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NEW COMPARISONS IN BIRNBAUM IMPORTANCE FOR THE CONSECUTIVE-*k*-OUT-OF-*n* SYSTEM

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We study the Birnbaum importance of the consecutive-k-out-of-n line and clear up confusion in some claimed but unproved results. We introduce some new techniques which not only prove these claimed results but also generalize them much further. Finally, we extend our results to the 2-out-of-m-out-of-n line.

1. INTRODUCTION

A consecutive-k-out-of-n system, or simply a consecutive line if k and n need not be emphasized, is a line of n components each working or failing such that the system

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is failed if and only if some k consecutive components all fail. For a system S, let $R(S)$ denote the reliability of S. Then the Birnbaum importance of a component i is defined as

$$I_i = R(S|i \text{ works}) - R(S|i \text{ fails}).$$

The Birnbaum importance of the consecutive line has been studied in the literature as a comparison of I_i and I_j and can provide information in the following two situations:

i. Suppose we can construct a consecutive line by an arbitrary permutation of n components with distinct reliabilities. To maximize the system reliability, the importance ranking of the n positions may provide guidance as to which component should go where.
ii. Suppose we want to improve the system reliability through improving the reliability of a component. Then the Birnbaum importance tells us that it is more effective to improve on a component with higher importance.

Unfortunately, comparing the Birnbaum importance for the consecutive line is not easy. The $k = 2$ case was completely solved by Zuo and Kuo [8]. For general n, only the following results were published (by symmetry, we only need to compare components in the first half plus the middle one if it exists):

a. $I_1 \le I_2 \le \cdots \le I_k$ for $n \ge 2k$ and $I_1 \le I_2 \le \cdots \le I_{n-k+1} = \cdots = I_{\lceil n/2 \rceil}$ for $n \le 2k$ (by Kuo, Zhang, and Zuo [3]);
b. $I_1 \le I_{k+1}$ (by Zuo [7]);
c. $I_{tk+1} \le I_{tk}$ for $t + 1 \le tk + 1 \le \lfloor (n+1)/2 \rfloor$ [7];
d. $I_k \ge I_{2k}$ [7].

However, the proof for result c is correct only for $t = 1$. In this paper, we point out where the proof of c is problematic and give a new proof of c.

In general, I_i is not comparable with all other I_j. The only exception is perhaps for I_1 and I_k. We prove in this paper

$$I_1 \le I_i \le I_k \quad \text{for all } 1 \le i \le \lceil n/2 \rceil,$$

which covers and generalizes b) and d).

2. COMPARING BIRNBAUM IMPORTANCE

For a given k, let $R(n)$ denote the reliability of a consecutive-k-out-of-n system. Let p denote the common component reliability and define $q = 1 - p$. Papastavridis [4] proved the following lemma.

LEMMA 2.1: $I_i = [R(i-1)R(n-1) - R(n)]/q$.

The proof of $I_1 \le I_k$ by Zuo can be extended in a straightforward manner to prove the following theorem.

THEOREM 2.2: *$I_1 \leq I_i$ for all $2 \leq i \leq \lceil n/2 \rceil$.*

PROOF: By Lemma 2.1,

$$I_1 = [R(n-1) - R(n)]/q \leq [R(i-1)R(n-i) - R(n)]/q = I_i$$

for $2 \leq i \leq \lceil n/2 \rceil$, since the $(n-1)$-line works only if both of its parts, an $(i-1)$-line and an $(n-i)$-line, work. ■

Recently, Hwang [2] defined a new kind of importance, I^H. A cutset is a set of components whose collective failures cause a system failure. Let $f(d,n)$ denote the number of d-cutsets, and let $f_i(d,n)$ the number of d-cutsets containing i. Then, $I_i^H \geq I_j^H$ if and only if

$$f_i(d,n) \geq f_j(d,n) \quad \text{for all } d.$$

Hwang also proved the following lemma.

LEMMA 2.3: *$I_i^H \geq I_j^H$ implies $I_i \geq I_j$.*

Although the I^H comparison is stronger than the I comparison, we show that it can be easier to prove the following theorem.

THEOREM 2.4: *$I_k^H \geq I_i^H$ for all $1 \leq i \leq \lceil n/2 \rceil$ and $n \geq 2k$.*

PROOF: Suppose $n = 2k$. Then Theorem 2.4 follows from the fact that every cutset containing i also contains k. We prove the general n case by induction.

