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Non-oscillation Criteria for Hypoelastic Models under Simple Shear Deformation

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Abstract. Ten objective rates, spinning or non-spinning, are critically examined from the viewpoint of Sturm's theorems in ordinary differential equations. Upon developing implication relations of oscillatory, non-oscillatory, and disconjugate behavior, we establish oscillation and non-oscillation criteria which pick out the objective stress rates that lead to oscillatory and non-oscillatory responses in simple shear deformation, respectively. Among the hypoelastic equations associated with the spinning objective rates examined, the Jaumann equation is an oscillatory minorant, the homogeneous Xiao–Bruhns–Meyers equation is a non-oscillatory majorant, and the homogeneous Green–Naghdi equation is a disconjugate majorant. If (Sturm comparable) non-spinning objective rates are also taken into consideration, then the Durban–Baruch equation becomes an oscillatory minorant, but the other two equations remain to play the same roles. The Jaumann equation is a Sturm majorant for all the other nine homogeneous hypoelastic equations, and the homogeneous Szabó–Balla-2 equation is a Sturm minorant for all the other nine homogeneous hypoelastic equations. Most of the solutions of the zeroth-grade hypoelastic equations at simple shear have already been published, except for those of Szabó and Balla, to which the closed-form solutions are derived here. Moreover, all solutions are extended to include the effect of initial stresses.

Key words: objective rates, logarithmic spin, hypoelasticity, non-oscillation criteria, comparison theorem.

1. Introduction

The study of objective rates is one of the major topics in constitutive modeling. In the last two decades, the problem of anomalously oscillatory stresses has been disclosed using finite deformation models; the Jaumann rate of the stress tensor employed in some constitutive models has been thought to be unsuitable. Dienes [1] was the first to reveal that the model of hypoelasticity (of grade zero) based on the Jaumann stress rate may result in an unstable response to simple shearing, that is, the response may be *shear oscillatory*. Due to the inappropriateness of the Jaumann stress rate for the rate-form constitutive model, considerable efforts have since been made to resolve the problem of choosing an appropriate objective stress rate in rate-form constitutive equations (e.g., [1, 2]). Several objective stress rates, such as the

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Green–Naghdi rate [3], the Sowerby–Chu rate [4], and the Xiao–Bruhns–Meyers rate [5, 6], have been suggested and shown to be plausible through the simple shear response of the constitutive models based on those rates. Although the hypoelastic equations had been successfully solved in closed-forms for most of the objective stress rates proposed in the literature, the questions of shear oscillation more than often could only be answered by using numerical cases with the aid of graphical displaying of the closed-form solutions. It is still not yet clear what criteria can be introduced to pick out an appropriate rate. To establish the criteria, it seems desirable to explore further the qualitative properties of all the objective rates in general, in addition to paying attention to the quantitative responses of each individual rate. In the present paper, we compare general spinning, general non-spinning, and ten individual hypoelastic equations (of grade zero) under simple shear deformation, with recourse to certain pertinent notions in qualitative theory of ordinary differential equations (ODEs). It is found that all the spinning hypoelastic equations can be recast as scalar ODEs of the second order (see (8.1) below), which are non-homogeneous and with variable or constant coefficients, and, moreover, all the hypoelastic equations, spinning or non-spinning, can be recast as tensorial ODEs of the first order (see (6.2) below), which are non-homogeneous (but with constant tensor forcing terms) and with variable coefficients, and whose components form 2×2 matrix ODEs. As such, Sturm's theorems for homogeneous second order ODEs can be invoked and further extended to compare the stress rates and to provide criteria for oscillatory and non-oscillatory responses.

Sections 2 and 3 introduce the rates and equations to be investigated; in particular, Section 3 outlines two procedures for finding the logarithmic spin. Section 4 limits the scope of study to simple shear deformation, listing the related kinematic quantities. Sections 5 and 6 present the closed-form solutions of ten hypoelastic equations with nonzero initial stresses taken into account. Sections 7 and 8 compare spinning hypoelastic equations, and Sections 9 and 10 extend the comparisons to non-spinning, in which the former (Sections 7 and 9) prepare mathematical apparatus whereas the latter (Sections 8 and 10) explore mechanical implications.

2. Objective Stress Rates

The integral representation of the hypoelastic equation

$$\overset{\circ}{\boldsymbol{\sigma}} = 2\mu\mathbf{D} + \lambda(\operatorname{tr}\mathbf{D})\mathbf{I}_3 \tag{2.1}$$

is

$$\begin{aligned}\boldsymbol{\sigma}(t) \;=\; & \boldsymbol{\Phi}(t,0)\boldsymbol{\sigma}(0)\boldsymbol{\Phi}^{\mathrm{T}}(t,0) \\ & + \int_0^t \boldsymbol{\Phi}(t,\tau)\big[2\mu\mathbf{D}(\tau) + \lambda\big(\operatorname{tr}\mathbf{D}(\tau)\big)\mathbf{I}_3\big]\boldsymbol{\Phi}^{\mathrm{T}}(t,\tau)\,\mathrm{d}\tau,\end{aligned} \tag{2.2}$$

where the initial stress $\boldsymbol{\sigma}(0)$ is prescribed at the initial time $t = 0$. The Lamé material parameters μ and λ are the only two property constants needed in the isotropic hypoelastic model. It is postulated that $0 < \mu < \infty$ and $-2\mu/3 < \lambda < \infty$. Throughout this paper, the superscript $^{\mathrm{T}}$ denotes the transpose, tr stands for the trace, and $\mathbf{I}_n$ is the identity tensor of order n. A surmounted circle "$\circ$" represents the objective rate, which will be discussed below.

The state transition $\boldsymbol{\Phi}(t,\tau)$ in (2.2) is related to the fundamental solution $\boldsymbol{\Psi}(t)$ in the following way

$$\boldsymbol{\Phi}(t,\tau) := \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(\tau), \tag{2.3}$$

where, by definition, $\boldsymbol{\Psi}(t)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Psi}(t) = \mathbf{A}(t)\boldsymbol{\Psi}(t), \qquad \boldsymbol{\Psi}(0) = \mathbf{I}_3, \tag{2.4}$$

for a specific $\mathbf{A}(t)$. Through this $\mathbf{A}$ one obtains the objective rate of the Cauchy stress tensor $\boldsymbol{\sigma}$ as follows [7]:

$$\overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - 2(\mathbf{A}\boldsymbol{\sigma})_{\mathrm{sym}}, \tag{2.5}$$

where the subscript "sym" denotes the symmetric part, and a surmounted dot denotes the material time derivative. In order to ensure the objectivity of the rate, the specific $\mathbf{A}$ must confirm that

$$\overline{\mathbf{A}} = \mathbf{Q}\mathbf{A}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}, \tag{2.6}$$

where $\overline{\mathbf{A}}$ is the observation of $\mathbf{A}$ from a frame rotated with respect to the original frame by an arbitrary orthogonal transformation $\mathbf{Q}$. When $\mathbf{A}$ takes different definitions, correspondingly $\overset{\circ}{\boldsymbol{\sigma}}$ stands for different objective rates as shown in Table I. References [3–12] are referred to for the first appearance of these rates.

The notation is summarized in the following: $\mathbf{L} := \dot{\mathbf{F}}\mathbf{F}^{-1}$ is the velocity gradient tensor, where $\mathbf{F}$ is the two-point tensor of deformation gradient; $\mathbf{D}$ and $\mathbf{W}$ are the

Table I. Ten objective stress rates

	Objective stress rates $\overset{\circ}{\boldsymbol{\sigma}}$	$\mathbf{A}$	$\overset{\circ}{\boldsymbol{\sigma}}$ spinning?
1.	Truesdell (T)	$\mathbf{L} - \frac{1}{2}(\mathrm{tr}\,\mathbf{D})\mathbf{I}_3$	non-spinning
2.	Oldroyd (O)	$\mathbf{L}$	non-spinning
3.	Cotter–Rivlin (CR)	$-\mathbf{L}^{\mathrm{T}}$	non-spinning
4.	Jaumann (J)	$\mathbf{W}$	spinning
5.	Durban–Baruch (DB)	$\frac{1}{2}\mathbf{D} + \mathbf{W} - \frac{1}{2}(\mathrm{tr}\,\mathbf{D})\mathbf{I}_3$	non-spinning
6.	Green–Naghdi (GN)	$\boldsymbol{\Omega}$	spinning
7.	Sowerby–Chu (SC)	$\boldsymbol{\Omega}_E$	spinning
8.	Szabó–Balla-1 (SB1)	$\mathbf{L}_E$	non-spinning
9.	Szabó–Balla-2 (SB2)	$-\mathbf{L}_E^{\mathrm{T}}$	non-spinning
10.	Xiao–Bruhns–Meyers (XBM)	$\boldsymbol{\Omega}^{\log}$	spinning

symmetric and skew-symmetric parts of $\mathbf{L}$, respectively. $\boldsymbol{\Omega} := \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$ is the rate of rotation, where $\mathbf{R}$ is the rotation, the orthogonal tensor in the polar decomposition $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$ of $\mathbf{F}$. $\boldsymbol{\Omega}_E := \dot{\mathbf{R}}_E\mathbf{R}_E^{\mathrm{T}}$ is known as the Eulerian spin tensor, where $\mathbf{R}_E$ is the diagonal transformation of $\mathbf{V}$, that is,

$$\mathbf{V} = \mathbf{R}_E\boldsymbol{\lambda}\mathbf{R}_E^{\mathrm{T}}, \tag{2.7}$$

with $\boldsymbol{\lambda} = \mathrm{diag}[\lambda_1, \lambda_2, \lambda_3]$ the diagonal tensor containing the eigenvalues, $\lambda_1, \lambda_2, \lambda_3$, of $\mathbf{V}$; $\mathbf{L}_E$ is defined by

$$\mathbf{L}_E := \dot{\mathbf{V}}\mathbf{V}^{-1} + \mathbf{V}\boldsymbol{\Omega}_E\mathbf{V}^{-1}.$$

The logarithmic spin $\boldsymbol{\Omega}^{\log}$ will be introduced in the next section.

The $\mathbf{A}$ is grossly referred to as the general spin if $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$. Accordingly, the objective rates can be grouped into two main classes, *spinning* and *non-spinning*. An objective stress rate $\mathring{\boldsymbol{\sigma}}$ is said to be spinning if $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$; otherwise, it is non-spinning. Table I lists ten objective stress rates and the corresponding $\mathbf{A}$'s. In the literature, the spinning rates are usually called the corotational rates.

3. The Logarithmic Spin

The logarithmic spin $\boldsymbol{\Omega}^{\log}$ was introduced recently in order that

$$\mathbf{D} = (\ln\mathbf{V})^{\cdot} - \boldsymbol{\Omega}^{\log}\ln\mathbf{V} + (\ln\mathbf{V})\boldsymbol{\Omega}^{\log}, \tag{3.1}$$

where $(\ln\mathbf{V})^{\cdot}$ denotes the material time derivative of the Eulerian logarithmic strain tensor $\ln\mathbf{V}$ [5, 6, 13, 14]. With the logarithmic spin $\boldsymbol{\Omega}^{\log}$, the logarithmic rate of any Eulerian symmetric tensor, say $\mathbf{G}$, is defined by

$$\mathring{\mathbf{G}}^{\log} := \dot{\mathbf{G}} - \boldsymbol{\Omega}^{\log}\mathbf{G} + \mathbf{G}\boldsymbol{\Omega}^{\log}, \tag{3.2}$$

which indeed generalizes the concept of the objective stress rates (2.5), but specializes $\mathbf{A}$ to be $\boldsymbol{\Omega}^{\log}$. In particular, $\mathring{\boldsymbol{\sigma}}^{\log}$ is referred to later as the stress rate of Xiao–Bruhns–Meyers, or simply as the Xiao–Bruhns–Meyers rate [5, 6]. Note that $\mathring{\boldsymbol{\sigma}} = \mathring{\boldsymbol{\sigma}}^{\log}$ if $\mathbf{A} = \Omega^{\log}$. Thus it is remarkable that (3.1) can be written as

$$\mathbf{D} = (\ln\mathring{\mathbf{V}})^{\log} \tag{3.3}$$

and, moreover,

$$\boldsymbol{\sigma} = 2\mu\,\ln\mathbf{V} + \lambda\big(\mathrm{tr}\,(\ln\mathbf{V})\big)\mathbf{I}_3 \tag{3.4}$$

is found to be the solution of the hypoelastic equation

$$\mathring{\boldsymbol{\sigma}}^{\log} = 2\mu\mathbf{D} + \lambda(\mathrm{tr}\,\mathbf{D})\mathbf{I}_3 \tag{3.5}$$

with the initial condition $\boldsymbol{\sigma}(0) = \mathbf{0}$.

If $\ln \mathbf{V}$ has three distinct eigenvalues, we obtain the solution of (3.1) based on the formulae of Guo et al. [15], in particular, on corollary 2.8 and equation (2.16) therein:

$$\begin{aligned} \mathbf{\Omega}^{\log} = \; & \beta_1\big[\ln \mathbf{V}\big(\mathbf{D}-(\ln \mathbf{V})^{\cdot}\big)-\big(\mathbf{D}-(\ln \mathbf{V})^{\cdot}\big)\ln \mathbf{V}\big] \\ & +\beta_2\big[(\ln \mathbf{V})^2\big(\mathbf{D}-(\ln \mathbf{V})^{\cdot}\big)-\big(\mathbf{D}-(\ln \mathbf{V})^{\cdot}\big)(\ln \mathbf{V})^2\big] \\ & +\beta_3\big[(\ln \mathbf{V})^2\big(\mathbf{D}-(\ln \mathbf{V})^{\cdot}\big)\ln \mathbf{V}-\ln \mathbf{V}\big(\mathbf{D}-(\ln \mathbf{V})^{\cdot}\big)(\ln \mathbf{V})^2\big], \end{aligned} \tag{3.6}$$

where

$$\beta_1 := \frac{6I\,III-5I^2II+I^4+4II^2}{\Delta_1}, \tag{3.7}$$

$$\beta_2 := \frac{4I\,II-I^3-9III}{\Delta_1}, \tag{3.8}$$

$$\beta_3 := \frac{I^2-3II}{\Delta_1}, \tag{3.9}$$

in which $I := \operatorname{tr}(\ln \mathbf{V})$, $II := [(\operatorname{tr}(\ln \mathbf{V}))^2-\operatorname{tr}(\ln \mathbf{V})^2]/2$, and $III := \det(\ln \mathbf{V})$ are the principal invariants of $\ln \mathbf{V}$, and $\Delta_1 := 18I\,II\,III+I^2II^2-4I^3III-4II^3-27III^2$. If $\ln \mathbf{V}$ has only two distinct eigenvalues, we obtain the solution of (3.1) as follows:

$$\mathbf{\Omega}^{\log} = \frac{1}{I^2-3II}\big[\ln \mathbf{V}\big(\mathbf{D}-(\ln \mathbf{V})^{\cdot}\big)-\big(\mathbf{D}-(\ln \mathbf{V})^{\cdot}\big)\ln \mathbf{V}\big]. \tag{3.10}$$

If the three eigenvalues of $\ln \mathbf{V}$ are coincident, i.e., $\lambda_1=\lambda_2=\lambda_3$, we may set $\mathbf{\Omega}^{\log}=\mathbf{\Omega}=\mathbf{W}$.

Formulae (3.6) and (3.10) for the logarithmic spin are expressed in terms of $\ln \mathbf{V}$, $(\ln \mathbf{V})^{\cdot}$ and $\mathbf{D}$. It seems difficult to calculate $(\ln \mathbf{V})^{\cdot}$ in more general cases, as pointed out in [5, 6, 16], and an alternative formula in terms of $\mathbf{B}$ and $\mathbf{D}$ has been derived in [5, 6, 17] as follows:

$$\mathbf{\Omega}^{\log} = \mathbf{W}+\mathbf{N}^{\log}, \tag{3.11}$$

where

$$\mathbf{N}^{\log} = \begin{cases} \mathbf{0} & \text{if } b_1=b_2=b_3, \\ \nu[\mathbf{BD}] & \text{if } b_1\neq b_2=b_3, \\ \nu_1[\mathbf{BD}]+\nu_2\big[\mathbf{B}^2\mathbf{D}\big]+\nu_3\big[\mathbf{B}^2\mathbf{DB}\big] & \text{if } b_1\neq b_2\neq b_3, \end{cases} \tag{3.12}$$

in which $\mathbf{B} := \mathbf{V}^2$ is the left Cauchy–Green tensor and $b_i=\lambda_i^2$, $i=1,2,3$, are its eigenvalues. The bracket is defined here by

$$\big[\mathbf{B}^r\mathbf{DB}^s\big] := \mathbf{B}^r\mathbf{DB}^s-\mathbf{B}^s\mathbf{DB}^r \tag{3.13}$$

for $r, s = 0, 1, 2$. The other parameters in (3.12) are

$$\nu = \frac{1}{b_1 - b_2}\left(\frac{1 + b_1/b_2}{1 - b_1/b_2} + \frac{2}{\ln b_1/b_2}\right), \tag{3.14}$$

$$\nu_k = -\frac{1}{\Delta}\sum_{i=1}^{3}(-b_i)^{3-k}\left(\frac{1 + \epsilon_i}{1 - \epsilon_i} + \frac{2}{\ln \epsilon_i}\right), \quad k = 1, 2, 3, \tag{3.15}$$

$$\Delta = (b_1 - b_2)(b_2 - b_3)(b_3 - b_1), \tag{3.16}$$

$$\epsilon_1 = \frac{b_2}{b_3}, \qquad \epsilon_2 = \frac{b_3}{b_1}, \qquad \epsilon_3 = \frac{b_1}{b_2}. \tag{3.17}$$

Notice that the minus sign before $1/\Delta$ in (3.15) was missing in [5, 6] and was corrected later in [17].

