

Analytics and Algorithms for Geometric Average Trigger Reset Options

Tian-Shyr Dai,*[©]I-Yuan Chen, Yuh-Yuan Fang, and [†]Yuh-Dauh Lyuu[†]

Department of Computer Science & Information Engineering, National Taiwan University.

[©]Department of Computer Science & Information Engineering, National Taiwan University.

Analyst, Investment & Research Division, Fuh-Hua Investment Trust Co., Taipei, Taiwan.

[†] Professor, Department of Finance and Department of Computer Science &

Information Engineering, National Taiwan University.

Emails: camel@turing.csie.ntu.edu.tw, r89075@csie.ntu.edu.tw, dollar@ms.fuhwatrust.com.tw

lyuu@csie.ntu.edu.tw

Abstract

The geometric average trigger reset option resets the strike price based on the geometric average of the underlying asset's prices over a monitoring window. This paper derives an analytic formula and two numerical methods for pricing this option with multiple resets. The analytic formula in fact is a corollary of a general formula that holds for a large class of path-dependent options: It prices any option whose payoff function can be written as $e^{b \cdot X} 1_{\{X \in A\}}$. For general American-style reset options, an $O(n^4 h^2)$ -time algorithm on n -period binomial lattice is presented. A much more efficient $O(n^3 h m)$ -time algorithm prices European-style reset options. Monte Carlo simulation suggests that the European-style geometric average trigger reset option and the arithmetic version have similar option values. This implies that results in this paper give tight prices for the difficult arithmetic version.

Keywords: Reset options, path-dependent options, analytics, algorithm

1 Introduction

A reset option is a path-dependent option whose strike price can be reset based on certain criteria. For example, the strike price of a reset call can be reset downward if the underlying asset's price falls below a predetermined value. Most reset features embedded in a reset call (put) protect the investors amid declines (increases, respectively) in underlying asset's

price. This makes a reset option useful to portfolio insurance, for example. To prevent price manipulation, many contracts use the average price of the underlying asset during a certain time period, the so-called monitoring window, as a reset trigger. At the end of each monitoring window lies a reset date. The advantages of using average price instead of the underlying asset's price alone, as in ordinary reset options, are (1) to mitigate the possibility of stock price manipulation, especially for thin markets, and (2) to provide a strike price correlated with a perceived price trend or fair value. These advantages make the products appealing to some investors. The price that comes with the above-mentioned advantages is complexity, as the option combines the features of Asian and reset options. Similar securities have been issued before in Taiwan, for example.

Ordinary reset options have been investigated earlier. Gray and Whaley [7, 8] analyze the S&P 500 bear market warrant with a single reset and derive an analytic solution for single-reset reset options. Heynen and Kat [9] discuss the discrete lookback options, which are closely related to reset options. For reset options with the average feature, Chang, Chung, and Shackleton [2] give a numerical approach for pricing arithmetic average trigger reset options by extending the approximation algorithm of [11]. This approach, however, lacks convergence guarantees as mentioned in [6].

Formally, a geometric average trigger reset option is an option that uses the geometric price average in a monitoring window to set the strike price at the end of that monitoring window, the reset date. Cheng and Zhang [3] derive an analytic formula for the option with a single monitoring window. However, their formula is erroneous, as will be confirmed later. This

*The author was supported in part by NSC grant 90-2213-E-002-081.

[†]The author was supported in part by NSC grant 90-2213-E-002-081.

paper will establish the formula for geometric average trigger reset options with an arbitrary number of monitoring windows, generalizing [3]. In fact, the analytic formula is a corollary of a general formula that is of independent interest. This general formula prices any option whose payoff function can be written as $e^{b \cdot X} 1_{\{X \in A\}}$, where X is a multinormal random vector and b is some constant vector. Such options include vanilla options, geometric Asian options, rainbow options, quanto options, and many others (see [5]). Discretely monitored barrier options also fall into this category, with the closed-form formula as a multiple integration. Because evaluating a multi-dimensional integral is a hard computational problem, the approximation solution of [1] remains important for such barrier options.

The American-style geometric average trigger reset call will not be exercised early if its underlying asset does not pay dividends. This makes our formula applicable to them. For general American-style geometric average trigger reset options, an $O(n^4 h^2)$ -time algorithm on an n -period binomial lattice is presented that handles multiple reset dates, where each monitoring window has the same duration of h periods. A much more efficient $O(n^3 h m)$ algorithm exists for European-style geometric average trigger reset options, where m denotes the number of reset dates. This algorithm makes novel use of generating functions. Numerical results cross-validate our formula and algorithms.