To count $f_i(d,n)$, we must assume that component i is failed. A d-set containing i and the last k components is certainly a d-cutset containing i. There are

$$\binom{n-k-1}{d-k-1}$$

ways of distributing the other $d-k-1$ failed components in the remaining $n-k-1$ positions (the number is zero if $d = k$). Otherwise, a d-set containing i and the last j components for some $0 \leq j \leq k-1$, but not the component $n-j$, is a d-cutset containing i if and only if the first $n-j-1$ components form a $(d-j)$-cutset containing i, whose number is $f_i(d-j,\, n-j-1)$. Therefore, we have,

$$f_i(d,n) = \sum_{j=1}^{k-1} f_i(d-j, n-j-1) + \binom{n-k-1}{d-k-1}.$$

Note that if $i > \lceil (n-j)/2 \rceil$, then we can substitute i with $i' = n-j-i$ by symmetry. Since $n-j-1 \geq 2(d-j)$, by the induction hypothesis,

$$f_k(d-j, n-j-1) \geq f_i(d-j, n-j-1) \quad \text{for all } j.$$

Hence,

$$f_k(d,n) - f_i(d,n) = \sum_{j=0}^{k-1} [\, f_k(d-j, n-j-1) - f_i(d-j, n-j-1)] \geq 0 \quad \text{for all } d.$$

Theorem 2.4 follows immediately. ■

COROLLARY 2.5: *$I_k \geq I_i$ for all $1 \leq i \leq \lceil n/2 \rceil$ and $n \geq 2k$.*

Zuo proved $I_k \geq I_{k+1}$. Then he claimed $I_{tk} \geq I_{tk+1}$ by arguing that tk can be treated as k', hence, $I_{tk} \geq I_{tk+1}$ follows from $I_{k'} \geq I_{k'+1}$. This is a false argument, a fact which can easily be seen if the parameter k (as in consecutive-k) is kept in the notation of I_i, say, $I_i(k)$. Then what is proved is $I_k(k) \geq I_{k+1}(k)$, and what is claimed is $I_{k'}(k) \geq I_{k'+1}(k)$. The latter clearly does not follow from the former. We now give a correct proof.

THEOREM 2.6: *$I_{tk} \geq I_{tk+1}$ for $k+1 \leq tk+1 \leq \lfloor (n+1)/2 \rfloor$.*

PROOF: Shanthikumar [6] and Hwang [1] independently gave the recursive equation

$$R(n) = R(n-1) - pq^k R(n-k-1).$$

Hence,

$$\frac{R(n-1)}{R(n)} = 1 + pq^k \frac{R(n-k-1)}{R(n)}. \tag{1}$$

Let $A \sim B$ mean that A and B have the same sign. We will also keep the parameter n in I_i, that is, $I_i(n)$. Using Lemma 2.1 and Eq. (1),

$$\begin{aligned} I_i(n) - I_{i+1}(n) &\sim R(i-1)R(n-i) - R(i)R(n-i-1) \\ &\sim \frac{R(i-1)}{R(i)} - \frac{R(n-i-1)}{R(n-i)} \\ &\sim \frac{R(i-k-1)}{R(i)} - \frac{R(n-i-k-1)}{R(n-i)} \\ &\sim R(i-k-1)R(n-i) - R(i)R(n-i-k-1) \\ &\sim I_{i-k}(n-k) - I_{i+1}(n-k) \quad \text{for } i > k. \end{aligned}$$

Thus the comparison on an n-line is reduced to a comparison on an $(n-k)$-line for $i > k$. Note that $i - k \leq (n-k+1)/2$. If $i+1 > (n-k+1)/2$, substitute $i+1$ with $i' = n-k-i$ by symmetry. Suppose $i-k > k$. Then we repeat the same reduction. When $i = tk$, then, after $t-1$ steps, tk is reduced to k (and n to $n-tk+k$). It is easily verified that $n - tk + k \geq 2k$. By Corollary 2.5,

$$I_k(n-tk+k) \geq I_{g(i+1)}(n-tk+k),$$

where $g(i+1)$ is $i+1$ after the possible reflections during the reduction. Theorem 2.6 follows immediately. ■

COROLLARY 2.7: $I_{tk}(2tk-1) > I_{tk-1}(2tk-1)$.

PROOF: After $t-1$ steps, $tk-1$ is reduced to $k-1$ and n to $tk+k-1$. So tk is reflected to k, which, by Corollary 2.5, is more important than any other i. ■

THEOREM 2.8: $I_{tk+1} \le I_{tk+2}$ *for* $k+2 \le tk+2 \le \lfloor (n+1)/2 \rfloor$.

PROOF: When $i = tk+1$, then after t steps, $tk+1$ is reduced to 1. By Theorem 2.2, $I_1 \le I_i$ for all $2 \le i \le \lceil n/2 \rceil$. Theorem 2.8 follows immediately. ■

3. THE 2-OUT-OF-*m*-OUT-OF-*n* SYSTEM

A k-out-of-m-out-of-n system fails if and only if there exists an m-window (a set of m consecutive components) which contains k or more failed components. The consecutive-k-out-of-n system is the special case $m = k$. In this section, we study the $k = 2$ case. Papastavridis and Sfakianakis [5] proved the following.

