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廣義容許區間之研究

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計畫主持人：廖振鐸

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主持人：廖振鐸　國立台灣大學農藝學研究所生統組

一、中文摘要

本研究針對隨機變數$W \sim N(\theta, \sum_{i=1}^{q} h_i \sigma_i^2)$，利用 Tsui and Weerahandi (1989)及 Weerahandi (1993)所提出的廣義檢定函數及廣義信賴區間的想法，來建立其合理的雙尾容許區間。給定信賴係數(confidence level) γ，構建一個隨機區間$[T_L(\hat{\theta}, S_1^2, \ldots, S_q^2), T_U(\hat{\theta}, S_1^2, \ldots, S_q^2)]$ 使其至少包含W之分佈的特定比例(content) β，其數學式可以表示如下：

$$P_{(\hat{\theta}, S_1^2, \ldots, S_q^2)}[P_W[T_L(\hat{\theta}, S_1^2, \ldots, S_q^2) \le W \le T_U(\hat{\theta}, S_1^2, \ldots, S_q^2)] \ge \beta] = \gamma$$

其中$\hat{\theta} \sim N(\theta, \sum_{i=1}^{q} c_i \sigma_i^2)$；$n_i S_i^2 / \sigma_i^2 \sim \chi_{n_i}^2$；$h_i, c_i$為已知的係數，$i = 1, 2, \ldots, q$。且由統計模擬結果中顯示我們所提出的方法對於解決本問題，有其實際應用價值。另外值得一提的是，此方法亦可應用到所有常態分配下的均衡混合線性模型之容許區間建構。

Abstract

A tolerance interval procedure is derived from the concept of generalized pivotal quantities which is usually used to obtain confidence intervals in situations where standard procedures do not lead to useful solutions. We apply the generalized confidence intervals approach and propose a two-sided tolerance interval for the distribution $N(\theta, \sum_{i=1}^{q} h_i \sigma_i^2)$ based on mutually independent statistics $\hat{\theta}$, S_1^2, S_2^2, …, S_q^2 where $\hat{\theta}$ is distributed with $N(\theta, \sum_{i=1}^{q} c_i \sigma_i^2)$, h_i and c_i are known constants, and $n_i S_i^2 / \sigma_i^2$ are independent chi-squared random variables with n_i df, for $i=1,2,..,q$. Some practical examples are given to illustrate the applications of the proposed procedure. A simulation study is conducted to evaluate its frequentist coverage probability. The results indicate that the proposed method may be recommended for use in practical applications. The procedure provided in this paper can be applied to tolerance interval questions arising in arbitrary normal balanced mixed linear model situations.

Keywords: chi-squared approximation; generalized P-values; generalized confidence

intervals; linear models; variance components.

二、緣由與目的

Let F denote the cumulative distribution of a random variable. An interval [L(***Y***), U(***Y***)] (or [L, U], for simplicity), based on the data vector ***Y***, is called a two-sided β-content, γ-confidence tolerance interval (or (β,γ)-tolerance interval, for short) for F if the following condition holds:

$$Pr[F(U(\mathbf{Y})) - F(L(\mathbf{Y})) \geq \beta] = \gamma.$$

Thus, we can state with confidence coefficient γ that at least a proportion β of the population modeled by F is contained in the interval [L, U].

Two-sided tolerance intervals are widely used in industrial applications where manufactured parts have to meet certain specifications. If the manufacturing process is capable, then a high proportion of the items manufactured will meet the specifications. Two-sided tolerance intervals give us L and U such that we can claim, with a specified degree of confidence γ, that a specified proportion β or more of the manufactured items lie between L and U.