The price of a geometric average trigger reset call is higher than that of an arithmetic one, and vice versa for a put (see [4]). Numerical results in Figure 1 suggest that geometric and arithmetic average trigger reset options have close values even when the volatility is as high as 150%. This implies that results in this paper also give tight prices for arithmetic average trigger reset options.

Volatility	50%	100%	120%	150%
Geometric	26.165	44.971	51.562	60.460
Arithmetic	26.105	44.784	51.424	60.162
Difference	0.23%	0.42%	0.27%	0.49%

Figure 1: **Geometric and Arithmetic Average Trigger Reset Call Values.** The underlying asset's initial price is 100, the strike price is 95, the risk-free rate is 5%, the time to maturity is 1 year, and the length of the monitoring window which ends at year 0.5 is 0.2 year. Monte Carlo simulation is based on 1,000,000 paths. "Geometric" and "Arithmetic" denote the geometric and arithmetic average trigger reset calls, respectively. "Difference" denotes relative price differences.

The paper is organized as follows. The geometric average trigger reset option is formally defined in Section 2. A general analytic pricing formula is introduced in Section 3. The analytic formula for pricing geometric average trigger reset options is also derived there. A numerical pricing approach based on the Cox-Ross-Rubinstein (CRR) lattice is discussed in Section 4. A much faster combinatorial method is developed for pricing European-style geometric average trigger reset options in Section 5. Section 6 concludes the paper.

2 Geometric Average Trigger Reset Options

An option starts at time 0 and matures at T . Let r denote the risk-free interest rate, $S(t)$ denote the underlying asset's price at time t , and σ denote the volatility of underlying asset S . We assume $S(t)$ follows $dS = rS dt + \sigma S dW$ in a risk-neutral economy.

Geometric average trigger reset options are reset options whose strike price can be reset to the geometric average of the underlying asset's prices over monitoring windows. Consider a general reset option with m reset dates: t_1, t_2, \dots, t_m , where $0 \leq t_1 < t_2 < \dots < t_{m-1} < t_m \leq T$. These m monitoring windows are $[t_1 - \ell_1, t_1], [t_2 - \ell_2, t_2], \dots, [t_m - \ell_m, t_m]$, where ℓ_i denotes the length of i th monitoring window. The geometric price average of the underlying asset during the i th monitoring window is defined as

$$\text{avg}(t_i) = \exp\left(\frac{1}{\ell_i} \int_{t_i - \ell_i}^{t_i} \ln S(t) dt\right).$$

Let $K(t_i)$ be the strike price prevailing at reset date t_i . Initially, $K(t_0) = K$, the original strike price. The reset procedure for the call at reset date t_i is

$$K(t_i) = \begin{cases} K(t_{i-1}), & \text{if } \text{avg}(t_i) \geq K(t_{i-1}) \\ \text{avg}(t_i), & \text{if } \text{avg}(t_i) < K(t_{i-1}) \end{cases}.$$

The terminal payoff of the call is $(S(T) - K(t_m))^+$. Similarly, the reset procedure for the put at time t_i is

$$K(t_i) = \begin{cases} K(t_{i-1}), & \text{if } \text{avg}(t_i) \leq K(t_{i-1}) \\ \text{avg}(t_i), & \text{if } \text{avg}(t_i) > K(t_{i-1}) \end{cases},$$

and the terminal payoff of the put is $(K(t_m) - S(T))^+$. If the option is American-style, the exercise value for the call at time t is $S(t) - K(t)$, where $K(t)$ is the prevailing strike price set at the most recent reset date. Similarly, the exercise value for the put at time t is $K(t) - S(t)$. We assume $\ell_1 = \ell_2 = \dots = \ell_m = \ell$ and that the monitoring windows are disjoint to simplify the presentation of the paper.

3 The Analytic Approach

A general analytic pricing formula for pricing a large class of path-dependent options is presented below. The analytic formula for geometric average trigger reset options will then be derived as a corollary of the formula.

