 34 (1997) 4281–4304.
[12] M.M. Mehrabadi, S.C. Cowin, J. Jaric, Six-dimensional orthogonal tensor representation of the rotation about an axis in three dimensions, Int. J. Solids Struct. 32 (1995) 439–449.
[13] A.A. Il'yushin, Plasticity, Foundations of general mathematical theory, Akad. Nauk S.S.S.R., Moscow, 1963 (in Russian).
[14] Y. Ohashi, M. Tokuda, H. Yamashita, Effect of third invariant of stress deviator on plastic deformation of mild steel, J. Mech. Phys. Solids 23 (1975) 295–323.
[15] J.F. Cornwell, Group Theory in Physics, vol. 2, Academic, London, 1984.
[16] C.F. Gerald, P.O. Wheatley, Applied Numerical Analysis, 3rd ed., Addison-Wesley, Reading, MA, 1984.