REMARKS. Of the above two procedures for finding the logarithmic spin $\mathbf{\Omega}^{\log}$, the latter procedure of (3.11)–(3.17) seems more suitable for calculation; however, the former procedure of (3.6)–(3.10) will be shown essential for the qualitative study to be performed in Section 8.3.

4. Simple Shear Deformation

In what follows, we shall study the above ten hypoelastic models under simple shear deformation (Figure 1):

$$\mathbf{x} = (X_1 + \gamma X_2)\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3, \quad \gamma \in [0, \infty), \tag{4.1}$$

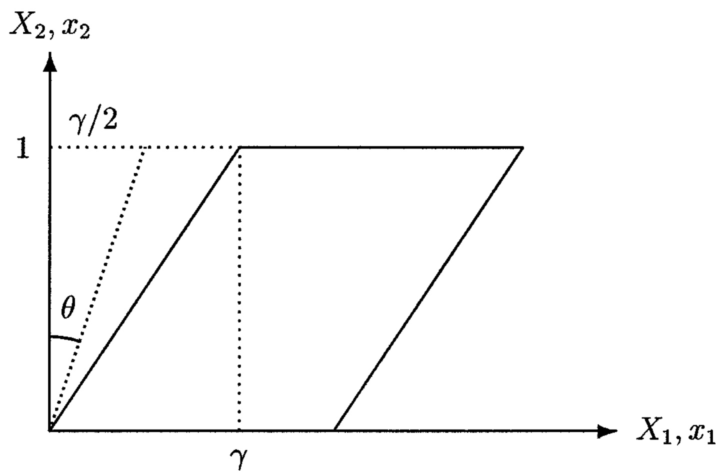

Figure 1. The simple shear deformation.

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a fixed orthonormal basis; $\mathbf{X} = \sum_{i=1}^{3} X_i \mathbf{e}_i$ and $\mathbf{x} = \sum_{i=1}^{3} x_i \mathbf{e}_i$ are the initial and the current position vectors of a material element, respectively. The deformation gradient of this deformed element is

$$\mathbf{F} = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}. \tag{4.2}$$

Let

$$\theta := \arctan\left(\frac{\gamma}{2}\right), \qquad \dot{\theta} = \frac{2\dot{\gamma}}{\gamma^2 + 4}, \qquad \theta \in \left[0, \frac{\pi}{2}\right). \tag{4.3}$$

See Figure 1. The related kinematic quantities are listed below:

$$\begin{aligned}
&\mathbf{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad \mathbf{U} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \dfrac{1 + \sin^2\theta}{\cos\theta} \end{pmatrix}, \\
&\mathbf{V} = \begin{pmatrix} \dfrac{1 + \sin^2\theta}{\cos\theta} & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \\
&\mathbf{L} = \begin{pmatrix} 0 & \dot{\gamma} \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{D} = \frac{\dot{\gamma}}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{W} = \frac{\dot{\gamma}}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
&\mathbf{\Omega} = \begin{pmatrix} 0 & \dot{\theta} \\ -\dot{\theta} & 0 \end{pmatrix}, \qquad \mathbf{L}_E = \frac{\dot{\gamma}}{\gamma^2 + 4}\begin{pmatrix} \gamma & 3 \\ 1 & -\gamma \end{pmatrix}, \\
&\mathbf{\Omega}_E = \begin{pmatrix} 0 & \dfrac{\dot{\theta}}{2} \\ -\dfrac{\dot{\theta}}{2} & 0 \end{pmatrix}, \qquad \mathbf{R}_E = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{1 + \sin\theta} & -\sqrt{1 - \sin\theta} \\ \sqrt{1 - \sin\theta} & \sqrt{1 + \sin\theta} \end{pmatrix}.
\end{aligned} \tag{4.4}$$

For saving space, the third column and third row of each 3×3 matrix are not written out explicitly here and in the sequel, since their components are either 0, 0, 1 or 0, 0, 0.

5. Responses for Spinning Objective Rates

Let us consider

$$\mathbf{A} = \dot{\alpha}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.1}$$

for the spinning rates, which include the rates of Jaumann, Green–Naghdi, Sowerby–Chu and Xiao–Bruhns–Meyers as special cases. Here 2α is the corotational angle induced by shear deformation. Since $\operatorname{tr}\mathbf{D} = 0$, from (2.1) and (2.5), we get

$$\begin{aligned}
&\dot{\gamma}\begin{pmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{12} & \sigma'_{22} \end{pmatrix} - \dot{\alpha}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \\
&\quad + \dot{\alpha}\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mu\begin{pmatrix} 0 & \dot{\gamma} \\ \dot{\gamma} & 0 \end{pmatrix},
\end{aligned} \tag{5.2}$$

where the prime denotes differentiation with respect to γ. The above differential equations can be rewritten as

$$\dot{\gamma}\sigma_{11}' - 2\dot{\alpha}\sigma_{12} = 0, \quad \dot{\gamma}\sigma_{12}' + \dot{\alpha}(\sigma_{11} - \sigma_{22}) = \mu\dot{\gamma}, \quad \dot{\gamma}\sigma_{22}' + 2\dot{\alpha}\sigma_{12} = 0. \tag{5.3}$$

The combination of the first and third equations leads to $\sigma_{11}' + \sigma_{22}' = 0$; thus, $\sigma_{11} + \sigma_{22} = \sigma_{11}(0) + \sigma_{22}(0)$. It follows that

$$\dot{\gamma}\sigma_{11}' = 2\dot{\alpha}\sigma_{12}, \qquad \dot{\gamma}\sigma_{12}' = \mu\dot{\gamma} - 2\dot{\alpha}\sigma_{11} + \dot{\alpha}\big[\sigma_{11}(0) + \sigma_{22}(0)\big]. \tag{5.4}$$

Let

$$2\dot{\alpha} := f(\gamma)\dot{\gamma}, \tag{5.5}$$

in which $f(\gamma) > 0$. Thus, (5.4) changes to

$$\sigma_{11}' = f(\gamma)\sigma_{12}, \qquad \sigma_{12}' = \mu - f(\gamma)\left[\sigma_{11} - \frac{\sigma_{11}(0) + \sigma_{22}(0)}{2}\right]. \tag{5.6}$$

The solutions to the above equations are

$$\begin{aligned}
\sigma_{11}(\gamma) \;=\; & \frac{\sigma_{11}(0) + \sigma_{22}(0)}{2} + \frac{\sigma_{11}(0) - \sigma_{22}(0)}{2}\cos H(\gamma) \\
& + \sigma_{12}(0)\sin H(\gamma) - \mu\cos H(\gamma)\int_0^\gamma \sin H(s)\,\mathrm{d}s \\
& + \mu\sin H(\gamma)\int_0^\gamma \cos H(s)\,\mathrm{d}s,
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
\sigma_{12}(\gamma) \;=\; & \frac{\sigma_{22}(0) - \sigma_{11}(0)}{2}\sin H(\gamma) + \sigma_{12}(0)\cos H(\gamma) \\
& + \mu\sin H(\gamma)\int_0^\gamma \sin H(s)\,\mathrm{d}s + \mu\cos H(\gamma)\int_0^\gamma \cos H(s)\,\mathrm{d}s,
\end{aligned} \tag{5.8}$$

where

$$H(\gamma) := \int_0^\gamma f(s)\,\mathrm{d}s. \tag{5.9}$$

When $f(\gamma)$ is given, the stress responses can be calculated using the above quadratures. The solutions to the Jaumann, Green–Naghdi, and Sowerby–Chu equations are thus obtained as listed below.

The Jaumann rate: $f_{\mathrm{J}}(\gamma) = 1$,

$$\begin{aligned}
\sigma_{11} &= \mu(1 - \cos\gamma) + \frac{\sigma_{11}(0) + \sigma_{22}(0)}{2} + \frac{\sigma_{11}(0) - \sigma_{22}(0)}{2}\cos\gamma \\
&\quad + \sigma_{12}(0)\sin\gamma, \\
\sigma_{12} &= \mu\sin\gamma + \sigma_{12}(0)\cos\gamma + \frac{\sigma_{22}(0) - \sigma_{11}(0)}{2}\sin\gamma.
\end{aligned} \tag{5.10}$$

The Green–Naghdi rate: $f_{\mathrm{GN}}(\gamma) = 4/(\gamma^2+4)$,

$$\begin{aligned}\sigma_{11} &= 4\mu\big[\cos 2\theta \ln\cos\theta + \theta\sin 2\theta - \sin^2\theta\big] + \frac{\sigma_{11}(0)+\sigma_{22}(0)}{2}\\ &\quad + \frac{\sigma_{11}(0)-\sigma_{22}(0)}{2}\cos 2\theta + \sigma_{12}(0)\sin 2\theta,\\ \sigma_{12} &= 2\mu\cos 2\theta\big[2\theta - 2\tan 2\theta\ln\cos\theta - \tan\theta\big]\\ &\quad + \sigma_{12}(0)\cos 2\theta + \frac{\sigma_{22}(0)-\sigma_{11}(0)}{2}\sin 2\theta.\end{aligned} \tag{5.11}$$

The Sowerby–Chu rate: $f_{\mathrm{SC}}(\gamma) = 2/(\gamma^2+4)$,

$$\begin{aligned}\sigma_{11} &= 2\mu\left[\sin\theta\ln\frac{1+\sin\theta}{\cos\theta} + \cos\theta - 1\right] + \frac{\sigma_{11}(0)+\sigma_{22}(0)}{2}\\ &\quad + \frac{\sigma_{11}(0)-\sigma_{22}(0)}{2}\cos\theta + \sigma_{12}(0)\sin\theta,\\ \sigma_{12} &= 2\mu\left[\cos\theta\ln\frac{1+\sin\theta}{\cos\theta} + \tan\theta - \sin\theta\right]\\ &\quad + \sigma_{12}(0)\cos\theta + \frac{\sigma_{22}(0)-\sigma_{22}(0)}{2}\sin\theta.\end{aligned} \tag{5.12}$$

They are plotted in Figure 2 under the initial conditions $\sigma_{11}(0) = \sigma_{12}(0) = \sigma_{22}(0) = 0$.

From (2.7) together with $\mathbf{R}_E$ in (4.4), we obtain

$$\ln\mathbf{V} = \ln\left(\frac{1+\sin\theta}{\cos\theta}\right)\begin{pmatrix}\sin\theta & \cos\theta\\ \cos\theta & -\sin\theta\end{pmatrix}. \tag{5.13}$$

By (4.4), $\mathbf{V}$ has three distinct eigenvalues: $\lambda_1 = \sec\theta + \tan\theta$, $\lambda_2 = \sec\theta - \tan\theta$, and $\lambda_3 = 1$; consequently, $\ln\mathbf{V}$ also has three distinct eigenvalues: $\ln(\sec\theta+\tan\theta)$, $\ln(\sec\theta - \tan\theta)$, and 0. From (5.13), it follows that

$$(\ln\mathbf{V})^2 = \left(\ln\frac{1+\sin\theta}{\cos\theta}\right)^2\mathbf{I}_3. \tag{5.14}$$

Noting that $I = III = 0$ from (3.7)–(3.9), we have

$$\beta_1 = \frac{1}{(\ln((1+\sin\theta)/\cos\theta))^2}, \quad \beta_2 = 0, \quad \beta_3 = \frac{3}{4(\ln((1+\sin\theta)/\cos\theta))^4},$$

which together with (5.14) and (3.6) lead to

$$\mathbf{\Omega}^{\log} = \left(\frac{\sin\theta}{4(\ln((1+\sin\theta)/\cos\theta))} + \frac{\cos^2\theta}{4}\right)\begin{pmatrix}0 & \dot\gamma\\ -\dot\gamma & 0\end{pmatrix}. \tag{5.15}$$

Letting the general spin $\mathbf{A}$ of (5.1) with (5.5) be $\mathbf{\Omega}^{\log}$ of (5.15), we have the exact form of f,

$$f_{\mathrm{XBM}}(\theta) = \frac{\sin\theta}{2(\ln((1+\sin\theta)/\cos\theta))} + \frac{\cos^2\theta}{2}. \tag{5.16}$$

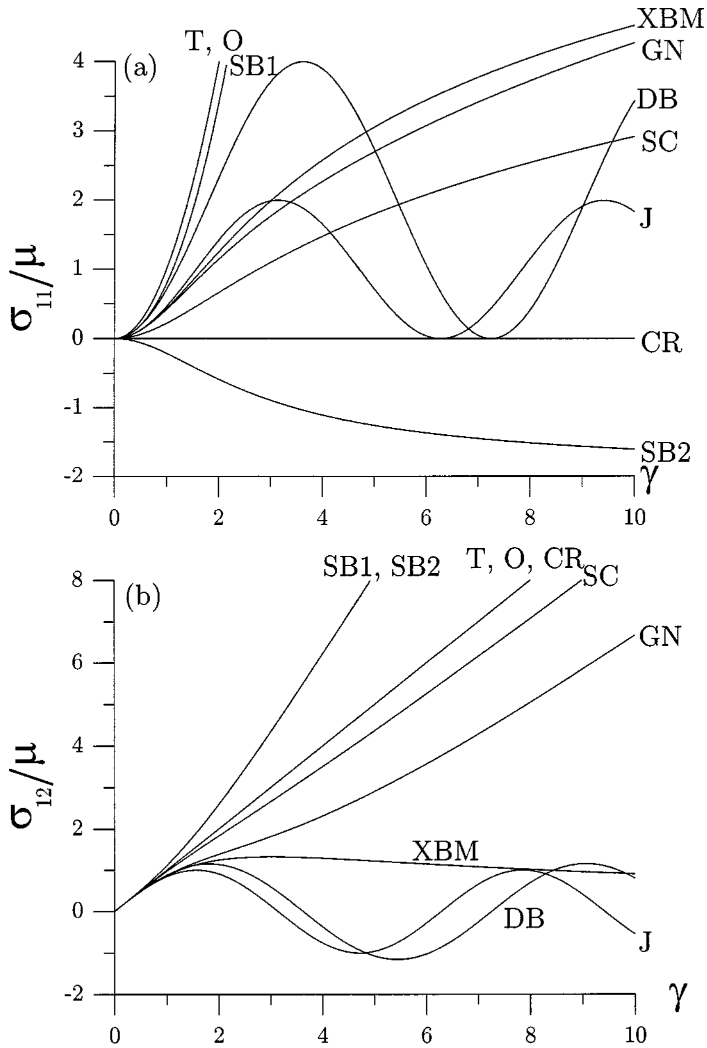

Figure 2. The axial and shear stress responses are compared for: 1. Truesdell, 2. Oldroyd, 3. Cotter–Rivlin, 4. Jaumann, 5. Durban–Baruch, 6. Green–Naghdi, 7. Sowerby–Chu, 8. Szabó–Balla-1, 9. Szabó–Balla-2, 10. Xiao–Bruhns–Meyers.

Specializing (3.11)–(3.17) to the simple shear deformation, we find that f can also be given by

$$f_{\mathrm{XBM}}(\gamma) = 1 + (\nu_1 + \nu_3)\gamma^2 + \nu_2\gamma^2\big(\gamma^2 + 2\big). \tag{5.17}$$

The above formula may be converted to formula (5.16), although the conversion is a rather lengthy, if not impossible, process. It is important to note that formula (5.16) is crucial to understanding the qualitative properties of the Xiao–Bruhns–Meyers equation, especially to proving Theorems 7 and 8 below.

It is easy to check that $\operatorname{tr}(\ln \mathbf{V}) = 0$, so the relevant components of (3.4) are as follows.

The Xiao–Bruhns–Meyers rate: f_{XBM} as given in (5.16),

$$\begin{aligned} \sigma_{11} &= 2\mu \sin\theta \ln \frac{1+\sin\theta}{\cos\theta} + \frac{\sigma_{11}(0)+\sigma_{22}(0)}{2} \\ &\quad + \frac{\sigma_{11}(0)-\sigma_{22}(0)}{2} \cos H_{\mathrm{XBM}}(\theta) + \sigma_{12}(0) \sin H_{\mathrm{XBM}}(\theta), \\ \sigma_{12} &= 2\mu \cos\theta \ln \frac{1+\sin\theta}{\cos\theta} + \sigma_{12}(0) \cos H_{\mathrm{XBM}}(\theta) \\ &\quad + \frac{\sigma_{22}(0)-\sigma_{11}(0)}{2} \sin H_{\mathrm{XBM}}(\theta), \end{aligned} \tag{5.18}$$

where

$$\begin{aligned} H_{\mathrm{XBM}}(\theta) &:= \int_0^\theta 2\sec^2 s\, f_{\mathrm{XBM}}(s)\,\mathrm{d}s \\ &= \theta + \int_0^\theta \frac{\sin s}{\cos^2 s \ln((1+\sin s)/\cos s)}\,\mathrm{d}s. \end{aligned} \tag{5.19}$$

From $(4.3)_1$ and the f_{XBM} in (5.16) above, it is not difficult to prove that (5.18) is the solution of (5.6). The two stresses are plotted in Figure 2 under $\sigma_{11}(0) = \sigma_{12}(0) = \sigma_{22}(0) = 0$ (see also Figure 1 in [3]). As Xiao et al. [5, 6] have observed, the shear stress-strain curve exhibits a yield-like phenomenon, the hypoelastic yield. It is also seen that there is only one zero at $\gamma = 0$ in each of the axial and shear response curves. However, it is cautioned that two or more zeros may emerge depending on the initial stresses, as illustrated in Figure 3.

