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飽和且直交的 2-變級複因子直交試驗之研究

Some Classes of Orthogonal and Saturated Two-Level Factorial Designs

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中文摘要

本研究探討飽和(saturated)且直交(orthogonal)的 2-變級(2-level)的複因子設計(factorial designs)之建構。根據兩個直交設計的直乘(Kronecker product)產生另一個直交設計及 Liao, Iyer and Vecchia (1996) 所提出有關於一個 parallel-flats 設計滿足某組特定的因子效應之直交性的充分必要條件，我們提出五組飽和且直交的 2-變級複因子設計。這些直交設計可分別用於估計平均(grand mean)，所有的主效應(main effects)，及某組特定的 2-因子效應(2-factor interactions)。這些新的直交設計更具一般性，涵蓋某些已經發表的 2-變級和 4-變級; 2-變級和 8-變級的混合變級之飽和且直交的複因子設計；在工業產品研發過程及其他實務上亦有其實用性存在。本研究成果已整理成英文論文，準備送到國際期刊審查。

Some Classes of Orthogonal and Saturated Two-Level Factorial Designs

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Abstract

In a two-level factorial experiment, we consider the orthogonal designs of median-resolution which can be used to estimate the grand mean, all main effects and certain classes of two-factor interactions involving at least one of few specified factors, assuming that the other factorial effects are negligible. We use two approaches to constructing the designs under consideration. The first one is to judiciously assign the nonnegligible factorial effects to the columns of the Kronecker product of two Hadamard matrices. The second one is to consider the construction from the class of parallel-flats designs. In this article, we propose some general classes of orthogonal and saturated designs that include some existing orthogonal main-effect plans of asymmetric factorials as special cases.
Key words: Fractional factorial designs; Hadamard matrices; Main-effect plans; Screening experiments.

1 Introduction

During the initial stage of experimentation, the two-level fractional factorial designs are commonly used to screen for the important factors. These influential factors usually involve larger main effects and most of low order interactions, particularly two-factor interactions. Therefore, from a practical viewpoint, it is required to construct a design to estimate some specified factorial effects, which might be the grand mean, all main effects and some two-factor interactions, assuming that the other factorial effects are negligible. A discussion how this problem occurs in industrial product designs and quality improvement processes, along with some examples, can be seen in Wu & Chen (1992). They discussed a graph-aided method designed to meet this requirement and suggested an approach for computer implementation of their method. Some of the earliest published works using the similar approach were the *linear graphs* presented in Taguchi (1959, 1960).

Greenfield (1976) tackled this problem from an algorithm point of view. His algorithm was later generalized by Franklin & Bailey (1977). The methods discussed so far only searched the desired designs within the class of the regular two-level fractional factorial designs. So these methods can only yield the designs with the run sizes that are powers of 2.

Srivastava & Li (1996) constructed the designs of *median-resolution* from a wider class of designs called *parallel-fltas designs* (PFDs). They presented three classes of orthogonal two-level factorial designs which can be used to estimate the grand mean, all main effects and some specified two-factor interactions which can be classified into the following three categories.

(i) Two-factor interactions within some subgroups of factors.

(ii) Two-factor interactions between some subgroups of factors.

(iii) Two-factor interactions within some subgroups of factors and two-factor interactions between some subgroups of factors.

Note that some of designs in Srivastava & Li (1996) are not the designs with the number of runs being a power of 2. Liao, Iyer & Vecchia (1996) further developed an algorithm for construction of orthogonal PFDs for user-specified resolution. Interestingly, their simulation study showed that the 48-run designs are plentiful in some median-resolution designs where each two-factor interaction are forced to include at least one of few specified factors. This motivates us to investigate the construction of some specified median-resolution designs of saturated type.

In the next section, we introduce the notation used in the article; review the definitions of the Hadamard matrices and PFDs; and describe a necessary and sufficient condition for a PFD to be orthogonal for any arbitrary set of factorial effects. In section 3, we propose some classes of orthogonal and saturated designs of median-resolution similar to the category (iii) described above. Finally, we point out that some existing orthogonal main-effects plans are special cases of the designs obtained in this paper.

