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APPLICATION OF INTEGRAL EQUATIONS WITH SUPERSTRONG SINGULARITY TO STEADY STATE HEAT CONDUCTION

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ABSTRACT

In this paper, we present the theory of dual integral equations for steady state heat conduction. There are four kernel functions with different orders of singularity in the two equations. Using the first equation with weaker singularity, the conventional direct boundary integral equation method (BIEM) was proposed long ago. An important characteristics of the first equation is that its kernels are of the Riemann and Cauchy types. The purpose of this paper is to present a method based on the second equation with stronger singularity kernels to solve the steady state heat conduction problems. Whereas the kernels of the second equation are of the Cauchy and Hadamard types. It is further shown that combination of the two equations can be used to solve problems with degenerate boundary which have long suffered from lack of a general formulation of the BIEM. For concreteness, an illustrative example is performed numerically to see the validity of the theory.

INTRODUCTION

The boundary integral equation method(BIEM) has been developed and applied quite successfully in various engineering problems such as in elasticity [1] and acoustics. Several attempts have been made for analysis of heat conduction in steady and nonsteady states [2,3,4]. But most BIEM literatures are on solving problems with simply-connected domain by evaluating the Riemann and Cauchy principal values. In this paper, we present a new application of superstrong singularity in the boundary element formulation. It is especially useful for those problems with degenerate boundary which may be a flux-guiding baffle, a heat-generating fin, or a

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nearly-insulating defect. In the conventional boundary element formulation, an artificial boundary was introduced to solve the problems [5] and it is known as the zone method. In fact, on the degenerate boundary, the numbers of unknown and prescribed boundary conditions are both doubled, therefore it is obvious that the number of independent equations should be doubled in order to accommodate the increasing known boundary data and to secure a unique solution. It is the reason why our dual equations work and no artificial boundaries are needed in our method. The superstrong singularity application in fracture mechanics and potential flow was developed by the authors [6,7,8,9,10]. In this paper we adopt the theory to the steady state heat conduction problem. First, we derive the second equation by applying the normal differential operator to Green's second identity, and then obtain the higher singularity boundary integral equation by pushing the point to the boundary. The two equations are called collectively dual boundary integral equations. They are described by means of the usual finite element technique. The alleged supersingularity can be explained by the concept of the Hadamard principal value [10]; although Hadamard [11] didn't treat exactly the same problem; similarities do exist, however.

INTEGRAL EQUATION FORMULATION

Consider the problem of the heat conduction in a homogeneous, isotropic medium with the governing equation

$$\frac{\partial u}{\partial t} = k \nabla^2 u \qquad \text{------------------------------} \quad (1)$$

where u is the temperature, k is diffusity, and ∇^2 denotes the Laplacian operator. The boundary and initial conditions are

B.C. Temperature $u = \bar{u}$

Normal flux $k \frac{\partial u}{\partial n} = \bar{q}$

Convection $k \frac{\partial u}{\partial n} = - h (u - u_f)$

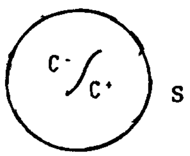

Fig.1 The considered Domain

I.C. $u(x,0) = u_0(x)$

where h is the heat transfer coefficient, u_f is the temperature of the surrounding medium, and n is the unit vector normal to the boundary and pointing out of the domain. For steady state, $\partial/\partial t = 0$, Eq.(1) reduces to the Laplace equation:

$$\nabla^2 u = 0. \qquad (2)$$

Consider a multi-connected region D (Fig.1) which is bounded by a regular boundary S and a degenerate boundary $C = C^+ + C^-$, which may be a baffle for quiding heat flow, a generating fin, or a defect-crack with little conductivity. The regular boundary S and the degenerate boundary C comprise the total boundary B = S + C of the domain D. An appropriate boundary condition must be prescribed everywhere on the boundary S + C; the boundary condition may be the temperature, normal flux, or convection types. It is noted that on the degenerate boundary $C = C^+ + C^-$, boundary conditions should be imposed on both C^+ and C^-, and the conditions imposed can be of different types. For the aforementioned problem a boundary integral equation can be obtained from Green's second identity [5] as

