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Boundary determination of a Riemannian metric by the localized boundary distance function

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Abstract

In this paper we give a reconstruction procedure for the Taylor series of a Riemannian metric on the boundary in boundary normal coordinates from the localized boundary distance function.

1. Introduction

Let (M, g) be a Riemannian manifold with smooth boundary $\Gamma := \partial M$. Define the boundary distance function

$$d_g(x, y) = \mathrm{dist}(x, y) \quad \text{for } x, y \in \Gamma$$

which is the geodesic distance between boundary points. An interesting inverse problem is whether one can determine g from its associated boundary distance function d_g. It is easy to see that uniqueness is not possible. Indeed, let $\psi : M \to M$ be a diffeomorphism that leaves Γ invariant, i.e., $\psi|_\Gamma = \mathrm{Id}$, then $d_{\psi^* g} = d_g$. Therefore, the inverse problem one would like to address is whether this is the only obstruction to uniqueness. This problem arose in geophysics in attempting to determine the inner structure of the Earth. The boundary distance function measures the travel times of seismic waves going through the Earth and the metric is the index of refraction [1,5]. This problem is also called the boundary rigidity problem in Riemannian geometry [2,3,9]. The boundary rigidity problem has been

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extensively studied in the last three decades. The reader is referred to the article [12] for a recent survey.

In this paper we would like to look at this rigidity problem from the boundary perspective. That is, we want to investigate what information of g on Γ can be determined from d_g. As indicated above, unique determination of $g|_\Gamma$ is not possible. Nevertheless, in this article we are able to show that one can reconstruct the Taylor series of the metric g on Γ in suitable coordinates from the *localized* boundary distance function. Precisely, we prove that for any $x \in \Gamma$ all derivatives of g at x in boundary normal coordinates can be determined from the knowledge of $d_g(x, y)$ for all $y \in \Gamma$ sufficiently close to x. More importantly, we will give explicit formulas for all the normal derivatives. To state the Main Theorem, we first introduce the notion of convexity. We say that the boundary Γ is locally convex at $p \in \Gamma$ if there exists an open neighborhood $\mathcal{O}$ of p such that for any two points $x, y \in \mathcal{O} \cap \Gamma$, $x \neq y$, there is a unique geodesic joining x and y and all inner points of this geodesic lie entirely in $\mathcal{O}$. Also, M is said to be extendible near p if we can extend M to another smooth manifold $\widetilde{M} \supset M$ across $\mathcal{O} \cap \Gamma$.

Main Theorem. *Let (M, g) be a Riemannian manifold of dimension n with smooth boundary Γ. Let $p \in \Gamma$ and Γ be locally convex there. Assume that g is smooth up to Γ near p and also M is extendible near p. Let $\{x^1, \dots, x^n\}$ be the boundary normal coordinates near p. Then we can reconstruct $\partial_x^\gamma g(p)$ for all $|\gamma| \geqslant 0$ from the knowledge of $d_g(q, r)$ for all $q, r \in \Gamma$ sufficiently near p.*

Previous results on the boundary determination of the Taylor series of g from d_g have been given in [9] (up to order 2), [10] (infinite order but $n = 2$) and recently in [7] (general case). But, none of these papers gave reconstruction formulas. The main tool of this paper is an identity which relates the metric g to the lengths of geodesics. This identity is similar to the one derived by Stefanov and Uhlmann in [11]. It should be noted that the identity derived in [11] relies on the hypotheses that two unknown metrics coincide with the Euclidean one up to certain order near the boundary and have the same boundary distance function. Since we are dealing with the reconstruction problem here, we have only one unknown metric with no boundary information available. To utilize Stefanov and Uhlmann's arguments, we therefore need to choose an appropriate reference metric to go with the unknown metric (see Section 2). With the key identity at hand, we are able to get the boundary information of the metric by differentiating the identity and letting the boundary distance go to zero (see Section 3).