3.1 A General Formula

Let b be an m -dimensional vector, X be a nondegenerate m -dimensional normal random variable, and A be a subspace of the m -dimensional space \mathfrak{R}^m . All vectors such as b and X will be row vectors throughout the rest of the paper. Superscript $*$ denotes the transpose of a vector or matrix. The general pricing formula holds for any derivative whose payoff is a linear combination of $E(e^{bX^*} 1_{\{X \in A\}})$. Many sophisticated derivatives fit this category.

Assume the mean vector and the covariance matrix of X are denoted by μ and Σ , respectively. Because X is not degenerate, $\det \Sigma \neq 0$. The probability density function of X is by definition

$$f(x) = \frac{1}{(\sqrt{2\pi})^m (\det \Sigma)^{1/2}} e^{-(x-\mu)\Sigma^{-1}(x-\mu)^*/2}.$$

Therefore,

$$\begin{aligned} & E(e^{bX^*} 1_{\{X \in A\}}) \\ &= \int_A \frac{1}{(\sqrt{2\pi})^m (\det \Sigma)^{1/2}} e^{bx^* - (x-\mu)\Sigma^{-1}(x-\mu)^*/2} dx. \end{aligned} \quad (1)$$

Let $a = b\Sigma$. Then $bx^* - (x-\mu)\Sigma^{-1}(x-\mu)^*/2$ equals

$$\mu\Sigma^{-1}a^* + (a\Sigma^{-1}a^*/2) - \{[x - (\mu + a)]\Sigma^{-1}[x - (\mu + a)]^*/2\}$$

(see [4]). Equation (1) can now be rewritten as

$$\frac{e^{\mu b^* + (b\Sigma b^*)/2}}{(\sqrt{2\pi})^m (\det \Sigma)^{1/2}} \int_A e^{-[x - (\mu + b\Sigma)]\Sigma^{-1}[x - (\mu + b\Sigma)]^*/2} dx. \quad (2)$$

Equation (2) reduces the problem of evaluating $E(e^{bX^*} 1_{\{X \in A\}})$ to the integration of a multi-dimensional normal distribution over a region A . If the region A is already a rectangular polyhedron or if X is a one-dimensional normal random variable, the result can be rewritten in terms of distribution functions. Otherwise, if A is a polyhedron, X can be multiplied by a matrix C to change A into a rectangular polyhedron, A' . This step transforms the pricing formula into one involving distribution functions. More precisely, with $Y^* = CX^*$, (2) becomes

$$\begin{aligned} & \frac{e^{\mu b^* + b\Sigma b^*/2}}{(\sqrt{2\pi})^m (\det \Sigma')^{1/2}} \\ & \times \int_{A'} e^{-[y - (\mu + b\Sigma)C^*]\Sigma'^{-1}[y - (\mu + b\Sigma)C^*]^*/2} dy, \end{aligned} \quad (3)$$

where $\Sigma' = C\Sigma C^*$.

3.2 Analytic Formula for Geometric Average Trigger Reset Options

We next apply (3) to derive the pricing formula for the geometric average trigger reset option with m monitoring windows. Define

$$\begin{aligned} X_i &\equiv \ln(\text{avg}(t_i)/S(0)), \quad i = 1, 2, \dots, m, \\ X_{m+1} &\equiv \ln(S(T)/S(0)), \\ X &\equiv [X_1, X_2, X_3, \dots, X_{m+1}]. \end{aligned}$$

X is clearly an $(m+1)$ -dimensional normal random variable. The expected future value of a reset call is

$$\begin{aligned} & S(0) \sum_{i=1}^m [E(e^{b_{m+1}X^*} 1_{\{X \in A_i\}}) - E(e^{b_i X^*} 1_{\{X \in A_i\}})] \\ & + S(0)E(e^{b_{m+1}X^*} 1_{\{X \in A_{m+1}\}}) - KE(1_{\{X \in A_{m+1}\}}), \end{aligned} \quad (4)$$

where

$$\begin{aligned} A_i &= \{X | K(t_m) = \text{avg}(t_i), S(T) \geq \text{avg}(t_i)\}, \\ & \quad \text{if } 1 \leq i \leq m, \\ A_{m+1} &= \{X | K(t_m) = K, S(T) \geq K\}, \\ b_i &= [0, \dots, 0, \overbrace{1, \dots, 1}^{m+1-i}], \quad 1 \leq i \leq m+1. \end{aligned}$$

X 's mean vector μ is equal to

$$[(r - \frac{\sigma^2}{2})(t_1 - \frac{\ell}{2}), \dots, (r - \frac{\sigma^2}{2})(t_m - \frac{\ell}{2}), (r - \frac{\sigma^2}{2})T].$$