2 Preliminaries

Let $F_1, F_2, \ldots, F_n$ denote the n two-level factors. As is common practice, $F_1, F_2, \ldots, F_n$ will also represent the main effects. The expression $F_1^{e_1} F_2^{e_2} \ldots F_n^{e_n}$ will represent a general factorial effect with e_is being 0 or 1. If e_i is 1 then F_i appears in the factorial effect, otherwise it does not. The vector $\boldsymbol{e} = [e_1, \ldots, e_n]$ is called the *defining vector* for the general factorial effect $F_1^{e_1} F_2^{e_2} \ldots F_n^{e_n}$. The defining vector $\boldsymbol{e} = [0, 0, \ldots, 0]$ is for the grand mean μ.

2.1 Linear model

Let $\boldsymbol{\beta}$ denote the vector of factorial effects that are not assumed to be zero. The corresponding linear model for the observations from an experiment using a design $\boldsymbol{D}$ of run size N may be written in the form

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \tag{2.1}$$

where the error vector $\boldsymbol{\epsilon}$ consists of random variables which are assumed to have zero means and are pairwise uncorrelated with a common variance σ^2. The entries of the *design matrix* $\boldsymbol{X}$ are $+1$ or -1 which can be described as

$$\boldsymbol{X} = [\mathbf{1} | \boldsymbol{X}_1 | \boldsymbol{X}_2]$$

where $\mathbf{1}$ is the $N \times 1$ column consisting of all $+1$ and corresponds to the grand mean; $\boldsymbol{X}_1$ is the $N \times n$ matrix, corresponding to the n main effects, whose (i, j) entry is $+1$ or -1 according to whether factor j occurs at its high level or low level, respectively, in treatment combination i; and the columns of $\boldsymbol{X}_2$, corresponding to the remaining nonnegligible low-order interactions, are determined by the Schur product of the columns of $\boldsymbol{X}_1$. The Schur product of $\boldsymbol{u} = (u_1, u_2, \ldots, u_N)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_N)$ is defined to be $\boldsymbol{w} = (u_1v_1, u_2v_2, \ldots, u_Nv_N)$. The Schur product of the columns corresponding to main effects F_i and F_j in $\boldsymbol{X}_1$ results in the column corresponding to the two-factor interaction F_iF_j in $\boldsymbol{X}_2$. Similarly, the columns corresponding to higher order interactions can be determined by the same way.

The design $\boldsymbol{D}$ is said to be *orthogonal* for $\boldsymbol{\beta}$ if the *information matrix* $\boldsymbol{X}^T\boldsymbol{X} = N\boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix. Obviously, for the specified set of factorial effects $\boldsymbol{\beta}$, the construction of an orthogonal design for the $\boldsymbol{\beta}$ is equivalent to obtaining a design matrix whose columns are pairwise orthogonal.

2.2 Hadamard matrices

A square matrix of order d, $\boldsymbol{H}_d$, is said to be a Hadamard matrix if its entries are $+1$ or -1 provided that its columns are pairwise orthogonal; in other words,

$$\boldsymbol{H}_d^T \boldsymbol{H}_d = d\boldsymbol{I}.$$

A necessary condition for the existence of a Hadamard matrix is its order must be 1, 2 or a multiple of 4. Also any Hadamard can be converted to the Hadamard matrix whose first column or first row consists of all +1. For more details regarding the construction of hadamard matrices and their applications refer to Heydayat & Wallis (1978). In this article, we shall only use the fact that the Kronecker (direct) product of two Hadamard matrices results in another Hadamard matrix.

2.3 Parallel-flats designs

A parallel-flats two-level design actually is the 'union' of several, say f, regular fractional factorial designs. Let $\boldsymbol{D}_i$ be the set consisting of the treatment combinations $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)$ satisfying $\boldsymbol{t} = \boldsymbol{z}_i + \boldsymbol{B}\boldsymbol{v}$ over GF[2], the *Galois Field* of order 2, where the matrix $\boldsymbol{B}$ is $n \times k$ of rank k, the vector $\boldsymbol{v}$ ranges over all possible vectors of length k (there are 2^k such vectors) and $\boldsymbol{z}_i$ are $n \times 1$ vectors, for $i = 1, 2, \ldots, f$. A PFD $\boldsymbol{D}$ determined by the $(\boldsymbol{B}, \boldsymbol{Z})$ is obtained by taking all the $\boldsymbol{D}_i$ together, where $\boldsymbol{Z}$ is the $n \times f$ matrix whose columns are $\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_f$. Clearly, the PFD consists of $N = f \times 2^k$ runs.