For the readers' convenience, we now outline the reconstruction procedure. It should be noted that we are working in boundary normal coordinates near any boundary point, say p $(\in \Gamma)$, throughout the whole reconstruction.

Step 1. Using Michel's arguments [9], we can determine $g(p)$ in the tangential directions.

Step 2. Differentiating the key identity (2.5), where the reference metric g_0 is given in (3.5), leads to a new formula whose one side is solely determined by d_g. We construct the first normal derivatives g^{-1} at p by taking $d_g \to 0$ in the new formula. Therefore, we can recover the first normal derivative of g at p.

Step 3. Inductively, by further differentiating the identity (2.5) and repeating the arguments in Step 2, we can determine all normal derivatives of g at p.

We remark that the result here is the pointwise boundary reconstruction of the metric by the knowledge of the boundary distance function. This problem is somehow related to the inverse problem of determining the metric by boundary measurements from the Dirichlet-to-Neumann map associated with the Laplace–Beltrami operator Δ_g. It is well known that the Dirichlet-to-Neumann map is a pseudo-differential operator of order one. By computing its full symbol (and this requires knowledge only of the local Dirichlet-to-Neumann map), one can determine the boundary value of the metric up to any order in boundary normal coordinates [8]. For another approach see [6].

2. Key identity

In this section we will derive an identity which plays a key role in the proof of the Main Theorem. Let H_g denote a Hamiltonian related to g and $H_g(x,\xi) = \frac{1}{2}(\sum_{i,j=1}^n g^{ij}\xi_i\xi_j - 1)$ in local coordinates, where $(g^{ij}) = (g_{ij})^{-1}$. Here we are interested in the integral curves associated with the Hamiltonian vector field induced by H_g. The projection of these integral curves are geodesics. Let $x^{(0)} \in \Gamma$ and $\{\mathcal{O}, x\}$ be a local chart near $x^{(0)}$. Assume that Γ is locally convex at $x^{(0)}$ and the Riemannian manifold (M,g) satisfies the assumptions near $x^{(0)}$ as in the Main Theorem. Let $\xi^{(0)} \in T^*_{x^{(0)}}M$ satisfy

$$\nu(x^{(0)}) \cdot g^{-1}\big(x^{(0)}\big)\xi^{(0)} < 0 \tag{2.1}$$

and

$$\xi^{(0)} \cdot g^{-1}\big(x^{(0)}\big)\xi^{(0)} = 1, \tag{2.2}$$

where $\nu(x^{(0)})$ is the unit outer normal (co)vector (relative to g^{-1}) to Γ at $x^{(0)}$. Considered in this local coordinates $\{x^1,\dots,x^n\}$ near $x^{(0)}$, let $x_g(s,x^{(0)},\xi^{(0)})$ and $\xi_g(s,x^{(0)},\xi^{(0)})$ solve the Hamiltonian system

$$\begin{cases} \dfrac{\mathrm{d}x^m}{\mathrm{d}s} = \displaystyle\sum_{j=1}^n g^{mj}\xi_j, \quad x|_{s=0} = x^{(0)}, \\[2ex] \dfrac{\mathrm{d}\xi_m}{\mathrm{d}s} = -\dfrac{1}{2}\displaystyle\sum_{i,j=1}^n \dfrac{\partial g^{ij}}{\partial x^m}\xi_i\xi_j, \quad \xi|_{s=0} = \xi^{(0)}, \end{cases} \qquad m = 1,\dots,n. \tag{2.3}$$

Note that $x_g(s)$ is the geodesic with initial condition $(x^{(0)}, g^{-1}(x^{(0)})\xi^{(0)})$ and s in (2.3) is the arc length parameter. Denote $X^{(0)} = (x^{(0)},\xi^{(0)})$ and $t := t(X^{(0)})$ the length of the geodesic issued from $X^{(0)}$ with endpoint on $\mathcal{O} \cap \Gamma$. Here we choose $\xi^{(0)}$ appropriately such that all inner points of $x_g(s)$ lie entirely in $\mathcal{O}$. This requirement is obviously true if $\xi^{(0)}$ is taken so that t is sufficiently small.