6. Responses for Non-spinning Objective Rates

Let us consider

$$\mathbf{A} = \dot{\gamma} \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix} \tag{6.1}$$

for the non-spinning rates, which include the rates of Truesdell, Oldroyd, Cotter–Rivlin, Durban–Baruch, Szabó–Balla-1 and Szabó–Balla-2 as special cases. The hypoelastic equation associated with this type of $\mathbf{A}$ is

$$\begin{aligned} &\begin{pmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{12} & \sigma'_{22} \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \\ &\quad - \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned} \tag{6.2}$$

or

$$\begin{cases} \sigma'_{11} - 2a\sigma_{11} - 2b\sigma_{12} = 0, \\ \sigma'_{12} - c\sigma_{11} - (a+d)\sigma_{12} - b\sigma_{22} = \mu, \\ \sigma'_{22} - 2c\sigma_{12} - 2d\sigma_{22} = 0. \end{cases} \tag{6.3}$$

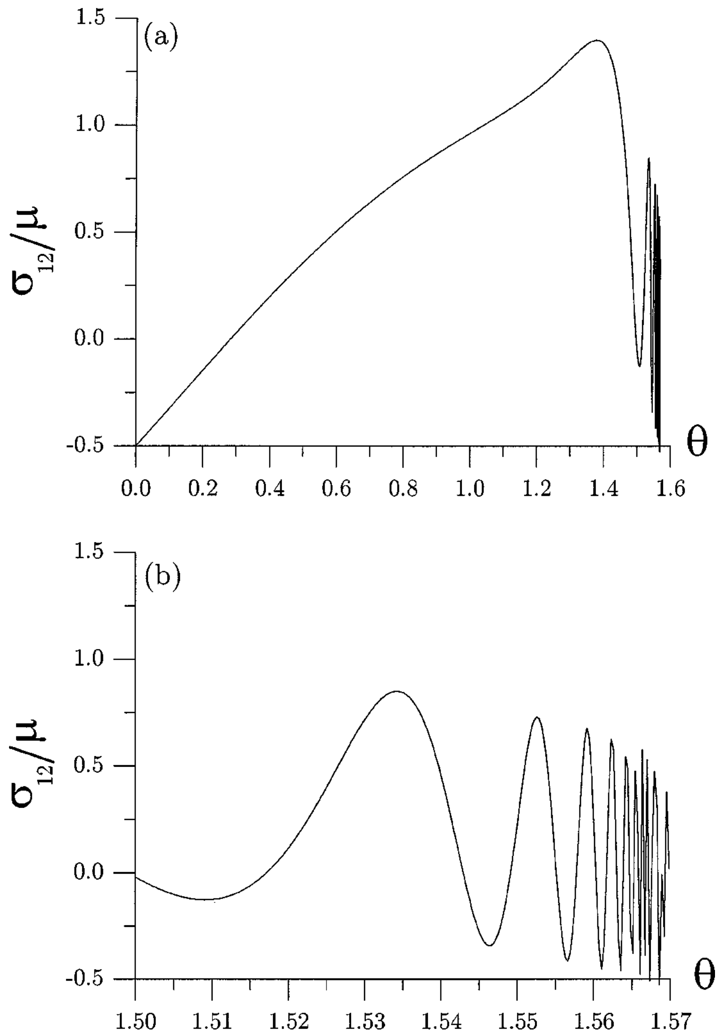

Figure 3. The effect of initial stresses on the Xiao–Bruhns–Meyers hypoelastic equation. The shear stress exhibits several zeros; (b) is a finer view of (a).

We remark in passing that (6.1)–(6.3) apply to the cases of the spinning objective rates as well and can be reduced to (5.1)–(5.3) by letting $a(\gamma) = d(\gamma) = 0$, and $b(\gamma) = -c(\gamma) = f(\gamma)/2$. The solutions to the hypoelastic equations (6.3) of Truesdell, Oldroyd, Cotter–Rivlin, and Durban–Baruch are obtained as listed below.

The Truesdell and Oldroyd rates:

$$\begin{aligned} &\sigma_{11} = \mu\gamma^2 + \sigma_{11}(0) + 2\gamma\sigma_{12}(0) + \gamma^2\sigma_{22}(0), \\ &\sigma_{12} = \mu\gamma + \sigma_{12}(0) + \gamma\sigma_{22}(0), \qquad \sigma_{22} = \sigma_{22}(0). \end{aligned} \tag{6.4}$$

The Cotter–Rivlin rate:

$$\begin{aligned}&\sigma_{11}=\sigma_{11}(0),\qquad \sigma_{12}=\mu\gamma+\sigma_{12}(0)-\gamma\sigma_{22}(0),\\&\sigma_{22}=-\mu\gamma^2+\gamma^2\sigma_{11}(0)-2\gamma\sigma_{12}(0)+\sigma_{22}(0).\end{aligned}\tag{6.5}$$

The Durban–Baruch rate:

$$\begin{aligned}\sigma_{11}={}&2\mu\left[1-\cos\left(\frac{\sqrt{3}}{2}\gamma\right)\right]+\sqrt{3}\sigma_{12}(0)\sin\left(\frac{\sqrt{3}}{2}\gamma\right)\\&+\sigma_{11}(0)\cos^2\left(\frac{\sqrt{3}}{4}\gamma\right)+3\sigma_{22}(0)\sin^2\left(\frac{\sqrt{3}}{4}\gamma\right),\\\sigma_{12}={}&\frac{2\mu}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\gamma\right)+\sigma_{12}(0)\cos\left(\frac{\sqrt{3}}{2}\gamma\right)-\frac{1}{2\sqrt{3}}\sigma_{11}(0)\sin\left(\frac{\sqrt{3}}{2}\gamma\right)\\&+\frac{\sqrt{3}}{2}\sigma_{22}(0)\sin\left(\frac{\sqrt{3}}{2}\gamma\right),\\\sigma_{22}={}&2\mu\left[1-\cos\left(\frac{\sqrt{3}}{2}\gamma\right)\right]-\frac{1}{\sqrt{3}}\sigma_{12}(0)\sin\left(\frac{\sqrt{3}}{2}\gamma\right)\\&+\frac{1}{3}\sigma_{11}(0)\sin^2\left(\frac{\sqrt{3}}{4}\gamma\right)+\sigma_{22}(0)\cos^2\left(\frac{\sqrt{3}}{4}\gamma\right).\end{aligned}\tag{6.6}$$

The axial stress σ_{11} and the shear stress σ_{12} are plotted in Figure 2 under the initial conditions $\sigma_{11}(0)=\sigma_{12}(0)=\sigma_{22}(0)=0$.

Except for the equations of Szabó–Balla-1 and Szabó–Balla-2, the closed-form solutions of the hypoelastic equations without considering the initial stresses were given by, for example, Dienes [1], Szabó and Balla [7], Moss [18] and Atluri [19].

Now, we give right below the closed-form solutions of the Szabó–Balla-1 and Szabó–Balla-2 equations. If the fundamental solution matrix $\mathbf{\Psi}(\gamma)$ of

$$x'=a(\gamma)x+b(\gamma)y,\qquad y'=c(\gamma)x+d(\gamma)y\tag{6.7}$$

is available, it is easy to solve (6.3) by using formula (2.2). For the rate of Szabó–Balla-1, (6.7) is specialized to be

$$x' = \frac{\gamma}{\gamma^2+4}x+\frac{3}{\gamma^2+4}y,\tag{6.8}$$

$$y' = \frac{1}{\gamma^2+4}x-\frac{\gamma}{\gamma^2+4}y.\tag{6.9}$$

Differentiating (6.9) with respect to γ and using (6.8) give

$$y''+\frac{2\gamma}{\gamma^2+4}y'+\frac{1}{(\gamma^2+4)^2}y=0,\tag{6.10}$$

which can be recast as

$$\left[(\gamma^2+4)y'\right]' + \frac{1}{\gamma^2+4}y = 0. \tag{6.11}$$

Similarly, differentiating (6.8) with respect to γ and using (6.9) give

$$\left[(\gamma^2+4)x'\right]' - \frac{2\gamma^2+7}{\gamma^2+4}x = 0. \tag{6.12}$$

Expressing (6.10) in terms of θ defined in $(4.3)_1$,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\theta^2} = \frac{-1}{4}y, \tag{6.13}$$

we readily find the closed-form solution

$$y(\theta) = x_0 \sin\frac{\theta}{2} + y_0 \cos\frac{\theta}{2}, \tag{6.14}$$

where x_0, y_0 are the initial values. Expressing (6.9) in terms of θ and using (6.14) yield

$$\begin{aligned} x(\theta) &= 2\frac{\mathrm{d}y}{\mathrm{d}\theta} + 2y\tan\theta \\ &= \left(\cos\frac{\theta}{2} + 2\tan\theta\sin\frac{\theta}{2}\right)x_0 + \left(2\tan\theta\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)y_0. \end{aligned} \tag{6.15}$$

From (6.15) and (6.14), the fundamental solution matrix is

$$\boldsymbol{\Psi}_{\mathrm{SB1}} = \begin{pmatrix} \cos\frac{\theta}{2} + 2\tan\theta\sin\frac{\theta}{2} & 2\tan\theta\cos\frac{\theta}{2} - \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}. \tag{6.16}$$

The inverse is

$$\boldsymbol{\Psi}_{\mathrm{SB1}}^{-1} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} - 2\tan\theta\cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + 2\tan\theta\sin\frac{\theta}{2} \end{pmatrix}. \tag{6.17}$$

Substituting (6.16) and (6.17) and carrying out the integrations in (2.2), we obtain the stress responses as follows.

The Szabó–Balla-1 rate:

$$\boldsymbol{\sigma} = 2\mu \begin{pmatrix} \cos\frac{\theta}{2} + 2\tan\theta\sin\frac{\theta}{2} & 2\tan\theta\cos\frac{\theta}{2} - \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$
\times \begin{pmatrix} 2 - \dfrac{1+\cos\theta}{\cos^2\theta} + \dfrac{\sigma_{11}(0)}{2\mu} & \dfrac{\sin\theta}{\cos^2\theta} + \dfrac{\sigma_{12}(0)}{2\mu} \\ \dfrac{\sin\theta}{\cos^2\theta} + \dfrac{\sigma_{12}(0)}{2\mu} & \dfrac{\cos\theta - 1}{\cos^2\theta} + \dfrac{\sigma_{22}(0)}{2\mu} \end{pmatrix}
$$

$$
\times \begin{pmatrix} \cos\dfrac{\theta}{2} + 2\tan\theta\sin\dfrac{\theta}{2} & \sin\dfrac{\theta}{2} \\ 2\tan\theta\cos\dfrac{\theta}{2} - \sin\dfrac{\theta}{2} & \cos\dfrac{\theta}{2} \end{pmatrix}. \tag{6.18}
$$

Because $\mathbf{A}_{\mathrm{SB2}} = -\mathbf{A}_{\mathrm{SB1}}$,

$$
\boldsymbol{\Psi}_{\mathrm{SB2}} = \boldsymbol{\Psi}_{\mathrm{SB1}}^{-\mathrm{T}} = \begin{pmatrix} \cos\dfrac{\theta}{2} & -\sin\dfrac{\theta}{2} \\ \sin\dfrac{\theta}{2} - 2\tan\theta\cos\dfrac{\theta}{2} & \cos\dfrac{\theta}{2} + 2\tan\theta\sin\dfrac{\theta}{2} \end{pmatrix}. \tag{6.19}
$$

Substitution of the above into (2.2) gives the solution below.

The Szabó–Balla-2 rate:

$$
\begin{aligned}
\boldsymbol{\sigma} \;=\; & 2\mu \begin{pmatrix} \cos\dfrac{\theta}{2} & -\sin\dfrac{\theta}{2} \\ \sin\dfrac{\theta}{2} - 2\tan\theta\cos\dfrac{\theta}{2} & \cos\dfrac{\theta}{2} + 2\tan\theta\sin\dfrac{\theta}{2} \end{pmatrix} \\
& \times \begin{pmatrix} \dfrac{1-\cos\theta}{\cos^2\theta} + \dfrac{\sigma_{11}(0)}{2\mu} & \dfrac{\sin\theta}{\cos^2\theta} + \ln\dfrac{\sec\theta + \tan\theta}{\tan(\pi/4+\theta/2)} + \dfrac{\sigma_{12}(0)}{2\mu} \\ \dfrac{\sin\theta}{\cos^2\theta} + \ln\dfrac{\sec\theta + \tan\theta}{\tan(\pi/4+\theta/2)} + \dfrac{\sigma_{12}(0)}{2\mu} & \dfrac{1+\cos\theta}{\cos^2\theta} - 2 + \dfrac{\sigma_{22}(0)}{2\mu} \end{pmatrix} \\
& \times \begin{pmatrix} \cos\dfrac{\theta}{2} & \sin\dfrac{\theta}{2} - 2\tan\theta\cos\dfrac{\theta}{2} \\ -\sin\dfrac{\theta}{2} & \cos\dfrac{\theta}{2} + 2\tan\theta\sin\dfrac{\theta}{2} \end{pmatrix}.
\end{aligned} \tag{6.20}
$$

Of the stress components of (6.18) and (6.20), the axial stresses σ_{11} and the shear stresses σ_{12} are plotted in Figure 2 under $\sigma_{11}(0) = \sigma_{12}(0) = \sigma_{22}(0) = 0$.

Although in Figure 2 all the responses were plotted under the initial conditions $\sigma_{11}(0) = \sigma_{12}(0) = \sigma_{22}(0) = 0$, we have derived the explicit formulae for the responses including the effect of nonvanishing initial stresses, since it has been observed, for example in Figure 3, that the numbers of zeros of the hypoelastic responses depend heavily on the initial conditions.

7. Comparison Theorems for the Sturm–Liouville Equations

To embark further analysis, we need to point out some facts about ordinary differential equations. First, let us consider the following two closely related differential equations:

$$\big(p(t)u'(t)\big)' + q(t)u(t) \;=\; 0, \tag{7.1}$$

$$\big(p(t)w'(t)\big)' + q(t)w(t) \;=\; r(t), \tag{7.2}$$

where $p(t)$, $q(t)$, and $r(t)$ are real-valued continuous functions of t on the interval $\mathbb{R}^+ = [0, \infty)$, $p(t) > 0$ on $\mathbb{R}^+$, and $r(t) \neq 0$ on $(0, \infty)$. The forcing term $r(t) \neq 0$ on $(0, \infty)$ means that either $r(t) > 0$ for all $t \in (0, \infty)$ or $r(t) < 0$ for all $t \in (0, \infty)$. If the second equation is multiplied by u, the first by w, and the results are subtracted and then integrated from 0 to t, one obtains

$$\big[p(uw' - u'w)\big]_0^t = \int_0^t r(s)u(s)\,\mathrm{d}s, \tag{7.3}$$

which may be written as

$$p(t)W(t) = p(0)\big[u(0)w'(0) - u'(0)w(0)\big] + \int_0^t r(s)u(s)\,\mathrm{d}s \tag{7.4}$$

in terms of the Wronskian of u and w,

$$W(t) = W\big(u(t), w(t)\big) := \begin{vmatrix} u(t) & w(t) \\ u'(t) & w'(t) \end{vmatrix}. \tag{7.5}$$

It is easy to verify the following lemma about the Wronskian of two arbitrary C^1 functions $x(t)$ and $y(t)$.

LEMMA 1. *If $x(t)$ and $y(t)$ are continuously differentiable on $\mathbb{R}^+$ and their Wronskian $W(x(t), y(t))$ is not equal to zero for sufficiently large t, then the zeros of $x(t)$ and $y(t)$ separate each other for sufficiently large t.*

A function defined on the interval $\mathbb{R}^+$ or a solution of the homogeneous equation (7.1) on $\mathbb{R}^+$ is oscillatory (respectively non-oscillatory) if it is real-valued and nontrivial ($\not\equiv 0$) and has an infinite (respectively a finite) number of zeros on $\mathbb{R}^+$. Specifically, a non-oscillatory solution of the homogeneous second order ODE (7.1) is said to be disconjugate if it has at most one zero. Equation (7.1) is said to be *oscillatory* on $\mathbb{R}^+$ if at least *one* real-valued nontrivial solution has an infinite number of zeros on $\mathbb{R}^+$. Conversely, when *every* real-valued nontrivial solution has at most a finite number of zeros on $\mathbb{R}^+$, it is said to be *non-oscillatory* on $\mathbb{R}^+$. Furthermore, in the latter case, (7.1) is said to be *disconjugate* on $\mathbb{R}^+$ if *every* real-valued nontrivial solution has at most one zero on $\mathbb{R}^+$.