For the PFD used for estimating the $\boldsymbol{\beta}$, the element of the information matrix $\boldsymbol{M} = \boldsymbol{X}^T\boldsymbol{X}$ of model (2.1) corresponding to the row indexed by the factorial effect whose defining vector is $\boldsymbol{e}_1 = [e_{11}, \ldots, e_{1n}]$ and the column indexed by the factorial effect whose defining vector is $\boldsymbol{e}_2 = [e_{21}, \ldots, e_{2n}]$ is $m(\boldsymbol{e}_1, \boldsymbol{e}_2)$ given by

$$m(\boldsymbol{e}_1, \boldsymbol{e}_2) = [\sum_{i=1}^{f}(-1)^{(\boldsymbol{e}_1+\boldsymbol{e}_2)\boldsymbol{z}_i}][\sum_{\boldsymbol{v}}(-1)^{(\boldsymbol{e}_1+\boldsymbol{e}_2)\boldsymbol{B}\boldsymbol{v}}]. \tag{2.2}$$

The following describing a necessary and sufficient condition for a PFD being orthogonal for the $\boldsymbol{\beta}$ can be directly verified by (2.2). See Liao et al. (1996).

Lemma 2.1. Let a PFD $\boldsymbol{D}$ be determined by the $(\boldsymbol{B}, \boldsymbol{Z})$. Also let $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ be two defining vectors for any two distinct factorial effects of $\boldsymbol{\beta}$. Then $\boldsymbol{D}$ is orthogonal for $\boldsymbol{\beta}$ if and only if $(\boldsymbol{e}_1 + \boldsymbol{e}_2)\boldsymbol{B} = \boldsymbol{0}$ implies that the vector $(\boldsymbol{e}_1 + \boldsymbol{e}_2)\boldsymbol{Z}$ consists of equal frequency of 0 and 1, i.e. $(\boldsymbol{e}_1 + \boldsymbol{e}_2)\boldsymbol{Z}$ is an *orthogonal array* of *strength* 1.

3 Orthogonal and saturated designs

In this section, we first consider the construction of some orthogonal and saturated designs of median-resolution based on the Kronecker product of two Hadamard matrices. Then we propose another class of designs constructed from PFDs whose rows of the $\boldsymbol{Z}$ matrix are appropriately chosen from the rows of Hadamard matrices. It is interesting to note that the presented class of PFDs cannot be generated directly from the Kronecker product of two Hadamard matrices.

Theorem 1. For Hadamard matrix $\boldsymbol{H}_d$, there is an orthogonal and saturated design with $N = 2d$ for the following $\boldsymbol{\beta}$.

$$\boldsymbol{\beta} = \{\mu; F_1, F_2, \ldots, F_n\} \cup \{F_1F_j | F_j \in G_1\}.$$

Let the number of factors in the set G_1 be $|G_1| = n_1$. Then $n_1 \leq d-1$ and $n + n_1 = 2d - 1$.

Proof. Consider the following Kronecker product of two Hadamard matrices.

$$\boldsymbol{H}_{2d} = \boldsymbol{H}_d \otimes \boldsymbol{H}_2 = \boldsymbol{H}_d \otimes \begin{bmatrix} +1 & +1 \\ +1 & -1 \end{bmatrix} = \begin{bmatrix} +\boldsymbol{H}_d & +\boldsymbol{H}_d \\ +\boldsymbol{H}_d & -\boldsymbol{H}_d \end{bmatrix} \tag{3.1}$$

where the first column of the $\boldsymbol{H}_d$ consists of all +1.