Now let us fix an $X^{(0)}$ satisfying (2.1) and (2.2). Assume that g_0 is a Riemannian metric in M such that (2.1) holds at $X^{(0)}$ with g being replaced by g_0. Notice that we do not require that $X^{(0)}$ satisfies (2.2) with respect to g_0. In the coordinate chart $\{\mathcal{O}, x\}$, we let $(x_{g_0}(s, x^{(0)}, \xi^{(0)}), \xi_{g_0}(s, x^{(0)}, \xi^{(0)}))$ be the solution to the Hamiltonian system (2.3) with respect to g_0 having initial data $X^{(0)}$. It is readily seen that $x_{g_0}(s, x^{(0)}, \xi^{(0)}) \subset \mathcal{O}$ for $0 < s \leqslant t$ provided that t is small. Denote $X = (x, \xi)$. The solutions to (2.3) related to g and g_0 can be written as $X_g(s, X^{(0)}) = X_g(s, x^{(0)}, \xi^{(0)})$ and $X_{g_0}(x, X^{(0)}) = X_{g_0}(s, x^{(0)}, \xi^{(0)})$, respectively. Let $F(s) = X_g(t - s, X_{g_0}(s, X^{(0)}))$, then

$$
\begin{aligned}
\int_0^t F'(s)\,\mathrm{d}s &= F(t) - F(0) = X_g\big(0, X_{g_0}\big(t, X^{(0)}\big)\big) - X_g\big(t, X_{g_0}\big(0, X^{(0)}\big)\big) \\
&= X_{g_0}\big(t, X^{(0)}\big) - X_g\big(t, X^{(0)}\big).
\end{aligned} \tag{2.4}
$$

It should be noted that x component of F may not be in $\mathcal{O}$. In order to make sense of F, we can extend g smoothly to $\widetilde{M}$. Thus, $\widetilde{M}$ becomes a Riemannian manifold carrying a metric which is a smooth extension of g. Nevertheless, the integral in (2.4) is independent of the extension of g. It is shown in [11] that

$$
F'(s) = \frac{\partial X_g}{\partial X^{(0)}}\big(t - s, X_{g_0}\big(s, X^{(0)}\big)\big)(V_{g_0} - V_g)\big(X_{g_0}\big(s, X^{(0)}\big)\big),
$$

where $V_{g_0} = (\partial H_{g_0}/\partial \xi, -\partial H_{g_0}/\partial x)$ and V_g is defined similarly. In conclusion, we obtain that

$$
\begin{aligned}
&\int_0^t \frac{\partial X_g}{\partial X^{(0)}}\big(t - s, X_{g_0}\big(s, X^{(0)}\big)\big)(V_{g_0} - V_g)\big(X_{g_0}\big(s, X^{(0)}\big)\big)\,\mathrm{d}s \\
&\quad = X_{g_0}\big(t, X^{(0)}\big) - X_g\big(t, X^{(0)}\big).
\end{aligned} \tag{2.5}
$$

Before leaving this section, we want to remark that the right-hand side of (2.5) is solely determined by d_g near $x^{(0)}$. This property will be verified in the following section.

3. Proof of Main Theorem

Assume that (M, g) satisfies the assumptions of Main Theorem near $x^{(0)} = 0 \in \Gamma$. Let $\widetilde{\mathcal{O}}$ be an open neighborhood of 0 in $\widetilde{M}$ and the metric g has been extended smoothly in $\widetilde{\mathcal{O}}$, still denoted by g. We now introduce the boundary exponential map $\exp_\Gamma(s, p) = \exp_p(s\mu(p))$ near 0, where $p \in \widetilde{\mathcal{O}} \cap \Gamma$ and $\mu(p) \in T_p\widetilde{M}$ is the unit inner normal to Γ with respect to g. It is clear that $\exp_\Gamma$ is a diffeomorphism if $\widetilde{\mathcal{O}}$ is small. By virtue of this map, we can introduce coordinates, still denoted by $\{x^1, \ldots, x^{n-1}, x^n\}$, in $\widetilde{\mathcal{O}}$ such that $\widetilde{\mathcal{O}}$