A solution of the non-homogeneous equation (7.2) or the stress response to simple shearing is oscillatory (respectively non-oscillatory) on $\mathbb{R}^+$ if it is real-valued and has an infinite (respectively a finite) number of zeros on $[t_1, \infty)$ for

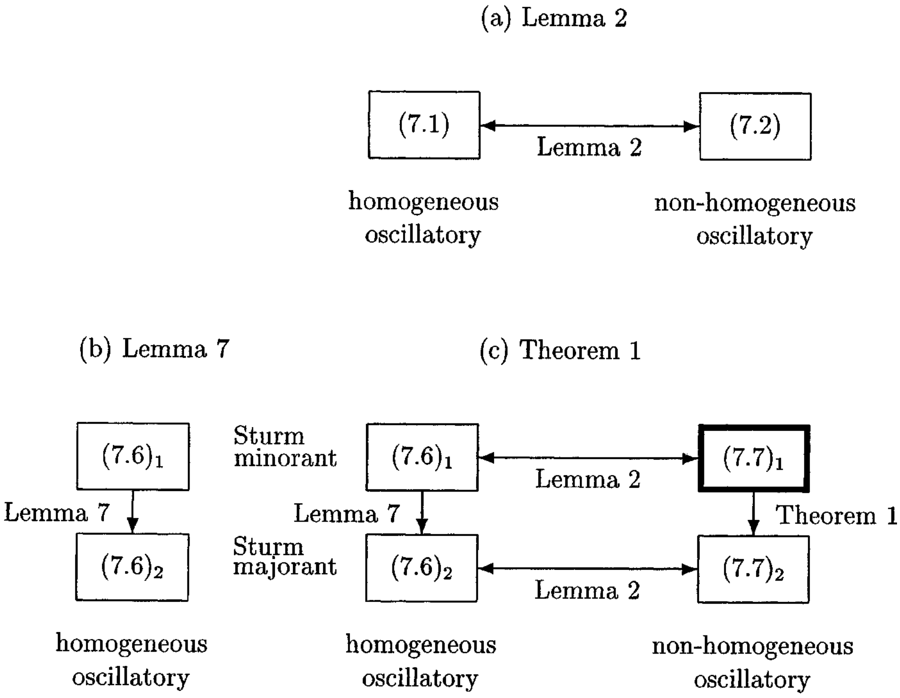

Figure 4. Implications of oscillatory behavior and an oscillatory minorant.

every (respectively *some*) $t_1 \geqslant 0$. Equation (7.2) is said to be oscillatory if it has at least *one* oscillatory solution and non-oscillatory if *all* solutions are non-oscillatory. Be cautious that for a non-homogeneous equation we speak of neither nontriviality nor disconjugacy due to the fact that the number of zeros of the solution depends heavily on initial conditions.

To proceed let us refer to the following lemma due to Tefteller [20]:

LEMMA 2 (Tefteller). *Suppose that $1/p(t)$, $q(t)$, and $r(t)$ are absolutely integrable on the interval $\mathbb{R}^+$, or suppose that the nontrivial solution $u(t)$ of* (7.1) *is bounded on $\mathbb{R}^+$ and $r(t)$ is absolutely integrable on $\mathbb{R}^+$. Then,* (7.2) *is oscillatory on $\mathbb{R}^+$ if and only if* (7.1) *is oscillatory on $\mathbb{R}^+$. (See Figure 4(a).)*

LEMMA 3. *If* (7.1) *is non-oscillatory on the interval $\mathbb{R}^+$, then* (7.2) *is non-oscillatory on the interval $\mathbb{R}^+$. (See Figure 5(a).)*

Proof. Since $p(t) > 0$ on $\mathbb{R}^+$ and $r(t) > 0$ on $(0, \infty)$ or $r(t) < 0$ on $(0, \infty)$, the Wronskian $W(u(t), w(t))$ of the non-oscillatory nontrivial solution $u(t)$ of (7.1) and the solution $w(t)$ of (7.2) is non-oscillatory by (7.4). Hence, the Wronskian $W(u(t), w(t))$ is not equal to zero after a sufficiently large time t_1. If $w(t)$ is assumed to be oscillatory on the interval $[t_1, \infty)$, by Lemma 1, $u(t)$ is also oscillatory on the interval $[t_1, \infty)$. This contradicts to the fact that $u(t)$ is non-oscillatory on $\mathbb{R}^+$. Therefore, $w(t)$ is non-oscillatory on $\mathbb{R}^+$. □

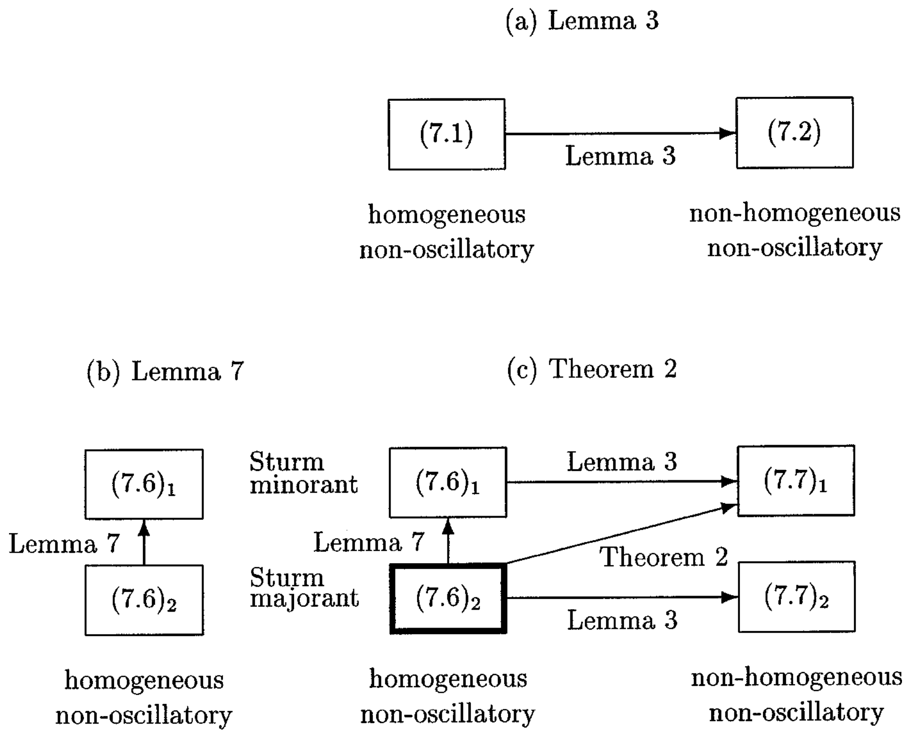

Figure 5. Implications of non-oscillatory behavior and a non-oscillatory majorant.

Obviously a stronger condition suffices to lead to the same result, as given by

LEMMA 4. *If* (7.1) *is disconjugate on the interval* $\mathbb{R}^+$, *then* (7.2) *is non-oscillatory on the interval* $\mathbb{R}^+$. *(See Figure 6(a).)*

Next, let us further consider the following four scalar ODEs of the second order:

$$\big(p_j(t)u'(t)\big)' + q_j(t)u(t) = 0, \qquad j = 1, 2, \tag{7.6}$$

$$\big(p_j(t)w'(t)\big)' + q_j(t)w(t) = r_j(t), \quad j = 1, 2, \tag{7.7}$$

where $p_j(t)$, $q_j(t)$, and $r_j(t)$ are real-valued continuous functions of t on the interval $\mathbb{R}^+$, $p_j(t) > 0$ on $\mathbb{R}^+$, and $r_j(t) \neq 0$ on $(0, \infty)$ for $j = 1, 2$. If on the entire interval $\mathbb{R}^+$,

$$p_1(t) \geqslant p_2(t), \qquad q_1(t) \leqslant q_2(t), \tag{7.8}$$

then $(7.6)_2$ is called a *Sturm majorant* for $(7.6)_1$ on $\mathbb{R}^+$, and $(7.6)_1$ is a *Sturm minorant* for $(7.6)_2$ on $\mathbb{R}^+$. The homogeneous equations $(7.6)_j$ have the celebrated theorems due to Sturm (see and compare with, for example, [21–23]) as follows:

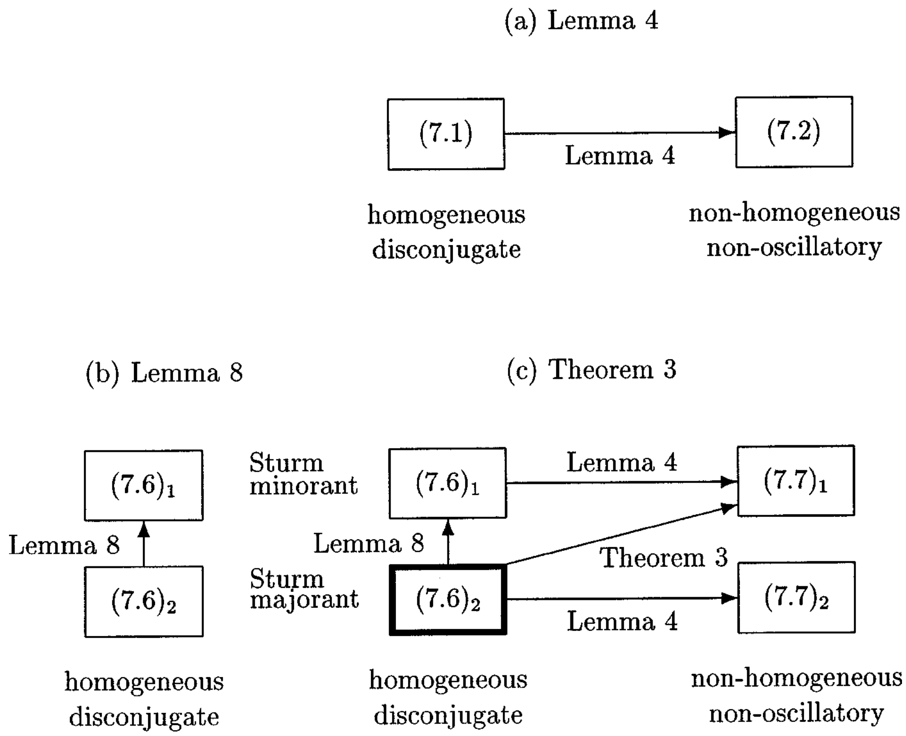

Figure 6. Implications of disconjugacy and a disconjugate majorant.

LEMMA 5 (Sturm's comparison theorem). *Suppose that* $(7.6)_1$ *is a Sturm minorant for* $(7.6)_2$. *Let* $u_j(t)$ *be a nontrivial solution of* $(7.6)_j$ *such that*

$$\frac{u_1(t)}{p_1(t)u_1'(t)} \leqslant \frac{u_2(t)}{p_2(t)u_2'(t)} \tag{7.9}$$

at the time instant $t = t_0 (< t_1) \in \mathbb{R}^+$. *If* $u_1(t)$ *has exactly* $n (\geqslant 1)$ *zeros at* $t = t_1 < t_2 < \cdots < t_n$ *on* $\mathbb{R}^+$, *then* $u_2(t)$ *has at least* n *zeros on* $t_0 < t \leqslant t_n$.

LEMMA 6 (Sturm's separation theorem). *Suppose that* $(7.6)_1$ *is a Sturm minorant for* $(7.6)_2$. *Let* $u_j(t)$ *be a nontrivial solution of* $(7.6)_j$. *If* $u_1(t)$ *vanishes at a pair of points* $t = t_1, t_2 (> t_1)$ *of* $\mathbb{R}^+$, *then* $u_2(t)$ *has at least one zero on* (t_1, t_2), *except when* $u_2 = ku_1$, *where* k *is a real nonzero constant. In particular, if* $p_1 \equiv p_2$, $q_1 \equiv q_2$ *and* u_1, u_2 *are two linearly independent solutions of* $(7.6)_1 \equiv (7.6)_2$, *then the zeros of* u_1 *separate and are separated by those of* u_2.

One direct consequence of the above lemma is

LEMMA 7 (Sturm's oscillation theorem). *Suppose that* $(7.6)_1$ *is a Sturm minorant for* $(7.6)_2$. *If* $(7.6)_1$ *is oscillatory, then* $(7.6)_2$ *is oscillatory. If* $(7.6)_2$ *is non-oscillatory, then* $(7.6)_1$ *is non-oscillatory. (See Figures 4(b) and 5(b).)*

Now, we prove below an extended version of Sturm's oscillation theorem suitable for the non-homogeneous equations $(7.7)_j$.

THEOREM 1. *For* $j = 1, 2$, *suppose that* (i) $1/p_j(t)$, $q_j(t)$, *and* $r_j(t)$ *are absolutely integrable on the interval* $\mathbb{R}^+$, *or suppose that* (ii) *the nontrivial solution* $u_j(t)$ *of* $(7.6)_j$ *is bounded on* $\mathbb{R}^+$ *and* $r_j(t)$ *is absolutely integrable on* $\mathbb{R}^+$. *Let* $(7.6)_1$ *be a Sturm minorant for* $(7.6)_2$. *If* $(7.7)_1$ *is oscillatory on* $\mathbb{R}^+$, *then* $(7.7)_2$ *is oscillatory on* $\mathbb{R}^+$. *(See Figure 4(c).)*

Proof. The proof is sketched diagrammatically in Figure 4(c). By Lemma 2, if $(7.7)_1$ is oscillatory, then $(7.6)_1$ is oscillatory. Since $(7.6)_1$ is a Sturm minorant for $(7.6)_2$, by Lemma 7 we observe that $(7.6)_2$ is also oscillatory. The conclusion then follows from Lemma 2. □

THEOREM 2. *Suppose that* $(7.6)_2$ *is a Sturm majorant for* $(7.6)_1$. *If* $(7.6)_2$ *is non-oscillatory on the interval* $\mathbb{R}^+$, *then* $(7.7)_1$ *is non-oscillatory on the interval* $\mathbb{R}^+$. *(See Figure 5(c).)*

Proof. The proof is sketched diagrammatically in Figure 5(c). Since $(7.6)_2$ is a Sturm majorant for $(7.6)_1$ and $(7.6)_2$ is non-oscillatory, by Lemma 7, $(7.6)_1$ is non-oscillatory. Then by Lemma 3, $(7.7)_1$ is non-oscillatory. □

LEMMA 8. *Suppose that* $(7.6)_2$ *is a Sturm majorant for* $(7.6)_1$. *If* $(7.6)_2$ *is disconjugate on the interval* $\mathbb{R}^+$, *then* $(7.6)_1$ *is disconjugate on the interval* $\mathbb{R}^+$. *(See Figure 6(b).)*

Proof. Let $u_2(t)$ be any nontrivial solution of $(7.6)_2$. The following proof is by contradiction. Assume that there exists a specific nontrivial solution $u_1(t)$ of $(7.6)_1$ that has exactly $n(\geqslant 2)$ zeros and $u_1 = u_2$ and $u_1' = u_2'$ at $t = t_0$. Since $(7.6)_2$ is a Sturm majorant for $(7.6)_1$, $p_1(t) \geqslant p_2(t) > 0$ on $\mathbb{R}^+$, so condition (7.9) is fulfilled at $t = t_0$. Then by Lemma 5, $u_2(t)$ has at least n zeros, and, therefore, is not disconjugate. Hence, the assumption is false, and $(7.6)_1$ is disconjugate. □

THEOREM 3. *Suppose that* $(7.6)_2$ *is a Sturm majorant for* $(7.6)_1$. *If* $(7.6)_2$ *is disconjugate on the interval* $\mathbb{R}^+$, *then* $(7.7)_1$ *is non-oscillatory on the interval* $\mathbb{R}^+$. *(See Figure 6(c).)*

Proof. The proof is sketched diagrammatically in Figure 6(c). Since $(7.6)_2$ is a Sturm majorant for $(7.6)_1$, by Lemma 8, the disconjugacy of $(7.6)_2$ implies the disconjugacy of $(7.6)_1$. Then by Lemma 4, $(7.7)_1$ is non-oscillatory. □

The implication relations of oscillatory, non-oscillatory, and disconjugate behavior are depicted in Figures 4–6, respectively. For heuristic purposes, they may be roughly summarized as follows:

$$\text{If the non-homogeneous minorant is oscillatory,}$$
$$\text{then the non-homogeneous majorant is oscillatory.}$$

If the homogeneous majorant is non-oscillatory,
then the non-homogeneous minorant is non-oscillatory.

If the homogeneous majorant is disconjugate,
then the non-homogeneous minorant is non-oscillatory.

Inspired by the three sets of implication relations (see Theorems 1–3 and Figures 4–6), we are led to the following three definitions. Equation $(7.7)_1$ is called an *oscillatory minorant* (see Figure 4(c)) for $(7.7)_2$ if $(7.6)_1$ is a Sturm minorant for $(7.6)_2$ and if both $(7.7)_1$ and $(7.7)_2$ are oscillatory. Equation $(7.6)_2$ is called a *non-oscillatory majorant* (see Figure 5(c)) for $(7.6)_1$ if $(7.6)_2$ is a Sturm majorant for $(7.6)_1$ and if both $(7.6)_1$ and $(7.6)_2$ are non-oscillatory. Equation $(7.6)_2$ is called a *disconjugate majorant* (see Figure 6(c)) for $(7.6)_1$ if $(7.6)_2$ is a Sturm majorant for $(7.6)_1$ and if both $(7.6)_1$ and $(7.6)_2$ are disconjugate. Therefore, it is significant for us to pick out an oscillatory minorant, a non-oscillatory majorant, or a disconjugate majorant from a certain list of hypoelastic equations and their associated homogeneous equations.