$$2\pi\ u(x) = \int_B T(s,x)\ u(s)\ dB(s) - \int_B U(s,x)\ t(s)\ dB(s) \qquad (3)$$

where $t(s) = \partial u/\partial n$, and u(x) is the temperature of the point x in the domain D bounded by the boundary B and, for the two-dimensional case,

$$U(s,x) = \ln(r) \qquad (4)$$

$$T(s,x) = \frac{\partial}{\partial n_s} \ln(r) \qquad (5)$$

in which r is the distance between the points s and x, and n_s is the outer unit normal to the boundary at the point s. After performing a normal differentiation to the above equation, the other equation emerges :

$$2\pi\ t(x) = \int_B M(s,x)\ u(s)\ dB(s) - \int_B L(s,x)\ t(s)\ dB(s) \qquad (6)$$

where

$$L(s,x) = \partial U(s,x)/\partial n_x \qquad (7)$$

$$M(s,x) = \partial T(s,x)/\partial n_x \qquad (8)$$

in which n_x is the outer unit normal to the boundary at the point x. Eqs.(3) and (6) are called herein the dual integral equations for the domain point x. In order to get a compatible relationship for the boundary unknowns, the point x of Eqs.(3) and (6) has to be on the boundary. This might induce the problem of singularity. Analogous to the treatment of complex contour integral involving poles, the boundary is detoured circularly or spherically around the point x of singularity and then shrinked back to the point. In this way the strong singularity of Cauchy kernel T(s,x) and L(s,x) leads to a Cauchy principal value and a jump term. Another superstrong singularity M(s,x) results in the interpretation of the Hadamard principal value. Accordingly, Eqs.(3) and (6) become

$$\pi\, u(x) = \text{R.P.V.}\int_B T(s,x)\; u(s)\; dB(s) - \text{C.P.V.}\int_B U(s,x)\; t(s)\; dB(s) \quad (9)$$

$$\pi\, t(x) = \text{C.P.V.}\int_B M(s,x)\; u(s)\; dB(s) - \text{H.P.V.}\int_B L(s,x)\; t(s)\; dB(s) \quad (10)$$

where R.P.V., C.P.V. and H.P.V. denote the Riemann , Cauchy and Hadamard principal values, respectively. We call the above equations dual boundary integral equations. After considering the degenerate boundary $B = S + C^{+} + C^{-}$, we can derive the following general equations.

For x on the S boundary,

$$\pi\, u(x) = \text{R.P.V.}\int_s T(s,x)\; u(s)\; dB(s) - \text{C.P.V.}\int_s U(s,x)\; t(s)\; dB(s)$$
$$+ \int_c T(s,x)\; \Sigma u(s)\; dB(s) - \int_c U(s,x)\; \Delta t(s)\; dB(s) \quad (11)$$

$$\pi\, t(x) = \text{C.P.V.}\int_S M(s,x)\; u(s)\; dB(s) - \text{H.P.V}\int_S L(s,x)\; t(s)\; dB(s)$$
$$+ \int_c M(s,x)\; \Sigma u(s)\; dB(s) - \int_c L(s,x)\; \Delta t(s)\; dB(s) \quad (12)$$

For x on C boundary,

$$\pi\, \Sigma u(x) = \int_s T(s,x)\; u(s)\; dB(s) - \int_s U(s,x)\; t(s)\; dB(s)$$
$$+ \text{R.P.V.}\int_c T(s,x)\; \Sigma u(s)\; dB(s) - \text{C.P.V.}\int_c U(s,x)\; \Delta t(s)\; dB(s) \quad (13)$$

$$\pi\, \Delta t(x) = \int_s M(s,x)\; u(s)\; dB(s) - \int_s L(s,x)\; t(s)\; dB(s)$$
$$+ \text{C.P.V.}\int_c M(s,x)\; \Sigma u(s)\; dB(s) - \text{H.P.V.}\int_c L(s,x)\; \Delta t(s)\; dB(s) \quad (14)$$

where $\Delta u(x) = u(x_o^{+}) - u(x_o^{-}) \qquad \Delta t(x) = t(x_o^{+}) - t(x_o^{-})$

$\Sigma u(x) = u(x_o^{+}) + u(x_o^{-}) \qquad \Sigma t(x) = t(x_o^{+}) + t(x_o^{-})$

With Eqs.(11), (12), (13) and (14) obtained, it is sufficient to secure a unique solution.