is mapped onto an open neighborhood U of 0 and the boundary Γ is determined by $x^n = 0$ and $x^n > 0$ in $\widetilde{\mathcal{O}} \cap M$. Moreover, in this coordinate system, the metric g is given by

$$g = \begin{pmatrix} & & & 0 \\ & g_{\alpha\beta} & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \quad \text{and} \quad g^{-1} = \begin{pmatrix} & & & 0 \\ & g^{\alpha\beta} & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Hereafter the indices α and β run from 1 to $n-1$.

First, we want to determine $g(0)$. Clearly, in boundary normal coordinates, it is enough to determine $g(v,v)|_{x=0}$ for any tangent vector v. We use here the following argument of Michel's paper [9]. Denote e_α the αth standard basis of $\mathbb{R}^n$. Let us define a set of vectors $V = \{v_{\alpha\beta},\ \alpha \leqslant \beta\}$, where $v_{\alpha\alpha} = e_\alpha$ and $v_{\alpha\beta} = e_\alpha + e_\beta$ for $\alpha < \beta$. Let $c : [0, \varepsilon) \to U$ be a curve on $\{x^n = 0\}$ with $c(0) = 0$ and $c'(0) = v$, where v is an element of V. Here we choose ε small enough so that all geodesics joining 0 and $c(\tau)$ for $0 < \tau \leqslant \varepsilon$ lie entirely in $U^+ = \{x \in U\colon\ x_n > 0\}$ except for the two endpoints. It is easy to see that

$$\lim_{\tau\to 0} \frac{d_g(0, c(\tau))}{\tau} = \|v\|_g = g(v,v)^{1/2}.$$

Now by repeating the arguments for all $v \in V$, we can determine $g_{\alpha\beta}(0)$ for all α, β and hence $g(0)$. Clearly, using the same method, we can find $g(x', 0)$ for $|x'| < \delta$ with δ sufficiently small, where $x' = (x^1, \ldots, x^{n-1})$.

Having found g on $\Gamma_\delta := \{(x', 0)\colon\ |x'| < \delta\}$ for small δ, we can determine the right-hand side of (2.5) by knowing $d_g(p, g)$ for all $p, g \in \Gamma_\delta$. This can be seen, using the notation of the previous section, by observing that d_g is the generating function of the canonical relation obtained by projecting the set $\{(X^{(0)}, X_g(t, X^{(0)})\}$ onto $T^*\Gamma \times T^*\Gamma$. This set is called the scattering relation in [4]. A more differential geometric way to see this is via the formula derived in [9]

$$\gamma'\big(t(p,q)\big) = i_*\big(\nabla'_q t(p,q)\big) - \sqrt{1 - \big\|\nabla'_q t(p,q)\big\|^2_{i^*g}}\,\mu(q), \tag{3.1}$$

where γ is the geodesic issued from p and parametrized by the arc length, $i : \Gamma_\delta \to U$ is the inclusion map and ∇' is the gradient operator on the boundary $x^n = 0$. Let $\xi^{(0)}(p,q)$ and $\xi_g(t(p,q), \xi^{(0)}(p,q)) =: \xi_g(p,q)$ be the initial and final covectors related to the geodesic connecting $p, q \in \Gamma_\delta$. Reinterpreting (3.1) in the covector setting, we can see that for the geodesic joining p and q, $p \neq q$, $\xi^{(0)}(p,q)$ and $\xi_g(p,q)$ satisfy

$$\xi^{(0)}(p,q) = g(p) i_*\big(\nabla'_p t(p,q)\big) - \sqrt{1 - \big\|\nabla'_p t(p,q)\big\|^2_{i^*g}}\, g(p)\mu(p) \tag{3.2}$$

and

$$\xi_g(p,q) = g(q) i_*\big(\nabla'_q t(p,q)\big) - \sqrt{1 - \big\|\nabla'_q t(p,q)\big\|^2_{i^*g}}\, g(q)\mu(q). \tag{3.3}$$