Interpret the Hadamard design $\boldsymbol{H}_{2d}$ of (3.1) as the design matrix in which

(a) the first column corresponds to the grand mean μ;

(b) the $(d+1)^{th}$ column corresponds to F_1;

(c) the 2^{nd}, 3^{rd}, ..., $(1+n_1)^{th}$ columns correspond to the main effects in G_1 where $n_1 \leq d-1$;

(d) the $(d+2)^{th}$, $(d+3)^{th}$, ..., $(d+n_1)^{th}$ columns correspond to the two-factor interactions F_1F_j for $F_j \in G_1$; and the remaining columns correspond to the other main effects.

The above is feasible since for $j = 2, 3, \ldots, d$, the Schur product of the j^{th} and $(d+1)^{th}$ columns equals to the $(d+j)^{th}$ column in the above construction.

Theorem 2. For Hadamard matrix $\boldsymbol{H}_d$, there exists an orthogonal and saturated design with $N = 4d$ for the following $\boldsymbol{\beta}$.

$$\boldsymbol{\beta} = \{\mu; F_1, F_2, \ldots, F_n\} \cup \{F_1F_2\} \cup \{F_1F_j | F_j \in G_1\} \cup \{F_2F_j | F_j \in G_2\}.$$

Let $|G_1| = n_1$ and $|G_2| = n_2$. Without loss of generality, it is assumed that $n_2 \leq n_1$. Also let $G_{12} = G_1 \cap G_2$ and $|G_{12}| = n_{12}$. Then $n_2 \leq d-1$, $n_1 \leq 2(d-1-n_2) + n_{12}$ and $n + n_1 + n_2 = 4d - 2$.

Proof. Consider the following Kronecker product of two Hadamard matrices.

$$\boldsymbol{H}_{4d} = \boldsymbol{H}_d \otimes \boldsymbol{H}_4 = \boldsymbol{H}_d \otimes \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & +1 & -1 \\ +1 & -1 & -1 & +1 \end{bmatrix} = \begin{bmatrix} +\boldsymbol{H}_d & +\boldsymbol{H}_d & +\boldsymbol{H}_d & +\boldsymbol{H}_d \\ +\boldsymbol{H}_d & +\boldsymbol{H}_d & -\boldsymbol{H}_d & -\boldsymbol{H}_d \\ +\boldsymbol{H}_d & -\boldsymbol{H}_d & +\boldsymbol{H}_d & -\boldsymbol{H}_d \\ +\boldsymbol{H}_d & -\boldsymbol{H}_d & -\boldsymbol{H}_d & +\boldsymbol{H}_d \end{bmatrix} \tag{3.2}$$

where the first column of the $\boldsymbol{H}_d$ consists of all $+1$.

Then we have the following observations on the above $\boldsymbol{H}_{4d}$.

(i) The Schur product of the j^{th} and $(d+1)^{th}$ columns equals to the $(d+j)^{th}$ column, for $j = 2, 3, \ldots, d$.

(ii) The Schur product of the j^{th} and $(2d+1)^{th}$ columns equals to the $(2d+j)^{th}$ column, for $j = 2, 3, \ldots, d$.

(iii) The Schur product of the $(d+1)^{th}$ and $(2d+j)^{th}$ columns equals to the $(3d+j)^{th}$ column, for $j = 1, 2, 3, \ldots, d$.

Based on the above observations, we successfully generate a desired orthogonal design by interpreting the $\boldsymbol{H}_{4d}$ of (3.2) as the design matrix in which

(a) the 1^{st} column corresponds to grand mean μ;

(b) the $(d+1)^{th}$ and $(2d+1)^{th}$ columns correspond to the main effects F_1 and F_2, respectively; the $(3d+1)^{th}$ column corresponds to the two-factor interaction F_1F_2;

(c) the 2^{nd}, 3^{rd}, ..., $(1+n_{12})^{th}$ columns correspond to the main effects of factors in G_{12}, where $n_{12} \leq n_2 \leq d-1$;

(d) the $(d+1+j)^{th}$ and $(2d+1+j)^{th}$ columns, for $j = 1, 2, \ldots, n_{12}$, correspond to the two-factor interactions F_1F_j and F_2F_j, respectively, for $F_j \in G_{12}$;

(e) the $(1+n_{12}+j)^{th}$ columns, for $j = 1, 2, \ldots, (n_2 - n_{12})$, correspond to the main effects of the factors in the set of $(G_2 - G_{12})$ which consists of the factors in G_2 but not in G_1; and the $(2d+1+n_{12}+j)^{th}$ columns, for $j = 1, 2, \ldots, (n_2 - n_{12})$, correspond to the two-factor interactions F_2F_j for $F_j \in (G_2 - G_{12})$.