LEMMA 3.1: $I_i = [R(n) - p^{x_i} R(i-m)R(n-i-m+1)]/p$, *where* $x_i = \min\{m-1, i-1\} + \min\{m-1, n-i\} \ge m-1$.

COROLLARY 3.2: *For* $n \ge 2m-1$ *and* $m \le i \le \lceil n/2 \rceil$, $I_i = [R(m) - p^{2(m-1)} \times R(i-m)R(n-i-m+1)]/p$.

They also obtained that for $1 \le i \le m-1$, every cutset containing i contains $i+1$. Hence,

LEMMA 3.3: $I_1 \le I_2 \le \cdots \le I_m (I_n \le I_{n-1} \le \cdots \le I_{n-m+1})$.

We have the following.

LEMMA 3.4: $R(x)p^{m-1}R(y) \le R(x+m-1+y)$.

PROOF: Break an $(x+m-1+y)$-line into three segments with x, $m-1$, y components in that order. If the middle segment consists of $m-1$ working components, and the other two segments are each a working line, then the $(x+m-1+y)$-line is working. But the latter can work in other scenarios. ■

THEOREM 3.5: $I_1 \le I_i$ *for all* i.

PROOF: By symmetry, we need only consider $i \le \lceil n/2 \rceil$. By Lemma 3.2, we may assume $i \ge m+1$ and $n \ge 2m+1$. Then, $x_i = 2(m-1)$. Note that

$$\begin{aligned} p^{x_i} R(i-m)R(n-i-m+1) &= p^{2(m-1)} R(i-m)R(n-i-m+1) \\ &= p^{m-1}[R(i-m)p^{m-1}R(n-1-m+1)] \\ &\le p^{m-1}R(n-m). \end{aligned}$$

Theorem 3.5 follows immediately. ■

THEOREM 3.6: *$I_m^H \geq I_i^H$ for all $1 \leq i \leq \lceil n/2 \rceil$ and $n \geq 2m$.*

PROOF: Suppose $n = 2m$. Then Theorem 3.6 follows from Lemma 3.2. We prove the general n case by induction.

$$f_i(d,n) = f_i(d,n-1) + f_i(d-1,n-m), \qquad 1 \leq i \leq \lceil n/2 \rceil.$$

Theorem 3.6 now follows by induction. ∎

COROLLARY 3.7: *$I_m \geq I_i$ for all $1 \leq i \leq \lceil n/2 \rceil$ and $n \geq 2m$.*

By simulating the proof of Theorem 2.6, we obtain

THEOREM 3.8: *$I_{tm} \geq I_{tm+1}$ for $k+1 \leq tk+1 \leq \lfloor (n+1)/2 \rfloor$.*

COROLLARY 3.9: *$I_{tm}(2tm-1) > I_{tm-1}(2tm-1)$.*

COROLLARY 3.10: *$I_{tm+1} \leq I_{tm+2}$ for $m+2 \leq tm+2 \leq \lceil n/2 \rceil$.*

References

1. Hwang, F.K. (1982). Fast solutions for consecutive-*k*-out-of-*n*: F system. *IEEE Transactions on Reliability* 31: 447–448.
2. Hwang, F.K. A new index of component importance. (to appear)
3. Kuo, W., Zhang, W., & Zuo, M. (1990). A consecutive-*k*-out-of-*n*: G system, the mirror image of a consecutive-*k*-out-of-*n*: F system. *IEEE Transactions on Reliability* 39: 244–253.
4. Papastavridis, S.G. (1987). The most important component in a consecutive-*k*-out-of-*n*: F system. *IEEE Transactions on Reliability* 36: 266–268.
5. Papastavridis, S.G. & Sfakianakis, M.E. (1991). Optimal-arrangement and importance of the components in a consecutive-*k*-out-of-*r*-from-*n*: F system. *IEEE Transactions on Reliability* 40: 277–279.
6. Shanthikumar, J.G. (1982). Recursive algorithm to evaluate the reliability of a consecutive-*k*-out-of-*n*: F system. *IEEE Transactions on Reliability* 31: 442–443.
7. Zuo, M. (1993). Component reliability—Importance in a linear consecutive-*k*-out-of-*n* system. *Journal of Microelectron Reliability* 33: 243–258.
8. Zuo, M. & Kuo, W. (1990). Design and performance analysis of consecutive-*k*-out-of-*n* structure. *Naval Research Logistics* 37: 203–230.