Next we would like to determine all derivatives of g at 0. Since $g^{-1}g = I$, it suffices to determine all derivatives of g^{-1} at 0. Let $v \in V$ and define $\tilde{c}(\tau)$, $0 \leqslant \tau \leqslant \tilde{\varepsilon}$, be a curve on $\{x^n = 0\}$ with $\tilde{c}(0) = 0$ and $\tilde{c}'(0) = g^{-1}(0)\eta$, where $\eta = v(v \cdot g^{-1}(0)v)^{-1/2}$. As before, we choose $\tilde{\varepsilon}$ sufficiently small so that all geodesics joining 0 and $\tilde{c}(\tau)$ for $0 < \tau \leqslant \tilde{\varepsilon}$ lie entirely in U^+ except for the two endpoints. Now we are at the position to choose our reference Riemannian metric g_0 in M. The goal here is to choose a g_0 such that $\nu(0) \cdot g_0^{-1}(0)\eta < 0$, i.e.,

$$\sum_{j=1}^{n} g_0^{nj}(0)v_j > 0 \tag{3.4}$$

for all $v = (v_1, \ldots, v_n) \in V$. One possible choice is that g_0^{-1} is of the form

$$g_0^{-1} = \begin{pmatrix} & & & \lambda_1 \\ & I_{n-1} & & \vdots \\ & & & \lambda_{n-1} \\ \lambda_1 & \cdots & \lambda_{n-1} & 1 \end{pmatrix}, \tag{3.5}$$

where I_{n-1} is the identity matrix of size $n-1$ and $\lambda_\alpha > 0$. Since $\det(g_0^{-1}) = 1 - \sum_{\alpha=1}^{n-1} \lambda_\alpha^2$, we can choose λ_α sufficiently small for all α to guarantee the positive-definiteness of g_0^{-1}. Here we want to point out that η does not satisfy the incoming direction (2.1) with respect to g, but it satisfies (2.1) in terms of g_0. In fact, we can see that $\nu(0) \cdot g^{-1}(0)\eta = 0$. With this choice of g_0, we obtain that the solution to the Hamiltonian system (2.3) with respect to g_0 can be written explicitly as

$$\big(x_{g_0}\big(s, 0, \xi^{(0)}\big), \xi_{g_0}\big(s, 0, \xi^{(0)}\big)\big) = \big(sg_0^{-1}\xi^{(0)}, \xi^{(0)}\big),$$

where the initial $(0, \xi^{(0)})$ satisfies (2.1) in terms of g_0. Note that the curve $x_{g_0}(s, 0, \xi^{(0)})$ lies entirely in U^+ for all small s.

Now consider the geodesic (relative to g) connecting 0 and $\tilde{c}(\tau)$ for $0 < \tau < \tilde{\varepsilon}$. In view of the formulas (3.2) and (3.3), it is readily seen that given $\tilde{c}(\tau)$ we can determine $\xi^{(0)} = \xi^{(0)}(\tau)$ and $X_g(t(\tau), X^{(0)}(\tau)) = X_g(\tau)$ from the boundary distance function $d_g(0, \tilde{c}(\tau)) = t(\tau)$. Notice that if $t(\tau)$ is sufficiently small (i.e., τ is small), then $\xi^{(0)}(\tau)$ is close to η and $(0, \xi^{(0)}(\tau))$ satisfies the incoming condition (2.1) related to g_0. Furthermore, we can see that $\xi^{(0)}(\tau) \to \eta$ as $\tau \to 0$. Expressing every variable in the identity (2.5) in terms of τ, we have that

$$\begin{aligned} &\int_0^{t(\tau)} \frac{\partial X_g}{\partial X^{(0)}}\big(t(\tau) - s, X_{g_0}\big(s, X^{(0)}(\tau)\big)\big)(V_{g_0} - V_g)\big(X_{g_0}\big(s, X^{(0)}(\tau)\big)\big)\,\mathrm{d}s \\ &\quad = X_{g_0}(\tau) - X_g(\tau) = \big(t(\tau)g_0^{-1}\xi^{(0)}(\tau) - x_g(\tau), \xi^{(0)}(\tau) - \xi_g(\tau)\big). \end{aligned} \tag{3.6}$$