(f) If $n_2 = d-1$, then this implies that $G_1 = G_2 = G_{12}$ $(n_1 = n_{12} = n_2)$. Let the remaining columns correspond to the other main effects. This completes the construction.

(g) If $n_2 < d-1$, then this implies that $(n_1 - n_{12}) \leq 2(d-1-n_2)$. We need to consider the following two cases.

(g1) When $(n_1 - n_{12}) \leq (d-1-n_2)$. Let the $(1+n_2+j)^{th}$ and $(d+1+n_2+j)^{th}$ columns, for $j = 1, 2, \ldots, (n_1 - n_{12})$, correspond to the main effects of F_j and the two-factor interactions F_1F_j, respectively, for $F_j \in (G_1 - G_{12})$. Then let the remaining columns correspond to the other main effects. This completes the construction.

(g2) When $(d-1-n_2) < (n_1 - n_{12}) \leq 2(d-1-n_2)$. Let the $(1+n_2+j)^{th}$ and $(d+1+n_2+j)^{th}$ columns, for $j = 1, 2, \ldots, (d-1-n_2)$, correspond to the main effects of F_j and the two-factor interactions F_1F_j, respectively, for $F_j \in (G_1 - G_{12})$; and the $(2d+1+n_2+j)^{th}$ and $(3d+1+n_2+j)^{th}$ columns, for $j = 1, 2, \ldots, (n_1-n_{12})-(d-1-n_2)$, correspond to the main effects of F_j and the two-factor interactions F_1F_j, respectively, for the remaining $F_j \in (G_1 - G_{12})$. Finally, let the remaining columns correspond to the other main effects. This completes the construction.

Theorem 3. For Hadamard matrix $\boldsymbol{H}_d$, there exists an orthogonal and saturated design with $N = 4d$ for the following $\boldsymbol{\beta}$.

$$\begin{aligned}\boldsymbol{\beta} &= \{\mu; F_1, F_2, F_3, \ldots, F_n\} \cup \{F_1F_2, F_1F_3, F_2F_3, F_1F_2F_3\} \\ &\cup \{F_1F_j | F_j \in G_1\} \cup \{F_2F_j | F_j \in G_2\}.\end{aligned}$$

Let $|G_1| = n_1$ and $|G_2| = n_2$. Without loss of generality, it is assumed that $n_2 \leq n_1$. Also let $G_{12} = G_1 \cap G_2$ and $|G_{12}| = n_{12}$. Then $n_2 \leq d-2$, $n_1 \leq 2(d-2-n_2) + n_{12}$ and $n + n_1 + n_2 = 4d - 2$.

Proof. This class of designs can be directly transformed from that of Theorem 2. Construct the designs of Theorem 2 and let the 2^{nd} column correspond to the main effect of F_3; and the $(d+2)^{th}$, $(2d+2)^{th}$ and $(3d+2)^{th}$ columns correspond to the two-factor interactions F_1F_3, F_2F_3 and the three-factor interaction $F_1F_2F_3$, respectively. This completes the construction.

Theorem 4. For Hadamard matrix $\boldsymbol{H}_d$, there exists an orthogonal and saturated design with $N = 4d$ for the following $\boldsymbol{\beta}$.

$$\boldsymbol{\beta} = \{\mu; F_1, F_2, F_3, \ldots, F_n\} \cup \{F_1F_j | F_j \in G_1\} \cup \{F_2F_j | F_j \in G_2\} \cup \{F_3F_j | F_j \in G_3\}$$

where none of F_1, F_2 and F_3 is in the sets of G_1, G_2 and G_3. Let $|G_1| = n_1$, $|G_2| = n_2$ and $|G_3| = n_3$. Also let $G^*_{123} = G_1 \cup G_2 \cup G_3$ and $|G^*_{123}| = n^*_{123}$. Then $n^*_{123} \leq d-1$ and $n + n_1 + n_2 + n_3 = 4d - 1$.