8. Comparison between Spinning Objective Rates

In this section let us compare the spinning objective rates with one another. From (5.6), we know that the solution properties of the axial stress σ_{11} relies on the solution properties of the shear stress σ_{12}, and vice versa. In particular, we can easily prove

THEOREM 4. *Suppose the axial stress σ_{11} and the shear stress σ_{12} are governed by* (5.6) *and $f(\gamma) > 0$ on the interval $\mathbb{R}^+$. Then σ_{11} is oscillatory (respectively non-oscillatory) if σ_{12} is oscillatory (respectively non-oscillatory).*

Proof. If σ_{12} is oscillatory on the interval $\mathbb{R}^+$, then its derivative σ'_{12} is oscillatory, and by $(5.6)_2$, σ_{11} is oscillatory. Conversely, by $(5.6)_1$ we have

$$\sigma_{11}(\gamma) = \sigma_{11}(0) + \int_0^\gamma f(s)\sigma_{12}(s)\,\mathrm{d}s.$$

Since $f(\gamma) > 0$, if σ_{12} is non-oscillatory on the interval $\mathbb{R}^+$, so is $f(s)\sigma_{12}(s)$ for all $s \in \mathbb{R}^+$, and then the above integral implies that σ_{11} is non-oscillatory on the interval $\mathbb{R}^+$. □

As a consequence, it suffices to consider only the equation for σ_{12}. Let $f(\gamma) \in C^1(\mathbb{R}^+)$, from (5.6) the shear stress σ_{12} is governed by

$$\left[\frac{1}{f(\gamma)}\sigma'_{12}\right]' + f(\gamma)\sigma_{12} = \mu\left(\frac{1}{f(\gamma)}\right)' \tag{8.1}$$

over the interval of $\gamma \in \mathbb{R}^+$. Hence, for the spinning objective rates, the function $f(\gamma)$ alone suffices to determine the response behavior, since $p(\gamma) = 1/f(\gamma)$, $q(\gamma) = f(\gamma)$, and $r(\gamma) = \mu(1/f(\gamma))'$.

8.1. THE JAUMANN RATE

THEOREM 5. *The Jaumann equation*

$$\sigma_{12}'' + \sigma_{12} = 0 \tag{8.2}$$

is oscillatory on the interval $\mathbb{R}^+$*, and is an oscillatory minorant for the hypoelastic equation (8.1) associated with any spinning objective rate with* $f(\gamma)$ *such that* $f(\gamma) \geqslant 1$ *on* $\mathbb{R}^+$ *and* $f'(\gamma)$ *is absolutely integrable on* $\mathbb{R}^+$.

Proof. It is easy to verify that the nontrivial solution $\sigma_{12}(\gamma) = \sigma_{12}(0)\cos\gamma + \sigma_{12}'(0)\sin\gamma$ of (8.2) has an infinite number of zeros, so the Jaumann equation (8.2) is oscillatory. First, we verify that (8.2) is a Sturm minorant for the homogeneous equation corresponding to (8.1) with $f(\gamma) \geqslant 1$. Second, from (5.8) we know that the homogeneous solution of (8.1) has the following form: $\cos H(\gamma)$ or $\sin H(\gamma)$ or a linear combination of both, which are all bounded on $\mathbb{R}^+$. Third, the forcing term in (8.1) satisfies the finiteness condition of the integral $\int_0^\infty |\mu(1/f(\gamma))'|\,\mathrm{d}\gamma$, because, when $f(\gamma) \geqslant 1$ and $f'(\gamma)$ is absolutely integrable, the integral equals $\mu\int_0^\infty (|f'|/f)\,\mathrm{d}\gamma \leqslant \mu\int_0^\infty |f'|\,\mathrm{d}\gamma < \infty$. Therefore, by Theorem 1 the equation (8.1) with $f(\gamma) \geqslant 1$ is oscillatory. Then the definition of oscillatory minorant completes the proof. □

8.2. THE GREEN–NAGHDI RATE

THEOREM 6. *The Green–Naghdi equation,*

$$\left[\frac{\gamma^2+4}{4}\sigma_{12}'\right]' + \frac{4}{\gamma^2+4}\sigma_{12} = \frac{\mu\gamma}{2}, \tag{8.3}$$

is non-oscillatory on the interval $\mathbb{R}^+$*, and the homogeneous Green–Naghdi equation is disconjugate on* $\mathbb{R}^+$*. The hypoelastic equation (8.1) associated with any spinning objective rate with* $f(\gamma) \leqslant 4/(\gamma^2+4)$ *is non-oscillatory on* $\mathbb{R}^+$*. In addition, the hypoelastic equations with spins* $\mathbf{\Omega}_\mathrm{D} := (1-\xi)\mathbf{\Omega} + \xi\mathbf{W}$*, where* $\mathbf{\Omega}$ *is the Green–Naghdi spin and* $\mathbf{W}$ *the Jaumann spin, are oscillatory on the interval* $\mathbb{R}^+$ *for any* $\xi \in (0, 1]$.

Proof. The homogeneous solution of (8.3) is

$$\sigma_{12}(\gamma) = \frac{\sigma_{12}(0)(4-\gamma^2) + 4\sigma_{12}'(0)\gamma}{\gamma^2+4},$$

of which the only zero occurs at

$$\gamma = \frac{2\sigma_{12}'(0)}{\sigma_{12}(0)} + 2\sqrt{\left(\frac{\sigma_{12}'(0)}{\sigma_{12}(0)}\right)^2 + 1}.$$

Hence, by definition, the homogeneous Green–Naghdi equation is disconjugate. By Lemma 4 the Green–Naghdi equation is non-oscillatory. We verify that the homogeneous Green–Naghdi equation is a Sturm majorant for any homogeneous hypoelastic equation with $f(\gamma) \leqslant 4/(\gamma^2+4)$. Consequently, by Theorem 3 the hypoelastic equation (8.1) with $f(\gamma) \leqslant 4/(\gamma^2+4)$ is non-oscillatory. By definition the homogeneous Green–Naghdi equation is a disconjugate majorant for the homogeneous equation corresponding to (8.1) with $f(\gamma) \leqslant 4/(\gamma^2+4)$.

Next, we consider the spin:

$$\mathbf{\Omega}_{\mathrm{D}} = (1-\xi)\mathbf{\Omega} + \xi\mathbf{W}, \quad 0 < \xi \leqslant 1. \tag{8.4}$$

The cases $\xi = 1$ and $\xi = 0$ correspond to the Jaumann spin and the Green–Naghdi spin, respectively. Then,

$$f_{\mathrm{D}}(\gamma) = \frac{\xi\gamma^2+4}{\gamma^2+4}. \tag{8.5}$$

The hypoelastic equation (8.1) with f_{D} above has one homogeneous solution $\cos(\xi\gamma + 2(1-\xi)\theta)$, where $\theta = \arctan(\gamma/2)$, which has an infinite number of zeros if $0 < \xi \leqslant 1$, since

$$\xi\gamma + 2(1-\xi)\arctan\left(\frac{\gamma}{2}\right) = \frac{(2n+1)\pi}{2}, \quad n \in \mathbb{Z}^+, \tag{8.6}$$

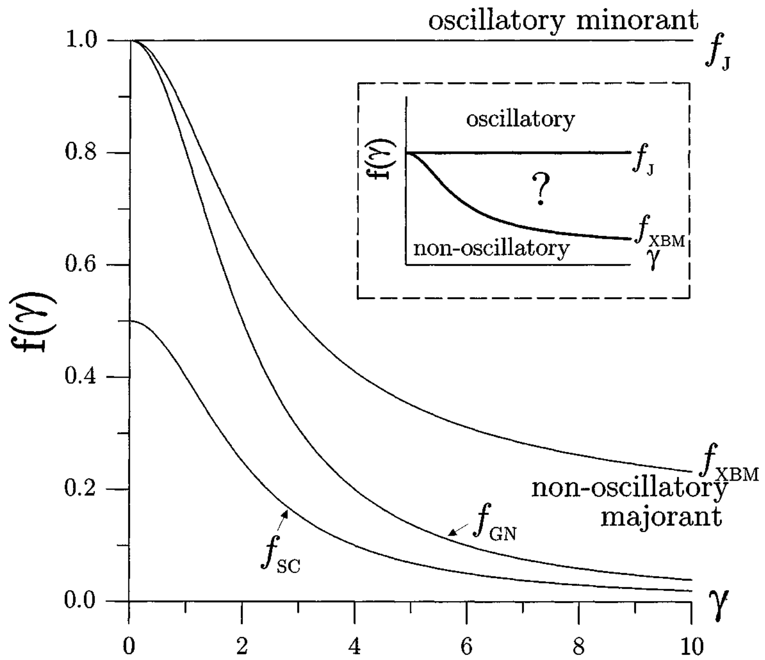

Figure 7. Comparison of the four f's of Jaumann, Green–Naghdi, Sowerby–Chu and Xiao–Bruhns–Meyers. Among the homogeneous hypoelastic equations associated with spinning objective rates, the homogeneous Xiao–Bruhns–Meyers equation is a non-oscillatory majorant, and the Jaumann equation is an oscillatory minorant.

has an infinite number of solutions. Thus, according to Lemma 6 the other homogeneous solution also has an infinite number of zeros. Because the homogeneous solution is bounded and the forcing term in (8.1) satisfies the finiteness condition:

$$\mu\int_0^\infty \left|\left(\frac{1}{f_{\mathrm{D}}(\gamma)}\right)'\right|\mathrm{d}\gamma \;=\; \mu\int_0^\infty \frac{-f_{\mathrm{D}}'(\gamma)}{f_{\mathrm{D}}^2(\gamma)}\,\mathrm{d}\gamma = \mu\left[\frac{1}{f_{\mathrm{D}}(\infty)}-\frac{1}{f_{\mathrm{D}}(0)}\right] \\ <\;\infty,\quad 0<\xi\leqslant 1, \tag{8.7}$$

where $f_{\mathrm{D}}'(\gamma)\leqslant 0$ on $\mathbb{R}^+$ is taken into account, by Lemma 2 the hypoelastic equations with spins $\mathbf{\Omega}_{\mathrm{D}}$, $0<\xi\leqslant 1$, are oscillatory. □

We have shown in Theorem 6 that the hypoelastic equations associated with spins $\mathbf{\Omega}_{\mathrm{D}}$ are oscillatory, no matter how small $\xi > 0$ is, although the Green–Naghdi equation is non-oscillatory. Henceforth we shall refer to this fact by saying figuratively that the non-oscillatory Green–Naghdi equation is the "infimum" for the oscillatory hypoelastic equations with spins $\mathbf{\Omega}_{\mathrm{D}}$ $(0<\xi\leqslant 1)$.

As an application of Theorem 6, the Sowerby–Chu rate is one of the spinning objective rate with $f(\gamma)\leqslant 4/(\gamma^2+4)$; indeed $f_{\mathrm{SC}} < f_{\mathrm{GN}}$, which implies that the homogeneous Sowerby–Chu equation is disconjugate and the Sowerby–Chu equation is non-oscillatory. Figure 7 compares the three f's of Jaumann, Green–Naghdi and Sowerby–Chu.

8.3. THE XIAO–BRUHNS–MEYERS RATE

By referring to (5.18), one nontrivial solution of the homogeneous Xiao–Bruhns–Meyers equation is given by

$$\cos H_{\mathrm{XBM}}(\theta),$$

the zeros of which are determined by

$$H_{\mathrm{XBM}}(\theta) = \frac{(2n+1)\pi}{2},\quad n\in\mathbb{Z}^+.$$

In view of (5.19) it is equivalent to solving the following equation:

$$\int_0^\theta \frac{\sin s}{\cos^2 s\ln((1+\sin s)/\cos s)}\,\mathrm{d}s = \frac{(2n+1)\pi}{2}-\theta,\quad n\in\mathbb{Z}^+.$$

The left-hand side is an increasing function of θ and the right-hand side is a straight line segment with slope -1. Figure 8(a) displays the above curves, the intersection points of which give the roots of the above equation. Utilizing the Gaussian quadrature we have calculated the above integral in the range $[0,\pi/2)$, giving the value around 16892. Thus we have

$$\int_0^{\pi/2} \frac{\sin s}{\cos^2 s\ln((1+\sin s)/\cos s)}\,\mathrm{d}s < \infty.$$

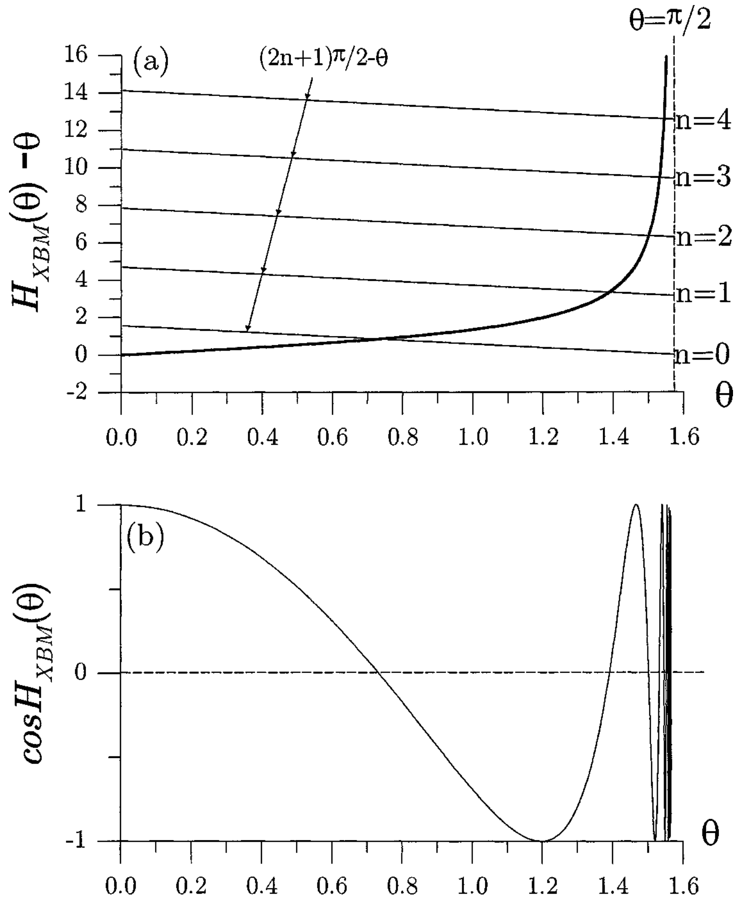

Figure 8. (a) The zeros and (b) the non-oscillatory behavior of the nontrivial solution of the homogeneous Xiao–Bruhns–Meyers equation.

Therefore, there are a finite number of zeros of $\cos H_{\mathrm{XBM}}(\theta)$ (see Figure 8(b)), and by definition the homogeneous Xiao–Bruhns–Meyers equation is non-oscillatory.

According to (5.16), the variation of f_{XBM} with respect to γ is plotted in Figure 7. It can be seen that $f_{\mathrm{GN}} \leqslant f_{\mathrm{XBM}}$, that is, the homogeneous Green–Naghdi equation is a Sturm minorant for the homogeneous Xiao–Bruhns–Meyers equation. In Theorem 6 we have shown that the Green–Naghdi equation is the "infimum" for the class of hypoelastic equations associated with spins $\mathbf{\Omega}_{\mathrm{D}} = (1-\xi)\mathbf{\Omega} + \xi\mathbf{W}$, $0 < \xi \leqslant 1$. Now we consider the parallel problem for a class of hypoelastic equations associated with spins $\mathbf{\Omega}_{\mathrm{L}} = (1-\xi)\mathbf{\Omega}^{\log} + \xi\mathbf{W}$, $0 < \xi \leqslant 1$, and prove that the Xiao–Bruhns–Meyers equation is the "infimum" for the class of hypoelastic equations associated with spins $\mathbf{\Omega}_{\mathrm{L}}$. Really, we can prove

THEOREM 7. *The Xiao–Bruhns–Meyers equation is non-oscillatory on the interval $\mathbb{R}^+$. The homogeneous Xiao–Bruhns–Meyers equation is non-oscillatory on* *the interval* $\mathbb{R}^+$*, and is a non-oscillatory majorant for the homogeneous hypoelastic equation (8.1) associated with any spinning objective rate with* $f(\gamma) \leqslant f_{\mathrm{XBM}}(\gamma)$ *on the interval* $\mathbb{R}^+$*, and hence for the homogeneous Green–Naghdi equation on the interval* $\mathbb{R}^+$*. The hypoelastic equations with spins* $\boldsymbol{\Omega}_{\mathrm{L}} := (1-\xi)\boldsymbol{\Omega}^{\log} + \xi\mathbf{W}$ *are oscillatory on the interval* $\mathbb{R}^+$ *for any* $\xi \in (0, 1]$.