The problem for computing a tolerance interval for the simple case in which F is the normal distribution with unknown mean μ and unknown variance σ^2 has been extensively studied; see, for example, Wald and Wolfowitz (1946), Howe (1969), and Odeh and Owen (1980). For more complex situations, only scattered results are available. The problem of setting a two-sided tolerance interval for the distribution $N(\theta,\sigma_1^2-\sigma_2^2)$ based on mutually independent statistics $\hat{\theta}$, S_1^2, S_2^2, where $\hat{\theta} \sim N(\theta, c\sigma_1^2)$, c is a known constant, $n_1 S_1^2/\sigma_1^2$, $n_2 S_2^2/\sigma_2^2$ are chi-squared random variables with n_1 and n_2 degrees of freedom (df), respectively, was considered in Wang and Iyer (1994). Also some practical examples were given in their paper to illustrate the applications of their proposed procedure. Brown et al. (1997) applied their results to evaluate the bioequivalence of two formulations of a drug using various cross-over designs for data collection. Liao and Iyer (2001) extended the results of Wang and Iyer (1994) and proposed tolerance intervals for the distribution $N(\theta,\sigma_1^2-\sigma_2^2)$ for the case where the distribution of the statistic $\hat{\theta}$ is $N(\theta,\sum_{i=1}^{q} c_i\sigma_i^2)$, with $q \geq 1$ and c_i being known constants. Their study was motivated by an actual application involving the assessment of the quality of a type of glucose monitoring meters.

In this paper, we generalize the problem as follows. We seek a two-sided tolerance interval for a random variable W which has a $N(\theta,\sum_{i=1}^{q} h_i\sigma_i^2)$ distribution. Suppose mutually independent statistics $\hat{\theta}$, S_1^2, S_2^2, …, S_q^2 are available, where $\hat{\theta}$

is normally distributed with mean θ and variance $c_i\sigma_i^2$, h_i and c_i are known constants, and $n_iS_i^2/\sigma_i^2$ are independent chi-squared random variables with n_i df, for $i=1,2,..,q$. The solution we propose is based on the concept of generalized confidence intervals, see Weerahandi (1993, 1995), and is different from the derivation based on the method given in Wang and Iyer (1994).

三、結果與討論

Let $\tau^2=\sum_{i=1}^{q}h_i\sigma_i^2$ and $\sigma^2=\sum_{i=1}^{q}c_i\sigma_i^2$. We first give generalized pivotal quantities for θ and τ. Let T represent the observable random variable $\hat{\theta}$ and t denote its observed value,

$$Z=\frac{T-\theta}{\sigma}$$

and for $i=1,2,..,q$,

$$U_i=n_iS_i^2/\sigma_i^2$$

Then $Z\sim N(0,1)$ and, for $i=1,2,..,q$, $U_i\sim\chi^2_{n_i}$. We define

$$R_\theta=t-Z\sqrt{\sum_{i=1}^{q}\frac{c_in_is_i^2}{U_i}} \tag{1}$$

which can also be written as

$$R_\theta=t-\left(\frac{T-\theta}{\sigma}\right)\sqrt{\sum_{i=1}^{q}\frac{c_in_is_i^2}{S_i^2}} \tag{2}$$

Likewise, let us define

$$R_\tau=\sqrt{\sum_{i=1}^{q}\frac{h_in_is_i^2}{U_i}} \tag{3}$$

which can also be written as

$$R_\tau=\sqrt{\sum_{i=1}^{q}\frac{h_in_is_i^2}{S_i^2}} \tag{4}$$

From (1) and (3), it follows that R_θ and R_τ have distributions that are free of model parameters. Furthermore, from (2) and (4) we see that, when the observed values t and s_i^2 are substituted for the observable random variables T and S_i^2, for $i=1,2,..,q$, R_θ and R_τ become $r_\theta=\theta$ and $r_\tau=\tau$. Therefore, R_θ and R_τ satisfy the requirements for being generalized pivotal quantities for θ and τ. See Weerahandi (1993) for the relevant definitions and details. Hence, a two-sided α confidence interval for θ and an upper α confidence bound for τ can be obtained, respectively, from the following two sets.

$$\left\{\theta\mid R_{\theta,(1-\alpha)/2}\le r_\theta\le R_{\theta,(1+\alpha)/2}\right\}$$

and

$$\left\{\tau \mid r_\tau \le R_{\tau,\alpha}\right\}$$

The required percentiles $R_{\theta,\alpha}$ and $R_{\tau,\alpha}$ may be determined by the following Monte-Carlo algorithm.