Proof. Consider the Hadamard matrix $\boldsymbol{H}_{4d}$ of (3.2). In addition to the observations of (i) and (ii) in Theorem 3, we need another observation on the $\boldsymbol{H}_{4d}$ given by

(iv) The Schur product of the j^{th} and $(3d+1)^{th}$ columns equals to the $(3d+j)^{th}$ column, for $j = 2, 3, \ldots, d$.

Based on these observations, we successfully generate a desired orthogonal design by interpreting the $\boldsymbol{H}_{4d}$ of (3.2) as the design matrix in which

(a) the 1^{st} column corresponds to μ;

(b) the $(d+1)^{th}$, $(2d+1)^{th}$ and $(3d+1)^{th}$ columns correspond to the main effects F_1, F_2 and F_3, respectively;

(c) the 2^{nd}, 3^{rd}, ..., $(1+n^*_{123})^{th}$ columns correspond to the main effects of factors in G^*_{123}, where $n^*_{123} \leq d-1$;

(d) for the k^{th} column corresponding to main effect of $F_j \in G_i$, $i = 1, 2, 3$, the $(i \times d+1+k)^{th}$ column simultaneously corresponds to the two-factor interaction F_iF_j for $F_j \in G_i$;

(e) the remaining columns correspond to the other main effects.

Theorem 5. For Hadamard matrix $\boldsymbol{H}_d$, there exists an orthogonal and saturated PFD with $N = 4d$ and $f = d$ for the following $\boldsymbol{\beta}$.

$$\begin{aligned}\boldsymbol{\beta} &= \{\mu; F_1, F_2, F_3, F_4, \ldots, F_n\} \cup \{F_1F_2, F_1F_3, F_2F_3, F_1F_2F_3\} \\ &\quad \cup\{F_1F_j|F_j \in G_1\} \cup \{F_2F_j|F_j \in G_2\} \cup \{F_3F_j|F_j \in G_3\}\end{aligned}$$

Let $|G_1| = n_1$, $|G_2| = n_2$ and $|G_3| = n_3$. Also let $G^*_{123} = G_1 \cup G_2 \cup G_3$ and $|G^*_{123}| = n^*_{123}$. Then $n^*_{123} \leq d-2$ and $n + n_1 + n_2 + n_3 = 4d - 5$.

Proof. This class of designs cannot be directly generated from the Kronecker product of two Hadamard matrices. We first obtain a PFD which is orthogonal and saturated for the following set of factorial effects.

$$\begin{aligned}\boldsymbol{\beta}^* &= \{\mu; F_1, F_2, F_3, F_4, \ldots, F_n\} \cup \{F_1F_2, F_1F_3, F_2F_3, F_1F_2F_3\} \\ &\quad \cup\{F_1F_j|F_j \in G_1\} \cup \{F_2F_j|F_j \in G_2\} \cup \{F_3F_j|F_j \in G_3\}\end{aligned}$$

where $n = d+1$ and $G_1 = G_2 = G_3 = \{F_4, F_5, \ldots, F_{d+1}\}$.

Recall that a PFD can be determined by the $(\boldsymbol{B}, \boldsymbol{Z})$, where $\boldsymbol{B}$ is $n \times k$ and $\boldsymbol{Z}$ is $n \times f$ over GF[2]. For convenience, let

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \vdots \\ \boldsymbol{b}_n \end{bmatrix} \text{ and } \boldsymbol{Z} = \begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_n \end{bmatrix},$$

where the vectors $\boldsymbol{b}_i$ and $\boldsymbol{a}_i$ are of size $1 \times k$ and $1 \times f$, respectively. Each pair of $(\boldsymbol{b}_i, \boldsymbol{a}_i)$ corresponds to factor i, for $i = 1, 2, \ldots, n$. By replacing $+1$ by 0 and -1 by 1 in the Hadamard matrices, we have the following

$$\boldsymbol{H}^*_d = \begin{bmatrix} \boldsymbol{h}^*_0 \\ \boldsymbol{h}^*_1 \\ \boldsymbol{h}^*_2 \\ \vdots \\ \boldsymbol{h}^*_{d-1} \end{bmatrix},$$

where $\boldsymbol{h}_0^*$ denotes the zero vector of size $1 \times d$ over GF[2] and the last $d-1$ rows consist of an orthogonal array of strength 2, briefly denoted by $\boldsymbol{L}_{4d}$. The $\boldsymbol{L}_{4d}$ indicates that any 2 rows of the last $d-1$ rows of the $\boldsymbol{H}_d^*$ contain all possible 2×1 columns of $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$ with the same frequency.