Proof. Since the homogeneous Xiao–Bruhns–Meyers equation is non-oscillatory, by Lemma 3 the Xiao–Bruhns–Meyers equation is non-oscillatory. If $f(\gamma) \leqslant f_{\mathrm{XBM}}(\gamma)$, then the homogeneous equation corresponding to (8.1) is a Sturm minorant for the homogeneous Xiao–Bruhns–Meyers equation, which is non-oscillatory on the interval $\mathbb{R}^+$. By definition the homogeneous Xiao–Bruhns–Meyers equation is a non-oscillatory majorant for the homogeneous hypoelastic equation (8.1) associated with any spinning objective rate with $f(\gamma) \leqslant f_{\mathrm{XBM}}(\gamma)$. In terms of θ the homogeneous Green–Naghdi equation can be written as

$$f_{\mathrm{GN}} = \cos^2\theta.$$

Let us investigate the difference $f_{\mathrm{XBM}} - f_{\mathrm{GN}}$ at $\theta \in [0, \pi/2)$,

$$f_{\mathrm{XBM}} - f_{\mathrm{GN}} = \frac{\cos^2\theta}{2\ln((1+\sin\theta)/\cos\theta)}\left(\frac{\sin\theta}{\cos^2\theta} - \ln\frac{1+\sin\theta}{\cos\theta}\right).$$

Because

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\sin\theta}{\cos^2\theta} - \ln\frac{1+\sin\theta}{\cos\theta}\right) = \frac{2\sin^2\theta}{\cos^3\theta} \geqslant 0$$

for $\theta \in [0, \pi/2)$, we have $f_{\mathrm{XBM}} \geqslant f_{\mathrm{GN}}$ for all $\theta \in [0, \pi/2)$ and hence $f_{\mathrm{XBM}} \geqslant f_{\mathrm{GN}}$ for all $\gamma \in \mathbb{R}^+$, that is, the homogeneous Xiao–Bruhns–Meyers equation is a Sturm majorant for the homogeneous Green–Naghdi equation. Now, let us consider the class of spins:

$$\boldsymbol{\Omega}_{\mathrm{L}} = (1-\xi)\boldsymbol{\Omega}^{\log} + \xi\mathbf{W}, \quad 0 < \xi \leqslant 1, \tag{8.8}$$

in which $\boldsymbol{\Omega}^{\log}$ is the logarithmic spin and $\mathbf{W}$ is the Jaumann spin. The cases $\xi = 1$ and $\xi = 0$ correspond to the Jaumann spin and the logarithmic spin, respectively. We have

$$f_{\mathrm{L}} - f_{\mathrm{D}} = (1-\xi)(f_{\mathrm{XBM}} - f_{\mathrm{GN}}),$$

where f_{L} denotes the f corresponding to $\boldsymbol{\Omega}_{\mathrm{L}} = (1-\xi)\boldsymbol{\Omega}^{\log} + \xi\mathbf{W}$ and f_{D} is defined in (8.5). Since $f_{\mathrm{XBM}} \geqslant f_{\mathrm{GN}}$, we obtain

$$f_{\mathrm{L}} \geqslant f_{\mathrm{D}},$$

no matter how small $\xi > 0$ is. Therefore, the homogeneous equation corresponding to (8.1) with f_{D} is a Sturm minorant for the homogeneous equation corresponding to (8.1) with f_{L}. Next, from (5.8) we know that the homogeneous solution of (8.1) with the above f_{L} is bounded on $\mathbb{R}^+$. Moreover, note that $f_{\mathrm{XBM}}(0) = f_{\mathrm{GN}}(0) = 1$, hence $f_{\mathrm{L}}(0) = f_{\mathrm{D}}(0) = 1$, and thus from the above inequality and the inequality in (8.7) it follows that the forcing term in (8.1) satisfies the finiteness condition,

$$\mu \int_0^\infty \left| \left(\frac{1}{f_{\mathrm{L}}(\gamma)} \right)' \right| \mathrm{d}\gamma = \mu \left[\frac{1}{f_{\mathrm{L}}(\infty)} - \frac{1}{f_{\mathrm{L}}(0)} \right] \leqslant \mu \left[\frac{1}{f_{\mathrm{D}}(\infty)} - \frac{1}{f_{\mathrm{D}}(0)} \right] < \infty, \quad 0 < \xi \leqslant 1,$$

where $f'_{\mathrm{L}}(\gamma) \leqslant 0$ on $\mathbb{R}^+$ has been taken into account. By Theorem 1 the hypoelastic equations with spins $\mathbf{\Omega}_{\mathrm{L}}$, $0 < \xi \leqslant 1$, are oscillatory. From this, we conclude that the Xiao–Bruhns–Meyers equation is the infimum for the oscillatory hypoelastic equations with spins $\mathbf{\Omega}_{\mathrm{L}}$. □

In the proof of Theorem 7 we have compared the f's in terms of θ. However, we have not yet derived the hypoelastic equation in terms of θ. In the following we recast (8.1) in terms of θ. In view of the homeomorphism between γ and θ (see $(4.3)_1$ and Figure 1), we set

$$f(\gamma)\dot{\gamma} = F(\theta)\dot{\theta}. \tag{8.9}$$

Thus

$$F(\theta) = \frac{2}{\cos^2\theta} f(2\tan\theta) > 0. \tag{8.10}$$

From (5.4) and (4.3) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{1}{F(\theta)} \frac{\mathrm{d}\sigma_{12}}{\mathrm{d}\theta} \right] + F(\theta)\sigma_{12} = 2\mu \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\sec^2\theta}{F(\theta)} \right) \tag{8.11}$$

at $\theta \in [0, \pi/2)$. Introduce $u = \sigma_{12}/\sqrt{F(\theta)}$, which transforms (8.11) into

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + S(\theta)u = 2\mu \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\sec^2\theta}{F(\theta)} \right)\sqrt{F(\theta)}, \tag{8.12}$$

where

$$S(\theta) := \frac{\mathrm{d}^2 F(\theta)/\mathrm{d}\theta^2}{2F(\theta)} - \frac{3(\mathrm{d}F(\theta)/\mathrm{d}\theta)^2}{4F^2(\theta)} + F^2(\theta). \tag{8.13}$$

For the rates of Jaumann, Green–Naghdi, and Sowerby–Chu, the values of $F(\theta)$ are $2/\cos^2\theta$, 2, and 1, respectively, and thus the values of $S(\theta)$ are $1 + 4/\cos^4\theta$, 4, and 1, respectively. Obviously, $1 + 4/\cos^4\theta > 4 > 1$. For the rate of Xiao–Bruhns–Meyers the explicit form of $F(\theta)$ and $S(\theta)$ are

$$F_{\mathrm{XBM}}(\theta) = 1 + \frac{\sin\theta}{\cos^2\theta \ln((1+\sin\theta)/\cos\theta)}$$

and

$$S_{\rm XBM} = \frac{\sin^2\theta + 2\sin\theta(1+2\cos^2\theta)\ln((1+\sin\theta)/\cos\theta)}{4\cos^2\theta\ln^2((1+\sin\theta)/\cos\theta)(\cos^2\theta\ln((1+\sin\theta)/\cos\theta)+\sin\theta)^2}$$
$$-\frac{\cos^2\theta(7+5\sin^2\theta)\ln^2((1+\sin\theta)/\cos\theta)}{4\cos^2\theta\ln^2((1+\sin\theta)/\cos\theta)(\cos^2\theta\ln((1+\sin\theta)/\cos\theta)+\sin\theta)^2}$$
$$+\frac{2\sin\theta\cos^2\theta(5+\sin^2\theta)\ln^3((1+\sin\theta)/\cos\theta)}{4\cos^2\theta\ln^2((1+\sin\theta)/\cos\theta)(\cos^2\theta\ln((1+\sin\theta)/\cos\theta)+\sin\theta)^2}$$
$$+\left(1+\frac{\sin\theta}{\cos^2\theta\ln((1+\sin\theta)/\cos\theta)}\right)^2,$$

respectively.

According to Theorem 2, a sufficient condition for (8.12) to be non-oscillatory is that the homogeneous equation corresponding to (8.12) possesses a non-oscillatory majorant. We give below another non-oscillation criteria for the hypoelastic equations, the applicability of which is broader than that of the criteria given in Theorem 6.

THEOREM 8. *If $F(\theta)$ is twice continuously differentiable on the interval $[0, \pi/2)$ of θ and $S(\theta)$ is given in (8.13), then the hypoelastic equation (8.12) with $F(\theta)$ satisfying $S(\theta) \leqslant S_{XBM}$ is non-oscillatory on the interval $[0, \pi/2)$ of θ.*

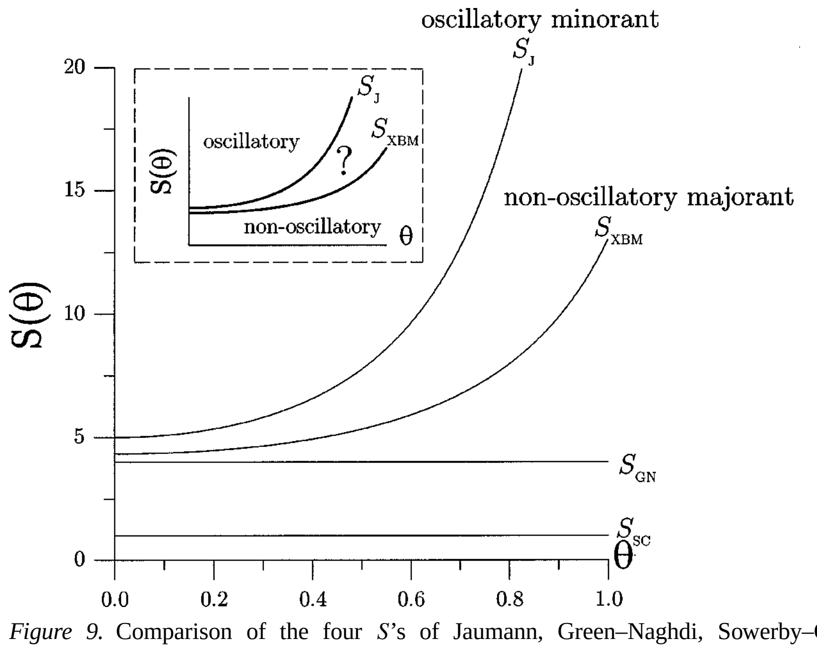

Figure 9. Comparison of the four S's of Jaumann, Green–Naghdi, Sowerby–Chu and Xiao–Bruhns–Meyers. Among the homogeneous hypoelastic equations associated with spinning objective rates, the homogeneous Xiao–Bruhns–Meyers equation is a non-oscillatory majorant, and the Jaumann equation is an oscillatory minorant.

Proof. If $S(\theta) \leqslant S_{\mathrm{XBM}}$, then the homogeneous equation corresponding to (8.12) is a Sturm minorant for the homogeneous Xiao–Bruhns–Meyers equation. Since the homogeneous Xiao–Bruhns–Meyers equation is non-oscillatory as proved in Theorem 7, by Theorem 2, the hypoelastic equation (8.12) with $F(\theta)$ satisfying $S(\theta) \leqslant S_{\mathrm{XBM}}$ is non-oscillatory on the interval $[0, \pi/2)$ of θ. □

Figure 9 compares the four S's of Jaumann, Green–Naghdi, Sowerby–Chu and Xiao–Bruhns–Meyers, showing that $S_{\mathrm{SC}} < S_{\mathrm{GN}} < S_{\mathrm{XBM}}$, from which we conclude that the Sowerby–Chu and Green–Naghdi equations are non-oscillatory because the Xiao–Bruhns–Meyers equation is non-oscillatory.

9. Comparison Theorems for the Hypoelastic Equations

In this section and the next, let us go on to compare the objective rates, no matter spinning or non-spinning, that is, let us compare various hypoelastic equations of the form (6.2). We observe that the trace of $\mathbf{A}$ at simple shear is zero for all the objective rates tabulated in Table I, and so we set the restriction $\operatorname{tr}\mathbf{A} = a + d = 0$ in the present section and the next one. In order to obtain the stress responses of (6.2) we need to know the fundamental solution $\boldsymbol{\Psi}(\gamma)$ of (6.2), which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\begin{pmatrix} \Psi_{11}(\gamma) & \Psi_{12}(\gamma) \\ \Psi_{21}(\gamma) & \Psi_{22}(\gamma) \end{pmatrix} = \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix}\begin{pmatrix} \Psi_{11}(\gamma) & \Psi_{12}(\gamma) \\ \Psi_{21}(\gamma) & \Psi_{22}(\gamma) \end{pmatrix} \tag{9.1}$$

and $\boldsymbol{\Psi}(0) = \mathbf{I}_3$. Since $\boldsymbol{\Psi}(0) = \mathbf{I}_3$, under the condition $\operatorname{tr}\mathbf{A} = 0$ it always follows that $\det \boldsymbol{\Psi}(\gamma) = 1$. If the above $\boldsymbol{\Psi}(\gamma)$ is obtained, in view of (2.2) the stress responses are given by

$$\begin{aligned} &\begin{pmatrix} \sigma_{11}(\gamma) & \sigma_{12}(\gamma) \\ \sigma_{12}(\gamma) & \sigma_{22}(\gamma) \end{pmatrix} \\ &\quad = \begin{pmatrix} \Psi_{11}(\gamma) & \Psi_{12}(\gamma) \\ \Psi_{21}(\gamma) & \Psi_{22}(\gamma) \end{pmatrix}\begin{pmatrix} \sigma_{11}(0) & \sigma_{12}(0) \\ \sigma_{12}(0) & \sigma_{22}(0) \end{pmatrix} \\ &\qquad \times \begin{pmatrix} \Psi_{11}(\gamma) & \Psi_{21}(\gamma) \\ \Psi_{12}(\gamma) & \Psi_{22}(\gamma) \end{pmatrix} + \mu \begin{pmatrix} \Psi_{11}(\gamma) & \Psi_{12}(\gamma) \\ \Psi_{21}(\gamma) & \Psi_{22}(\gamma) \end{pmatrix} \\ &\qquad \times \begin{pmatrix} -2\displaystyle\int_0^{\gamma} \Psi_{12}(s)\Psi_{22}(s)\,\mathrm{d}s & \gamma + 2\displaystyle\int_0^{\gamma} \Psi_{12}(s)\Psi_{21}(s)\,\mathrm{d}s \\ \gamma + 2\displaystyle\int_0^{\gamma} \Psi_{12}(s)\Psi_{21}(s)\,\mathrm{d}s & -2\displaystyle\int_0^{\gamma} \Psi_{11}(s)\Psi_{21}(s)\,\mathrm{d}s \end{pmatrix} \\ &\qquad \times \begin{pmatrix} \Psi_{11}(\gamma) & \Psi_{21}(\gamma) \\ \Psi_{12}(\gamma) & \Psi_{22}(\gamma) \end{pmatrix}. \end{aligned} \tag{9.2}$$

We call a fundamental solution $\boldsymbol{\Psi}$ satisfying (9.1) disconjugate (respectively non-oscillatory, oscillatory) on the interval $\mathbb{R}^+$ if every (respectively every, at least one) component of $\boldsymbol{\Psi}$ is disconjugate (respectively non-oscillatory, oscillatory) on the interval $\mathbb{R}^+$. Similarly, we call the stress $\boldsymbol{\sigma}$ non-oscillatory (respectively oscillatory) on the interval $\mathbb{R}^+$ if every (respectively at least one) component of $\boldsymbol{\sigma}$ is non-oscillatory (respectively oscillatory) on the interval $\mathbb{R}^+$. The following results are the extensions of Lemmas 2–4; Lemmas 9–11 are for the tensorial ODEs (9.1) and (6.2) just as Lemmas 2–4 are for the scalar ODEs (7.1) and (7.2).

LEMMA 9. *Suppose that the coefficients $a(\gamma)$, $b(\gamma)$, $c(\gamma)$, and $d(\gamma)$ of (6.2) are bounded on the interval $\mathbb{R}^+$ and the fundamental solution $\boldsymbol{\Psi}$ of (6.2) is bounded and absolutely integrable on $\mathbb{R}^+$. Then the fundamental solution $\boldsymbol{\Psi}$ is oscillatory on the interval $\mathbb{R}^+$ if and only if the stress $\boldsymbol{\sigma}$ of (6.2) is oscillatory on the interval $\mathbb{R}^+$.*

Proof. Let us consider the Wronskian of Ψ_{ij} and σ_{ij},

$$W_{ij} = \begin{vmatrix} \Psi_{ij} & \sigma_{ij} \\ \Psi'_{ij} & \sigma'_{ij} \end{vmatrix} = \Psi_{ij}\sigma'_{ij} - \Psi'_{ij}\sigma_{ij}, \quad i, j = 1, 2, \ i, j \text{ not summed.} \tag{9.3}$$

Denote the integrals in (9.2) by $\mathbf{C}$, i.e.,

$$\mathbf{C} := \mu \begin{pmatrix} -2\int_0^\gamma \Psi_{12}(s)\Psi_{22}(s)\,\mathrm{d}s & \gamma + 2\int_0^\gamma \Psi_{12}(s)\Psi_{21}(s)\,\mathrm{d}s \\ \gamma + 2\int_0^\gamma \Psi_{12}(s)\Psi_{21}(s)\,\mathrm{d}s & -2\int_0^\gamma \Psi_{11}(s)\Psi_{21}(s)\,\mathrm{d}s \end{pmatrix}. \tag{9.4}$$

Substituting (9.2) into (9.3), we obtain

$$\begin{aligned} W_{ij} \;=\; & \Psi_{ij}\big[\Psi'_{im}\sigma_{mn}(0)\Psi_{jn} + \Psi_{im}\sigma_{mn}(0)\Psi'_{jn} \\ & + \Psi'_{im}C_{mn}\Psi_{jn} + \Psi_{im}C'_{mn}\Psi_{jn} + \Psi_{im}C_{mn}\Psi'_{jn}\big] \\ & - \Psi'_{ij}\big[\Psi_{im}\sigma_{mn}(0)\Psi_{jn} + \Psi_{im}C_{mn}\Psi_{jn}\big]. \end{aligned} \tag{9.5}$$