Step1:Let M be a large positive integer, say 100,000. For i equal to 1 through M, carry out the following steps 2 and 3.

Step2:Generate a standard normal random deviate Z_i and chi-squared random deviates $U_{1,i}, U_{2,i}, \ldots, U_{q,i}$ with $n_1, n_2, \ldots, n_q$ degrees of freedom, respectively. The random deviates are required to be jointly independent.

Step3:Calculate $R_{\theta,i}$ and $R_{\tau,i}$ using the expressions (1) and (3) for R_θ and R_τ, respectively.

Let $\hat{\theta}_{(1-\gamma)/2}$ and $\hat{\theta}_{(1+\gamma)/2}$ be the $(1-\gamma)/2$ and $(1+\gamma)/2$ sample percentiles of the collection of values $R_{\theta,1}, R_{\theta,2}, \ldots, R_{\theta,M}$. Then $\left[\hat{\theta}_{(1-\gamma)/2}, \hat{\theta}_{(1+\gamma)/2}\right]$ may be used as a two-sided generalized confidence interval for θ with nominal confidence coefficient γ. Similarly, we may use $\hat{\tau}_\gamma$ as an upper generalized confidence bound for τ with nominal confidence coefficient γ.

Then we seek a (β,γ)-tolerance interval for a random variable $W \sim N(\theta,\tau^2)$. We have $\hat{\theta} \sim N(\theta,\sigma^2)$ with σ^2 given by $\sigma^2 = \sum_{i=1}^{q} c_i \sigma_i^2$, where h_i and c_i are known constants. Furthermore, mutually independent statistics S_1^2, S_2^2, …, S_q^2 are available such that they are independent of $\hat{\theta}$ and $n_i S_i^2 / \sigma_i^2 \sim \chi^2_{n_i}$, for $i=1,2,..,q$. We need to find a margin of error statistic $D = D(S_1^2, S_2^2, \ldots, S_q^2)$ such that

$$\Pr_{\hat{\theta}, S_1^2, S_2^2, \ldots, S_q^2}\{\Pr_W[\hat{\theta} - D \le W \le \hat{\theta} - D] \ge \beta\} = \gamma$$

Define

$$Q(\beta, D) = \Pr_{\hat{\theta}, S_1^2, S_2^2, \ldots, S_q^2}\{\Pr_W[\hat{\theta} - D \le W \le \hat{\theta} - D] \ge \beta\}$$

Also let $Z = (\hat{\theta} - \theta)/\sigma$ which is a standard normal random variable. We thus have

$$Q(\beta, D) = \Pr_{Z, S_1^2, S_2^2, \ldots, S_q^2}\{[\Phi(Z\frac{\sigma}{\tau} + \frac{D}{\tau}) - \Phi(Z\frac{\sigma}{\tau} - \frac{D}{\tau})] \ge \beta\},$$

where $\Phi(.)$ is the standard normal distribution function. Therefore, the problem is to find D such that $Q(\beta, D) = \gamma$. As in the work of Wald and Wolfowitz (1946), we can first compute $k = k(z, \sigma, \tau, \beta)$ that satisfies

$$\Phi(z\frac{\sigma}{\tau}+k)-\Phi(z\frac{\sigma}{\tau}-k)=\beta \tag{5}$$

Then

$$Q(\beta,D\,|\,Z=z)=\Pr_{S_1^2,S_2^2,\ldots,S_q^2}\{\frac{D}{\tau}\geq k\}=\Pr_{S_1^2,S_2^2,\ldots,S_q^2}\{\frac{D}{k}\geq \tau\}=\gamma$$

Obviously, *D/k* is an upper γ confidence bound for τ. So the value of *D* may be estimated by the $k\hat{\tau}_\gamma$, where $\hat{\tau}_\gamma$ is given in the previous subsection. Finally, the value of *k* must be computed from (5) which satisfies

$$E\left[\Phi(z\frac{\sigma}{\tau}+k)-\Phi(z\frac{\sigma}{\tau}-k)\right]=\beta.$$