Differentiating both sides of (3.6) in τ yields

$$\begin{aligned}
&\frac{\partial X_g}{\partial X^{(0)}}\big(0, X_{g_0}\big(t(\tau), X^{(0)}(\tau)\big)\big)(V_{g_0} - V_g)\big(X_{g_0}\big(t(\tau), X^{(0)}(\tau)\big)\big)t'(\tau) \\
&\quad + \int_0^{t(\tau)} \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\frac{\partial X_g}{\partial X^{(0)}}\big(t(\tau) - s, X_{g_0}\big(s, X^{(0)}(\tau)\big)\big)(V_{g_0} - V_g)\big(X_{g_0}\big(s, X^{(0)}(\tau)\big)\big)\right\}\mathrm{d}s \\
&\quad = \big(t'(\tau)g_0^{-1}\xi^{(0)}(\tau) + t(\tau)g_0^{-1}\xi^{(0)\prime}(\tau) - x_g'(\tau), \xi^{(0)\prime}(\tau) - \xi_g'(\tau)\big). \qquad (3.7)
\end{aligned}$$

Taking $\tau \to 0$ in (3.7), we obtain that

$$\begin{aligned}
&\frac{\partial X_g}{\partial X^{(0)}}\big(0, X_{g_0}\big(0, X^{(0)}(0)\big)\big)(V_{g_0} - V_g)\big(X_{g_0}\big(0, X^{(0)}(0)\big)\big)t'(0) \\
&\quad = I_{2n\times 2n}\cdot(V_{g_0} - V_g)(0,\eta)t'(0) = \big(t'(0)g_0^{-1}\eta - x_g'(0), \xi^{(0)\prime}(0) - \xi_g'(0)\big), \qquad (3.8)
\end{aligned}$$

where $I_{2n\times 2n}$ is the identity matrix of size $2n$. It has been shown previously that $t'(0) = \|\tilde{c}'(0)\|_g = 1$. Writing out the formula (3.8), we conclude that

$$g^{-1}(0)\eta = g_0^{-1}(0)\eta - t'(0)g_0^{-1}\eta - x_g'(0) \tag{3.9}$$

and

$$\frac{1}{2}\eta\cdot\partial_x g^{-1}(0)\eta = \xi^{(0)\prime}(0) - \xi_g'(0). \tag{3.10}$$

It should be noted that we do not use (3.9) as the reconstruction formula for $g(0)$ since we need to choose the curve $\tilde{c}$ before proceedings with the arguments. The curve $\tilde{c}$ has already used the information $g(0)$. It follows from (3.10) that

$$v\cdot\partial_x g^{-1}(0)v = 2\big(v\cdot g^{-1}(0)v\big)\big(\xi^{(0)\prime}(0) - \xi_g'(0)\big). \tag{3.11}$$

By repeating the above arguments for each one element of V, we can derive (3.11) for all $v \in V$. In turn we are able to determine $\partial_x g^{-1}(0)$. Using the same method, we can find $\partial_x g^{-1}(x',0)$ for $(x',0)$ near 0. Thus, $\partial_{x^\alpha}^k\partial_{x^n} g^{-1}(0)$ is also determined for all positive integer k.