Substituting (9.1) and (9.4) into (9.5) yields

$$\begin{aligned} W_{ij} \;=\; & \big\{\Psi_{ij}A_{i\ell}\Psi_{\ell m}\sigma_{mn}(0)\Psi_{jn} + \Psi_{ij}\Psi_{im}\sigma_{mn}(0)A_{j\ell}\Psi_{\ell n} \\ & + \Psi_{ij}A_{i\ell}\Psi_{\ell m}\widetilde{C}_{mn}\Psi_{jn} \\ & + \Psi_{ij}\Psi_{im}C'_{mn}\Psi_{jn} + \Psi_{ij}\Psi_{im}\widetilde{C}_{mn}A_{j\ell}\Psi_{\ell n} \\ & - A_{i\ell}\Psi_{\ell j}\Psi_{im}\sigma_{mn}(0)\Psi_{jn} - A_{i\ell}\Psi_{\ell j}\Psi_{im}\widetilde{C}_{mn}\Psi_{jn}\big\} \\ & + \big\{\Psi_{ij}A_{i\ell}\Psi_{\ell m}\widehat{C}_{mn}\Psi_{jn} + \Psi_{ij}\Psi_{im}\widehat{C}_{mn}A_{j\ell}\Psi_{\ell n} \\ & - A_{i\ell}\Psi_{\ell j}\Psi_{im}\widehat{C}_{mn}\Psi_{jn}\big\}, \end{aligned} \tag{9.6}$$

where

$$\begin{aligned} \mathbf{C}' &= \frac{\mathrm{d}\mathbf{C}}{\mathrm{d}\gamma} = \mu \begin{pmatrix} -2\Psi_{12}(\gamma)\Psi_{22}(\gamma) & 1 + 2\Psi_{12}(\gamma)\Psi_{21}(\gamma) \\ 1 + 2\Psi_{12}(\gamma)\Psi_{21}(\gamma) & -2\Psi_{11}(\gamma)\Psi_{21}(\gamma) \end{pmatrix}, \\ \widetilde{\mathbf{C}} &:= \mu \begin{pmatrix} -2\int_0^\gamma \Psi_{12}(s)\Psi_{22}(s)\,\mathrm{d}s & 2\int_0^\gamma \Psi_{12}(s)\Psi_{21}(s)\,\mathrm{d}s \\ 2\int_0^\gamma \Psi_{12}(s)\Psi_{21}(s)\,\mathrm{d}s & -2\int_0^\gamma \Psi_{11}(s)\Psi_{21}(s)\,\mathrm{d}s \end{pmatrix}, \\ \widehat{\mathbf{C}} &:= \mu \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}. \end{aligned} \tag{9.7}$$

Because the fundamental solution $\mathbf{\Psi}$ is bounded and absolutely integrable on $\mathbb{R}^+$, $\mathbf{C}'$ and $\widetilde{\mathbf{C}}$ are bounded, which together with the boundedness of $\mathbf{A}$ render the boundedness of the first braced term on the right-hand side of (9.6). Moreover, because $\widehat{\mathbf{C}}$ includes γ, when γ is sufficiently large the Wronskian W_{ij} is not equal to zero. Hence, by Lemma 1 the zeros of Ψ_{ij} and σ_{ij} separate each other for sufficiently large γ. Thus, the fundamental solution $\mathbf{\Psi}$ is oscillatory on the interval $\mathbb{R}^+$ if and only if the stress $\boldsymbol{\sigma}$ is oscillatory on the interval $\mathbb{R}^+$. This completes the proof. □

LEMMA 10. *If the fundamental solution $\mathbf{\Psi}$ of (6.2) is non-oscillatory on the interval $\mathbb{R}^+$, then the stress response $\boldsymbol{\sigma}$ of the hypoelastic equation (6.2) is non-oscillatory on the interval $\mathbb{R}^+$.*

Proof. If the fundamental solution $\mathbf{\Psi}$ is non-oscillatory, by definition the components Ψ_{11}, Ψ_{12}, and Ψ_{21} are non-oscillatory. Thus, the four integral terms in (9.2) are also non-oscillatory, rendering non-oscillatory the stress $\boldsymbol{\sigma}$, which is composed by adding and multiplying non-oscillatory terms as indicated in (9.2). □

Obviously a stronger condition suffices to lead to the same result, as given by

LEMMA 11. *If the fundamental solution $\mathbf{\Psi}$ of (6.2) is disconjugate on the interval $\mathbb{R}^+$, then the stress $\boldsymbol{\sigma}$ of the hypoelastic equation (6.2) is non-oscillatory on the interval $\mathbb{R}^+$.*

Let us consider the following four tensorial ODEs of the first order:

$$\begin{pmatrix} \Psi_{11}' & \Psi_{12}' \\ \Psi_{21}' & \Psi_{22}' \end{pmatrix} = \begin{pmatrix} a_j(\gamma) & b_j(\gamma) \\ c_j(\gamma) & d_j(\gamma) \end{pmatrix} \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix}, \qquad j = 1, 2, \tag{9.8}$$

$$\begin{aligned} &\begin{pmatrix} \sigma_{11}' & \sigma_{12}' \\ \sigma_{12}' & \sigma_{22} \end{pmatrix} - \begin{pmatrix} a_j(\gamma) & b_j(\gamma) \\ c_j(\gamma) & d_j(\gamma) \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \\ &\quad - \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \begin{pmatrix} a_j(\gamma) & b_j(\gamma) \\ c_j(\gamma) & d_j(\gamma) \end{pmatrix} = \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad j = 1, 2. \end{aligned} \tag{9.9}$$

These tensorial ODEs extend the scalar ODEs $(7.6)_j$ and $(7.7)_j$.

If $b_j \neq 0$, combining the two equations of (6.7) with coefficients $a_j(\gamma)$, $b_j(\gamma)$, $c_j(\gamma)$, and $d_j(\gamma)$ yields

$$\left(\frac{1}{b_j} x'\right)' - \frac{1}{b_j}\left[a_j' - \frac{a_j b_j'}{b_j} + b_j c_j + a_j^2\right] x = 0, \qquad \text{if } b_j > 0, \quad j \text{ not summed}, \tag{9.10}$$

$$\left(\frac{1}{-b_j} x'\right)' + \frac{1}{b_j}\left[a_j' - \frac{a_j b_j'}{b_j} + b_j c_j + a_j^2\right] x = 0, \qquad \text{if } b_j < 0, \quad j \text{ not summed}. \tag{9.11}$$

Let

$$p_j = \frac{1}{b_j} \quad \text{and} \quad q_j = -\frac{a_j' - a_j b_j'/b_j + b_j c_j + a_j^2}{b_j} \quad \text{for } b_j > 0,$$

or let

$$p_j = -\frac{1}{b_j} \quad \text{and} \quad q_j = \frac{a_j' - a_j b_j'/b_j + b_j c_j + a_j^2}{b_j} \quad \text{for } b_j < 0.$$

Similarly, if $c_j \neq 0$, we may let

$$p_j = \frac{1}{c_j} \quad \text{and} \quad q_j = -\frac{-a_j' + a_j c_j/c_j + b_j c_j + a_j^2}{c_j} \quad \text{for } c_j > 0,$$

or let

$$p_j = -\frac{1}{c_j} \quad \text{and} \quad q_j = \frac{-a_j' + a_j c_j'/c_j + b_j c_j + a_j^2}{c_j} \quad \text{for } c_j < 0.$$

If on the entire interval $\mathbb{R}^+$,

$$p_1(t) \geqslant p_2(t), \qquad q_1(t) \leqslant q_2(t), \tag{9.12}$$

then $(9.8)_2$ is called a *Sturm majorant* for $(9.8)_1$ on $\mathbb{R}^+$, and $(9.8)_1$ is a *Sturm minorant* for $(9.8)_2$ on $\mathbb{R}^+$. Now we are in a position to further extend the Sturm theorems, namely to extend Theorems 1–3, so as to be able to compare the hypoelastic equations $(9.9)_1$ with $(9.9)_2$.

THEOREM 9. *For $j = 1, 2$, suppose that the coefficients $a_j(\gamma)$, $b_j(\gamma)$, $c_j(\gamma)$, and $d_j(\gamma)$ of $(9.9)_j$ are bounded on the interval $\mathbb{R}^+$ and the fundamental solution Ψ_j of $(9.8)_j$ is bounded and absolutely integrable on $\mathbb{R}^+$. Let $(9.8)_1$ be a Sturm minorant for $(9.8)_2$. If $(9.9)_1$ is oscillatory on $\mathbb{R}^+$, then $(9.8)_2$ is oscillatory on $\mathbb{R}^+$.*

Proof. By Lemma 9, if $(9.9)_1$ is oscillatory, then $(9.8)_1$ is oscillatory. Since $(9.8)_1$ is a Sturm minorant for $(9.8)_2$, we observe from Lemma 7 that $(9.8)_2$ is oscillatory, which by Lemma 9 implies that $(9.9)_2$ is oscillatory. This completes the proof. □

THEOREM 10. *Suppose that $(9.8)_2$ is a Sturm majorant for $(9.8)_1$. If $(9.8)_2$ is non-oscillatory on the interval $\mathbb{R}^+$, then $(9.9)_1$ is non-oscillatory on the interval $\mathbb{R}^+$.*

Proof. Since $(9.8)_2$ is a Sturm majorant for $(9.8)_1$, by Lemma 7, if $(9.8)_2$ is non-oscillatory, then $(9.8)_1$ is non-oscillatory. Then by Lemma 10, $(9.9)_1$ is non-oscillatory. □

THEOREM 11. *Suppose that $(9.8)_2$ is a Sturm majorant for $(9.8)_1$. If $(9.8)_2$ is disconjugate on the interval $\mathbb{R}^+$, then $(9.9)_1$ is non-oscillatory on the interval $\mathbb{R}^+$.*

Proof. Since $(9.8)_2$ is a Sturm majorant for $(9.8)_1$, by Lemma 8, the disconjugacy of $(9.8)_2$ implies the disconjugacy of $(9.8)_1$. Then by Lemma 11, $(9.9)_1$ is non-oscillatory. □

The above three theorems are the counterparts of Theorems 1–3, which have been so modified as to apply to the hypoelastic equations associated with the non-spinning objective rates. The definitions of oscillatory minorant, non-oscillatory majorant, and disconjugate majorant originally designed for coining the qualitative relations between the four scalar ODEs $(7.6)_j$ and $(7.7)_j$ can be carried over and tailored to the needs of coining the qualitative relations between the four tensorial ODEs $(9.8)_j$ and $(9.9)_j$. Figures similar to Figures 4–6 can readily be drawn by retaining all the structures, boxes, lines, and arrows in Figures 4–6, while modifying the contents of Figures 4–6 as follows: replacing Lemmas 2–4 by Lemmas 9–11, Theorems 1–3 by Theorems 9–11, (7.1) by $\boldsymbol{\Psi}$, (7.2) by $\boldsymbol{\sigma}$, $(7.6)_1$ in (b) by $(9.10)_1$ or $(9.11)_1$, $(7.6)_2$ in (b) by $(9.10)_2$ or $(9.11)_2$, $(7.6)_1$ in (c) by $(9.8)_1$, $(7.6)_2$ in (c) by $(9.8)_2$, $(7.7)_1$ by $(9.9)_1$, $(7.7)_2$ by $(9.9)_2$, respectively, and deleting homogeneous and non-homogeneous, but Lemmas 7 and 8 remain unchanged.

10. Comparison between Objective Rates

The stress responses of the hypoelastic equations of Szabó–Balla-1 and Szabó–Balla-2 were already obtained in (6.18) and (6.20), respectively. These two formulae are rather lengthy, and so it is hard to judge whether their behavior is oscillatory or not, unless they are explicitly calculated with the aid of graphical display to the extent that all possible ranges of parameters have been exhausted. Hence, it seems desirable to prove the following theorem, which asserts that the hypoelastic equations of Szabó–Balla-1 and Szabó–Balla-2 are non-oscillatory.

THEOREM 12. *The homogeneous Green–Naghdi equation is a Sturm majorant for the homogeneous equations of Szabó–Balla-1 and Szabó–Balla-2. The fundamental solutions of these three equations are disconjugate on the interval* $\mathbb{R}^+$, *and the stress responses are non-oscillatory on the interval* $\mathbb{R}^+$.

Proof. In view of (6.11) we set $p_{\mathrm{SB1}} = \gamma^2 + 4$ and $q_{\mathrm{SB1}} = 1/(\gamma^2 + 4)$. Since $\gamma^2 + 4 > (\gamma^2 + 4)/4$ and $1/(\gamma^2 + 4) < 4/(\gamma^2 + 4)$, i.e., $p_{\mathrm{SB1}} > p_{\mathrm{GN}}$ and $q_{\mathrm{SB1}} < q_{\mathrm{GN}}$, the homogeneous Green–Naghdi equation is a strict Sturm majorant for the homogeneous Szabó–Balla-1 equation. Since the nontrivial solution of (8.3) is disconjugate as proved in Theorem 6, the fundamental solution of (6.11) is also disconjugate by Lemma 8. Similarly, since $\gamma^2 + 4 > (\gamma^2 + 4)/4$ and $-(2\gamma^2 + 7)/(\gamma^2 + 4) < 4/(\gamma^2 + 4)$, the homogeneous Green–Naghdi equation is a strict Sturm majorant for (6.12). By the same token, the fundamental solution of (6.12) is also disconjugate. Hence, the fundamental solution matrix $\boldsymbol{\Psi}$, composed of the fundamental solutions of (6.8) and (6.9), is also disconjugate. Thus, by Lemma 11, the solution of (6.3) is non-oscillatory. In view of (6.19), namely $\boldsymbol{\Psi}_{\mathrm{SB2}} = \boldsymbol{\Psi}_{\mathrm{SB1}}^{-\mathrm{T}}$, the disconjugacy of Szabó–Balla-1 implies the disconjugacy of Szabó–Balla-2, and vice versa. □

Note that (6.1)–(6.3) apply to the cases of the spinning objective rates as well and can be reduced to (5.1)–(5.3) by letting $a(\gamma) = d(\gamma) = 0$, and $b(\gamma) = -c(\gamma) = f(\gamma)/2$. So the following oscillation and non-oscillation criteria, which are applicable to the hypoelastic equation (6.3) associated with any non-spinning objective rate, are also applicable to the hypoelastic equations (5.3) associated with any spinning objective rate.

THEOREM 13 (Oscillation criteria). *Suppose that the coefficients* $a(\gamma)$, $b(\gamma)$, $c(\gamma)$, *and* $d(\gamma)$ *of (6.2) are continuously differentiable and bounded on the interval* $\mathbb{R}^+$, $b^2 + c^2 \neq 0$, $a + d = 0$, *and the fundamental solution* $\mathbf{\Psi}$ *of (6.2) is bounded and absolutely integrable on* $\mathbb{R}^+$. *If*

$$b \neq 0, \quad \frac{1}{|b|} \leqslant p_{\mathrm{DB}} = \frac{4}{3}, \qquad \frac{-(a' - ab'/b + bc + a^2)}{|b|} \geqslant q_{\mathrm{DB}} = \frac{1}{4}$$

on the interval $\mathbb{R}^+$, *or if*

$$c \neq 0, \quad \frac{1}{|c|} \leqslant p_{\mathrm{DB}}, \qquad \frac{-(ac'/c + bc + a^2 - a')}{|c|} \geqslant q_{\mathrm{DB}}$$

on the interval $\mathbb{R}^+$, *then the stress responses of (6.2) are oscillatory on the interval* $\mathbb{R}^+$.

Proof. If $b \neq 0$, combining the two equations of (6.7) yields

$$\left(\frac{1}{b}x'\right)' - \frac{1}{b}\left[a' - \frac{ab'}{b} + bc + a^2\right]x = 0, \qquad \text{if } b > 0, \tag{10.1}$$

$$\left(\frac{1}{-b}x'\right)' + \frac{1}{b}\left[a' - \frac{ab'}{b} + bc + a^2\right]x = 0, \quad \text{if } b < 0. \tag{10.2}$$

Let $p = 1/b$ and $q = -[a' - ab'/b + bc + a^2]/b$ for $b > 0$, or let $p = -1/b$ and $q = [a' - ab'/b + bc + a^2]/b$ for $b < 0$. If the conditions $1/|b| \leqslant p_{\mathrm{DB}}$ and $-(a' - ab'/b + bc + a^2)/|b| \geqslant q_{\mathrm{DB}}(\gamma)$ are satisfied, i.e., $p \leqslant p_{\mathrm{DB}}$ and $q \geqslant q_{\mathrm{DB}}$, the comparison made between the above two equations and the Sturm–Liouville equation with p_{DB} and q_{DB} reveals that the latter is a Sturm minorant for the above two equations. Because the responses of the Durban–Baruch equation as indicated in (6.6) are oscillatory, by Theorem 9 the stress responses of (6.2) are oscillatory. □

THEOREM 14 (Non-oscillation criteria). *Suppose that the coefficients* $a(\gamma)$, $b(\gamma)$, $c(\gamma)$, *and* $d(\gamma)$ *of (6.2) are continuously differentiable on the interval* $\mathbb{R}^+$, $b^2+c^2 \neq 0$, *and* $a + d = 0$. *If*

$$b \neq 0, \quad |b| \leqslant f_{\mathrm{XBM}}, \qquad \frac{-(a' - ab'/b + bc + a^2)}{|b|} \leqslant f_{\mathrm{XBM}}$$

on the interval $\mathbb{R}^+$, *or if*

$$c \neq 0, \quad |c| \leqslant f_{\mathrm{XBM}}, \qquad \frac{-(ac'/c + bc + a^2 - a')}{|c|} \leqslant f_{\mathrm{XBM}}$$

on the interval $\mathbb{R}^+$, then the fundamental solutions of (6.2) are non-oscillatory on the interval $\mathbb{R}^+$, and the stress responses of (6.2) are non-oscillatory on the interval $\mathbb{R}^+$.