Using the Wald-Wolfowitz (1946) approximation which states that *k* is closely approximated by the root of the nonlinear equation

$$\Phi(\frac{1}{\phi}+k)-\Phi(\frac{1}{\phi}-k)=\beta$$

where $\phi=\tau/\sigma$. Another approximation is given in Howe (1969), which uses

$$k=\sqrt{1+\frac{1}{\phi^2}}z_{(1+\beta)/2}. \tag{6}$$

The parameter $\phi^2=\tau^2/\sigma^2$ can be estimated by $\hat{\phi}^2=\hat{\tau}^2/\hat{\sigma}^2$, where $\hat{\tau}^2=\sum_{i=1}^q h_i s_i^2$ and $\hat{\sigma}^2=\sum_{i=1}^q c_i s_i^2$. There is a possibility that the estimated value $\hat{\tau}$ is a negative number for some situations. If this happens, we replace $\hat{\tau}$ by $\hat{\tau}_\gamma$ in $\hat{\phi}$. In case that $\hat{\tau}_\gamma\leq 0$, we may use the two-sided γ generalized confidence interval $\left[\hat{\theta}_{(1-\gamma)/2},\hat{\theta}_{(1+\gamma)/2}\right]$ as the (β,γ)-tolerance interval.

Based on the above discussion, we propose the following as the (β,γ)-tolerance interval for the distribution $N(\theta,\tau^2)$:

1. When $\hat{\tau}_\gamma>0$, the required tolerance interval is computed as

   $$\hat{\theta}\pm\hat{k}\hat{\tau}_\gamma$$

   where $\hat{k}$ is obtained from (6) in which ϕ is estimated by $\hat{\phi}=\hat{\tau}/\hat{\sigma}$ if $\hat{\tau}>0$, and by $\hat{\phi}=\hat{\tau}_\gamma/\hat{\sigma}$ otherwise.
2. When $\hat{\tau}_\gamma\leq 0$, the required tolerance interval is taken to be the two-sided γ-level generalized confidence interval $\left[\hat{\theta}_{(1-\gamma)/2},\hat{\theta}_{(1+\gamma)/2}\right]$.

In the simulation studies, Liao and Iyer (2001) conducted the same simulation study to evaluate their procedure. For most of the parameter combinations the coverage probabilities of the two methods are nearly the same. This indicates that they have similar performance. Nonetheless, both Wang-Iyer (1994) and Liao-Iyer (2001) methods are problem specific and are derived for a specific family of distributions under consideration. Clearly, the procedure provided in this study can be applied to tolerance interval questions arising in arbitrary normal balanced mixed linear model situations. We note that the construction of tolerance intervals using the concept of the generalized confidence intervals can also be applied to the one-sided (β,γ)-tolerance interval for the random variable $W \sim N(\theta,\tau^2)$, where $\tau^2 = \sum_{i=1}^{q} h_i\sigma_i^2$, which corresponds to a statistic U satisfying $Pr\{\ Pr\ [W \leq U] \geq\ \beta\} = \gamma$. It is obvious that U is simply equal to the upper γ-level generalized confidence bound for $\theta + \tau z_\beta$.

Finally, to the best of our knowledge, there appear to be no satisfactory two-sided tolerance interval procedures available in the literature for general unbalanced data situations. Bagui, et al. (1996) do discuss procedures for one-sided tolerance limits in *m*-way random effects ANOVA models. However, their approach is based on the `plug-in' method whereby tolerance intervals are derived assuming various parameters to be known and then estimates for these parameters are substituted in the results. The coverage probabilities of intervals based on the `plug-in' method have not been satisfactorily evaluated in general mixed-model situations. In the context of one-way random effects models, our own simulation studies (unpublished) indicate poor performance for the `plug-in' methods. We are currently investigating other approaches for obtaining satisfactory tolerance intervals in unbalanced mixed models.

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