Then we consider the following PFD. For the $\boldsymbol{B}$ matrix, let $\boldsymbol{b}_1 = [1,0]$, $\boldsymbol{b}_2 = [0,1]$, $\boldsymbol{b}_3 = [1,1]$ and $\boldsymbol{b}_i = [0,0]$, for $i = 4, 5, \ldots, d+1$. For the $\boldsymbol{Z}$ matrix, let $\boldsymbol{a}_1 = \boldsymbol{a}_2 = \boldsymbol{h}_0^*$, $\boldsymbol{a}_3 = \boldsymbol{h}_1^*$ and $\boldsymbol{a}_i = \boldsymbol{h}_{i-2}^*$, for $i = 4, 5, \ldots, d+1$. It can be easily verified, by using Lemma 2.1, that the PFD is orthogonal for the $\boldsymbol{\beta}^*$.

Suppose $\boldsymbol{X}$ is the design matrix determined by the orthogonal PFD for the $\boldsymbol{\beta}^*$. Compare the elements between $\boldsymbol{\beta}$ and $\boldsymbol{\beta}^*$. Let the columns of $\boldsymbol{X}$, corresponding to those two-factor interactions of F_1F_j, F_2F_j, F_3F_j which are in $\boldsymbol{\beta}^*$ but not in $\boldsymbol{\beta}$, be changed to correspond to those main effects which are in $\boldsymbol{\beta}$ but not in $\boldsymbol{\beta}^*$. This gives the design matrix for the $\boldsymbol{\beta}$ and automatically an orthogonal and saturated design for the $\boldsymbol{\beta}$.

4 Concluding remarks

It is obvious that the designs provided in the previous section are those of inter-effect orthogonality for the hierarchical models discussed in Dey & Mukerjee (1999). As proved in their paper, these designs are universally optimal, and hence in particular, are A-optimal, D-optimal and E-optimal for estimating $\boldsymbol{\beta}$ within the class of designs with the fixed N.

Since the treatment combinations of the PFDs satisfying the equation $\boldsymbol{t} = \boldsymbol{z}_i + \boldsymbol{B}\boldsymbol{v}$, by taking $\boldsymbol{B}$ to be the zero matrix, the PFDs can be regarded as a general class of designs including all possible designs. Therefore, the classes of designs of Theorem 1 to Theorem 4 might also be constructed from the PFDs. But this can make the presentation somewhat unnecessarily complicated. However, the classes of designs provided in this paper should prove useful in practice and the study also sheds light on how the orthogonal designs of median-resolution with run size $N = 48$ are plentiful, pointed out in Liao et al. (1996). Note that it should be fruitful to investigate the designs with the larger run size, e.g. $N = 8d$ or $N = 16d$. But these larger run size designs may not be practical since they can lead to an uneconomical experimentation.

Some existing orthogonal main-effect plans (OMEPs) of asymmetrical factorial designs in Dey (1985) are special cases of the designs obtained in this paper. Let the main effects of a 4-level factor and an 8-level factor be represented by the main effects and all interactions involving 2 and 3 two-level pseduo-factors, respectively. Then the OMEPs of $4 \cdot 2^{2d-4}$ factorials are exactly the designs of Theorem 1 for $n_1 = 1$. Clearly, both the OMEPs of $4^2 \cdot 2^{4d-7}$ and $4^3 \cdot 2^{4d-10}$ factorials belong to the class of designs of Theorem 4. Finally, the OMEPs of $8 \cdot 2^{4d-8}$ factorials are the designs of Theorem 3 for $n_1 = n_2 = 0$ or of Theorem 5 for $n_1 = n_2 = n_3 = 0$.

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