To continue the proof, we differentiate (3.7) in τ and set $\tau = 0$. To end, we get that

$$\begin{aligned}
&I_{2n\times 2n}\sum_{j=1}^n \partial_{x^j}(V_{g_0} - V_g)\big(g_0^{-1}\eta\big)_j\big(t'(0)\big)^2 \\
&\quad = \big(t''(0)g_0^{-1}\eta + 2t'(0)g_0^{-1}\xi^{(0)\prime}(0) - x_g''(0), \xi^{(0)\prime\prime}(0) - \xi_g''(0)\big) + \Psi, \qquad (3.12)
\end{aligned}$$

where $(g_0^{-1}\eta)_j$ is the jth component of $g_0^{-1}\eta$ and Ψ is a $2n$ vector which consists of terms containing only $g^{-1}(0)$ and $\partial_x g^{-1}(0)$. Therefore, Ψ is a known vector-valued function. It is easily observed that only the last n components of $\sum_{j=1}^n \partial_{x^j}(V_{g_0} - V_g)(g_0^{-1}\eta)_j$ contain the second derivatives of g^{-1} at 0. Moreover, since we have found $\partial_{x^\alpha}\partial_{x^n} g^{-1}(0)$, the only term yet to be determined is $\partial^2_{x^n} g^{-1}(0)$. Singling out the last component of (3.12), we have

$$\frac{1}{2}\eta \cdot \left[\sum_{j=1}^{n} \partial_{x^j}\partial_{x^n} g^{-1}(0)\big(g_0^{-1}\eta\big)_j\right]\eta = \big(\xi^{(0)\prime\prime}(0) - \xi_g''(0)\big)_n + (\Psi)_{2n}$$

from which we get that

$$\begin{aligned}
&\big\{v \cdot \partial^2_{x^n} g^{-1}(0)v\big\}\big(g_0^{-1}v\big)_n \\
&\quad = 2\big(v \cdot g^{-1}(0)v\big)^{3/2}\left\{\eta \cdot \left[\sum_{\alpha=1}^{n-1} \partial_{x^\alpha}\partial_{x^n} g^{-1}(0)\big(g_0^{-1}\eta\big)_\alpha\right]\eta \right. \\
&\qquad \left. + \big(\xi^{(0)\prime\prime}(0) - \xi_g''(0)\big)_n + (\Psi)_{2n}\right\}.
\end{aligned} \tag{3.13}$$

Since $(g_0^{-1}v)_n$ is not zero (see (3.4)), we can determine $v \cdot \partial^2_{x^n} g^{-1}(0)v$ from (3.13). Once again, repeating the arguments for all $v \in V$ and noting that $(g_0^{-1}v)_n$ is never zero for any $v \in V$, we can determine $v \cdot \partial^2_{x^n} g^{-1}(0)v$ for all $v \in V$ and hence $\partial^2_{x_n} g^{-1}(0)$. Using the same procedure, we can determine $\partial^2_{x^n} g^{-1}(x', 0)$ for $|x'| < \delta$ with δ sufficiently small. In turn we can find $\partial^{\alpha'}_{x'}\partial^2_{x^n} g^{-1}(0)$ for any multi-index α'.

Inductively, assume that we have determined $\partial^{\alpha'}_{x'}\partial^{l}_{x^n} g^{-1}(0)$ with $0 < l < \ell$ and arbitrary α'. Now by differentiating (3.6) ℓ times in τ and setting $\tau = 0$, we single out the term containing $\partial^{\ell}_{x^n} g^{-1}(0)$ and find that

$$\big\{v \cdot \partial^{\ell}_{x^n} g^{-1}(0)v\big\}\big(g_0^{-1}v\big)_n^{\ell-1} = \mathcal{R}, \tag{3.14}$$

where $\mathcal{R}$ is a known value which is determined by the induction assumption. Deriving (3.14) for each $v \in V$ and noting that $(g_0^{-1}v)_n$ is never zero, we can determine $v \cdot \partial^{\ell}_{x^n} g^{-1}(0)v$ for all $v \in V$ and therefore $\partial^{\ell}_{x^n} g^{-1}(0)$.

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