NUMERICAL IMPLEMENTATION

For the numerical solution of Eqs.(11), (12),(13) and (14) the boundary S and C is divided into a series of constant boundary elements. In real calculation, we evaluate the following coefficients for constant element,

$$M_{ij} = \int M(s_j, x_i)\, dB(s_j)\, u_j \qquad T_{ij} = \int T(s_j, x_i)\, dB(s_j)\, u_j$$
$$L_{ij} = \int L(s_j, x_i)\, dB(s_j)\, \partial u_j/\partial n, \quad U_{ij} = \int U(s_j, x_i)\, dB(s_j)\, \partial u_j/\partial n\ .$$

we combine the two algebraic equation $M\underset{\sim}{u} = L\underset{\sim}{t}$ and $T\underset{\sim}{u} = U\underset{\sim}{t}$ to solve for the boundary unknowns. After we have all the boundary data, the temperature and heat flux at any interior point can be evaluated by Eqs.(3) and (6).

RESULTS AND DISCUSSION

Consider the problem as in Fig.2 with degenerate boundary. The boundary condition is also shown in Fig.2. Using 30 elements on the regular boundary and 5 elements on the degenerate boundary as the mesh shown in Fig.2. Fig.3 shows the temperature contour and

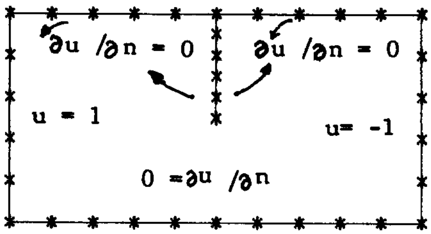

Fig.2 The illustrative example and the BEM mesh.

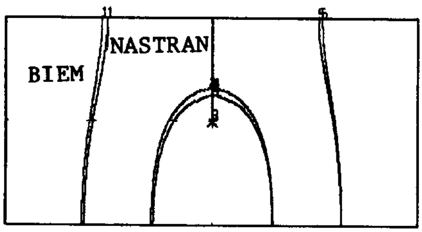

Fig.3 Temperature contour.

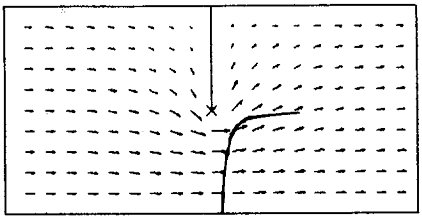

Fig.4 Temperature gradient

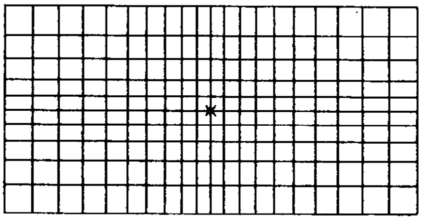

Fig.5 NASTRAN FEM mesh.

Fig.4 the heat flux distribution. In order to see the validity of the present theory a NASTRAN FEM mesh is also presented in Fig.5. For further comparison a result obtained from a scheme based on the Schwarz-Christoffel transformation [5] is also shown in Fig.4. Note that in the vicinity of the tip where singularity is expected and most schemes suffer much there the agreement between solutions of the present theory and the Schwarz-Christoffel transformation solution is remarkable.

CONCLUSION

The dual boundary integral equations have been derived to give a complete description of the compatible relation of the boundary data (temperature and normal flux). The singularity problem has been solved by a careful derivation which leads naturally to a convergent formulation. The formulation is written in compact form since the notions and notations of the Cauchy and Hadamard principal values are taken advantage of, as was done in the present work. The adoption of the derived dual equations to the boundary element method has resulted in a powerful numerical scheme suitable for solution of a wide class of problems. For illustration we have presented one numerical example, the results of which were found encouraging. By the same algorithm, one can apply the theory to transient heat conduction problems except for different kernels and an additional time integration has to be utilized.It is obvious that the present method, based on the second equation, is particularly suitable for the problem of extremely localized and concentrated heat flux.

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