Proof. If $b \neq 0$, according to (10.1) and (10.2) we set $p = 1/b$ and $q = -[a' - ab'/b+bc+a^2]/b$ for $b > 0$, or set $p = -1/b$ and $q = [a' - ab'/b+bc+a^2]/b$ for $b < 0$. If $|b| \leqslant f_{\mathrm{XBM}}$ and $-(a' - ab'/b+bc+a^2)/|b| \leqslant f_{\mathrm{XBM}}(\gamma)$, i.e., $p \geqslant 1/f_{\mathrm{XBM}}$ and $q \leqslant f_{\mathrm{XBM}}$, the comparison made between the above two equations and the homogeneous equation corresponding to (8.1) with f_{XBM} reveals that the latter is a Sturm majorant for the above two equations. Because the homogeneous Xiao–Bruhns–Meyers equation is non-oscillatory, by Lemma 3, x is non-oscillatory. From $(6.7)_1$ we have $y = (x' - ax)/b$, hence y is also non-oscillatory. If $c \neq 0$, a similar procedure leads to the results that y and x are non-oscillatory under the conditions $|c| \leqslant f_{\mathrm{XBM}}$ and $-(ac'/c + bc + a^2 - a')/|c| \leqslant f_{\mathrm{XBM}}$. Accordingly, the fundamental solutions of (6.2) are non-oscillatory. Therefore, by Lemma 10 the stress responses of (6.2) are non-oscillatory. □

Further discussion of Figure 2. The responses of the ten hypoelastic models at simple shear are compared in Figure 2. The axial stress σ_{11} of Cotter–Rivlin is zero (see also Table II) and both the axial and shear responses of Jaumann and Durban–Baruch are oscillatory (see also Table II). Both the axial and shear stress responses of Sowerby–Chu and Szabó–Balla-1 are monotonically increasing. However, for Szabó–Balla-2, the shear stress σ_{12} is monotonically increasing, while the axial stress σ_{11} is monotonically decreasing and compressive (see also Table II). It is interesting to observe the shear stress response of Xiao–Bruhns–Meyers exhibits the hypoelastic yield-like behavior. It can be seen that the response curves of the axial stresses of Truesdell, Oldroyd and Szabó–Balla-1 and those of the shear stress of Szabó–Balla (the two rates of SB1 and SB2 lead to the same shear stress) are all very steep.

Discussion of Table II. In Table II the p's and q's for the ten hypoelastic equations are listed, and the oscillatory properties are also indicated. It can be seen that among the ten q's, only the q of Szabó–Balla-2 is negative. This is why the axial stress σ_{11} of Szabó–Balla-2 is compressive, as shown in Figure 2(a) and Table II. The p's and q's of the first five rates in Table II are very simple; it seems that they are too simple to give good responses. In Figure 10, we plot the ten p's and q's. It is seen that $p_{\mathrm{SB2}} = p_{\mathrm{SB1}} > p_{\mathrm{SC}} > p_{\mathrm{GN}} \geqslant p_{\mathrm{XBM}}$ and $q_{\mathrm{SB2}} < q_{\mathrm{SB1}} < q_{\mathrm{SC}} < q_{\mathrm{GN}} \leqslant q_{\mathrm{XBM}}$, which mean that the homogeneous Xiao–Bruhns–Meyers equation is a Sturm majorant for the homogeneous equations of Green–Naghdi, Sowerby–Chu and Szabó–Balla. Since the homogeneous Xiao–Bruhns–Meyers equation is non-oscillatory, the hypoelastic equations of Green–Naghdi, Sowerby–Chu and Szabó–Balla are non-oscillatory, and so the responses pertaining to the models using these rates are non-oscillatory.

Discussion of Table III. The Sturm majorant-minorant relations of the ten objective rates at simple shear are summarized in Table III. It provides a clear picture of

Table II. Characteristic properties of the hypoelastic equations at simple shear. See Theorem 1 for (ii), (8.4) for (D) and (8.8) for (L)

Equations	p	q	Homogeneous eqs.	r satisfies (ii)?	Hypoelastic eqs.	σ_{11} tensile?	σ_{12} yield?
1 (T)	1	0	disconjugate	no	non-oscillatory	tensile	no
2 (O)	1	0	disconjugate	no	non-oscillatory	tensile	no
3 (CR)	1	0	disconjugate	no	non-oscillatory	vanishing	no
4 (J)	1	$1 = f_{\mathrm{J}}$	oscillatory	yes	oscillatory	tensile	no
5 (DB)	$4/3$	$1/4$	oscillatory	no	oscillatory	tensile	no
6 (GN)	$(\gamma^2+4)/4$	$4/(\gamma^2+4) = f_{\mathrm{GN}}$	disconjugate	no	non-oscillatory	tensile	no
7 (SC)	$(\gamma^2+4)/2$	$2/(\gamma^2+4) = f_{\mathrm{SC}}$	disconjugate	no	non-oscillatory	tensile	no
8 (SB1)	γ^2+4	$1/(\gamma^2+4)$	disconjugate	no	non-oscillatory	tensile	no
9 (SB2)	γ^2+4	$-(2\gamma^2+7)/(\gamma^2+4)$	disconjugate	no	non-oscillatory	compressive	no
10 (XBM)	$1/f_{\mathrm{XBM}}$	f_{XBM}; see (5.16)	non-oscillatory	no	non-oscillatory	tensile	yes
11 (D)	$(\gamma^2+4)/(\xi\gamma^2+4)$	$(\xi\gamma^2+4)/(\gamma^2+4)$	oscillatory	yes	oscillatory		
12 (L)	$1/((1-\xi)f_{\mathrm{XBM}}+\xi)$	$(1-\xi)f_{\mathrm{XBM}}+\xi$	oscillatory	yes	oscillatory		

Table III. Sturm majorant–minorant relations of the homogeneous hypoelastic equations at simple shear. M: Sturm majorant, m: Sturm minorant. Example: (T) is a Sturm majorant for (SB2)

Homogeneous eqs.	1 (T)	2 (O)	3 (CR)	4 (J)	5 (DB)	6 (GN)	7 (SC)	8 (SB1)	9 (SB2)	10 (XBM)
1 (T)	M&m	M&m	M&m	m	×	×	×	×	M	×
2 (O)	M&m	M&m	M&m	m	×	×	×	×	M	×
3 (CR)	M&m	M&m	M&m	m	×	×	×	×	M	×
4 (J)	M	M	M	M&m	M	M	M	M	M	M
5 (DB)	×	×	×	m	M&m	×	×	M	M	×
6 (GN)	×	×	×	m	×	M&m	M	M	M	m
7 (SC)	×	×	×	m	×	m	M&m	M	M	m
8 (SB1)	×	×	×	m	m	m	m	M&m	M	m
9 (SB2)	m	m	m	m	m	m	m	m	M&m	m
10 (XBM)	×	×	×	m	×	M	M	M	M	M&m

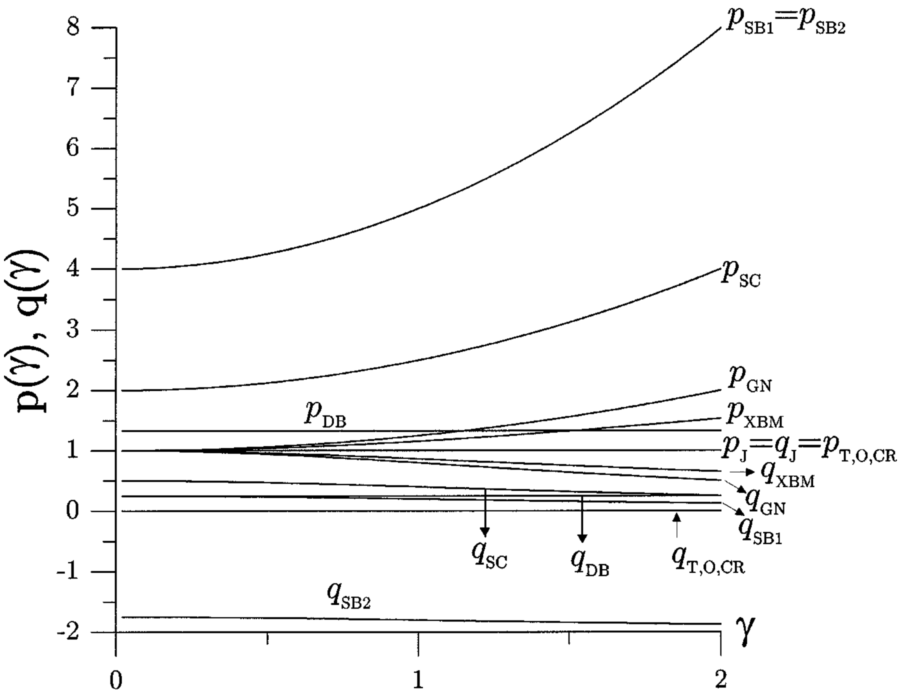

Figure 10. Comparison of the p's and q's of the ten hypoelastic equations. It is seen that $p_{SB2} = p_{SB1} > p_{SC} > p_{GN} \geqslant p_{XBM}$ and $q_{SB2} < q_{SB1} < q_{SC} < q_{GN} \leqslant q_{XBM}$. $p_J = p_T = p_O = p_R$ and $q_J > q_T = q_O = q_{CR}$. $p_{SB2} > p_T = p_O = p_{CR}$ and $q_{SB2} < q_T = q_O = q_{CR}$.

the Sturm majorant-minorant order, as also shown in the left side of Figure 11. In this table the Sturm non-comparable stress rates in the sense of (7.8) are marked by the symbol $\times$, and M and m stand for the Sturm majorant and Sturm minorant relations, respectively, and M&m is the identity relation. The hypoelastic equations of Truesdell, Oldroyd and Cotter–Rivlin are coincident at simple shear, and are not Sturm comparable with the other equations except for those of Jaumann and Szabó–Balla-2. Remarkably, the Jaumann equation is a Sturm majorant for all the other nine homogeneous equations, and the homogeneous Szabó–Balla-2 equation is a Sturm minorant for all the other nine homogeneous equations. It is hardly surprising that both the extreme hypoelastic equations show anomalous behavior.

11. Concluding Remarks

Most of the closed-form solutions of the hypoelastic equations at simple shear have been obtained in the literature, although without considering the initial stresses. Here, we have derived the closed-form solutions with any specified initial stresses

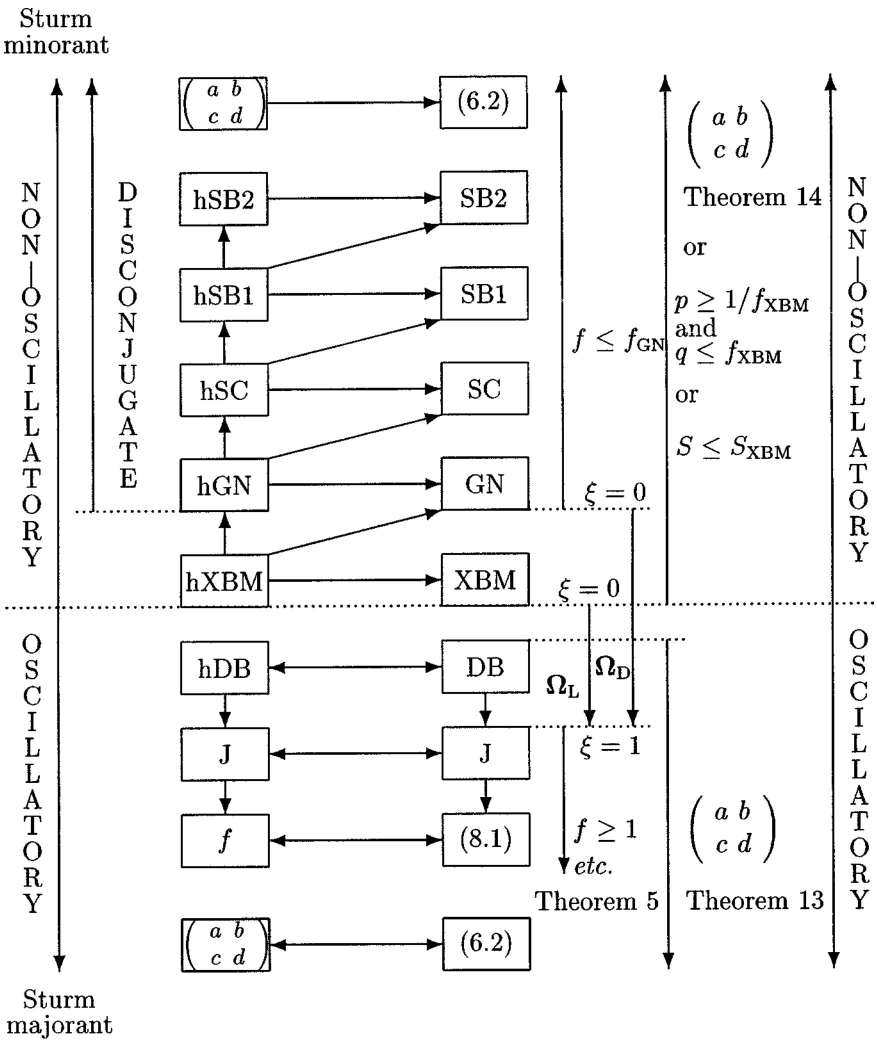

Figure 11. According to Theorems 5–14, general and individual hypoelastic equations are compared on the right side and classified as oscillatory or non-oscillatory. The corresponding homogeneous (abbreviated to h) equations are arranged on the left side in the Sturm minorant–majorant order when connected with vertical lines and classified as oscillatory, non-oscillatory, or disconjugate. The arrows between boxes have the same meaning as those in Figures 4–6 or in the similar figures described only in writing in Section 9.

for the ten hypoelastic equations at simple shear. One can not overemphasize the effect of the initial conditions on the oscillatory properties of the responses.

Sturm's theorems in ordinary differential equations have been extended as in Theorems 1–3 and further extended as in Theorems 9–11 to investigate the qualitative behavior of the hypoelastic models of grade zero under simple shear deformation.

For the ten individual hypoelastic equations examined, the following may be concluded: The Szabó–Balla-2 equation is non-oscillatory, but its axial stress response is anomalously compressive. The homogeneous Szabó–Balla-2 equation is disconjugate and is a Sturm minorant for all the other nine homogeneous hypoelastic equations. The Sowerby–Chu equation is non-oscillatory, and the homogeneous Sowerby–Chu equation is a disconjugate majorant for the *two* homogeneous equations of Szabó–Balla-1 and Szabó–Balla-2. The Green–Naghdi equation is non-oscillatory, and is the "infimum" for the oscillatory behavior of the hypoelastic equations associated with the class of spins $\mathbf{\Omega}_{\mathrm{D}}$ defined in (8.4). The homogeneous Green–Naghdi equation is a disconjugate majorant for the *three* homogeneous equations of Sowerby–Chu, Szabó–Balla-1, and Szabó–Balla-2. The Xiao–Bruhns–Meyers equation is non-oscillatory with the shear stress response exhibiting hypoelastic yield-like phenomenon, and is the "infimum" for the oscillatory behavior of the hypoelastic equations associated with the class of spins $\mathbf{\Omega}_{\mathrm{L}}$ defined in (8.8). The homogeneous Xiao–Bruhns–Meyers equation is a non-oscillatory majorant for the *four* homogeneous equations of Green–Naghdi, Sowerby–Chu, Szabó–Balla-1, and Szabó–Balla-2. The Durban–Baruch equation is oscillatory, and is an oscillatory minorant for the Jaumann equation. The Jaumann equation is oscillatory; moreover, it is a Sturm majorant for all the other nine homogeneous hypoelastic equations. Fortunately, the Jaumann equation is not a Sturm minorant for all the other nine homogeneous hypoelastic equations; otherwise, all the other nine hypoelastic equations would be oscillatory. The Truesdell, Oldroyd, and Cotter–Rivlin equations are Sturm comparable with the Szabó–Balla-2 and Jaumann equations, but are not Sturm comparable with the other five equations.

Besides the ten individual hypoelastic equations examined, general hypoelastic equations have also be investigated. The main results (for both general and individual equations) are summarized in Figure 11. Theorem 5 provides a set of sufficient conditions (i.e., $f \geqslant 1$, etc.) for the hypoelastic equation (and hence for the stress response to simple shearing) associated with any spinning objective rate to be oscillatory. Theorem 6 (respectively Theorems 7 and 8) asserts that the hypoelastic equation (and hence, the stress response to simple shearing) associated with any spinning objective rate with $f \leqslant f_{\mathrm{GN}}$ (respectively $f \leqslant f_{\mathrm{XBM}}$ and $S \leqslant S_{\mathrm{XBM}}$) is non-oscillatory. Theorem 13 provides a set of sufficient conditions about a, b, c, and d (and hence, about p and q) for the hypoelastic equation (and hence, for the stress response to simple shearing) associated with any (spinning or non-spinning) objective rate to be oscillatory. Theorem 14 provides a set of sufficient conditions about a, b, c, and d (and hence, about p and q) for the hypoelastic equation (and hence, for the stress response to simple shearing) associated with any (spinning or non-spinning) objective rate to be non-oscillatory.

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