Let $\Sigma_{i,j}$ be the covariance of X_i and X_j . Then the elements of matrix $\Sigma = [\Sigma_{i,j}]$ are

$$\begin{aligned} \Sigma_{i,i} &= \sigma^2(t_i - \frac{2\ell}{2}), \quad \text{if } 1 \leq i \leq m \\ \Sigma_{i,j} = \Sigma_{j,i} &= \sigma^2(t_i - \frac{\ell}{2}), \quad \text{if } 1 \leq i < j \leq m+1 \\ \Sigma_{m+1,m+1} &= \sigma^2 T \end{aligned}$$

(see [4]). Consequently, each term in (4) can be reduced to the integration of a multinormal distribution by (2). To state the formula in terms of distribution functions, a matrix is needed for each integration to transform the polyhedral integration area into a rectangular one. This transformation can be divided in two cases.

Case 1: Area A_k ($1 \leq k \leq m$)

Each point $X = (X_1, \dots, X_{m+1})$ in A_k satisfies these $m+1$ inequalities:

$$\begin{aligned} X_k &\leq \ln(K/S(0)), \\ X_k - X_i &\leq 0 \text{ for } 1 \leq i \leq m+1 \text{ and } i \neq k. \end{aligned}$$

As A_k is not rectangular, a matrix C_k is needed to linearly transform it into a rectangular one. Define random variables Y_i by

$$Y_i = \begin{cases} X_k, & \text{for } i = k \\ X_k - X_i, & \text{for } 1 \leq i \leq m+1 \text{ and } i \neq k \end{cases}$$

The matrix C_k that transforms X into $(Y_1, Y_2, \dots, Y_{m+1})$ is

$$\begin{aligned} C_k(i, k) &= 1, \quad \text{if } 1 \leq i \leq m+1, \\ C_k(i, i) &= -1, \quad \text{if } 1 \leq i \leq m+1 \text{ and } i \neq k, \\ C_k(i, j) &= 0, \quad \text{otherwise,} \end{aligned}$$

where $C_k(i, j)$ denotes the element at the i th row and the j th column of C_k . Area A_k is hence transformed into

$$A'_k = \{Y | Y_k \leq \ln(K/S(0)), Y_i \leq 0 \text{ for } 1 \leq i \leq m+1 \text{ and } i \neq k\}.$$

Case 2: Area A_{m+1}

Each point $X = (X_1, \dots, X_{m+1})$ in A_{m+1} satisfies these $m+1$ inequalities:

$$X_i \geq \ln(K/S(0)) \quad \text{for } 1 \leq i \leq m+1.$$

Although they define a rectangular polyhedron, a matrix C_{m+1} is required to transform it so the desired formula can be expressed in terms of a distribution function. Let

$$Y_i = -X_i \quad \text{for } 1 \leq i \leq m+1.$$

Then the desired C_{m+1} is $-I$ where I is the $(m+1) \times (m+1)$ identity matrix. Area A_{m+1} is now transformed into

$$A'_{m+1} = \{Y | Y_i \leq -\ln(K/S(0)) \text{ for } 1 \leq i \leq m+1\}.$$

To simplify the notation, define $\Sigma_i \equiv C_i \Sigma C_i^*$. The expected payoff now equals

$$\begin{aligned} & S(0) \sum_{i=1}^m \\ & \{ e^{rT} \int_{A'_i} \frac{e^{-[y - (\mu + b_{m+1} \Sigma) C_i^*] \Sigma_i^{-1} [y - (\mu + b_{m+1} \Sigma) C_i^*]^* / 2}}{(\sqrt{2\pi})^{m+1} |\det \Sigma_i|^{1/2}} dy \\ & - e^{r(t_i - \ell/2) - \sigma^2 \ell / 12} \\ & \times \int_{A'_i} \frac{e^{-[y - (\mu + b_i \Sigma) C_i^*] \Sigma_i^{-1} [y - (\mu + b_i \Sigma) C_i^*]^* / 2}}{(\sqrt{2\pi})^{m+1} |\det \Sigma_i|^{1/2}} dy \} \\ & + S(0) e^{rT} \times \int_{A'_{m+1}} \\ & \frac{e^{-[y - (\mu + b_{m+1} \Sigma) C_{m+1}^*] \Sigma_{m+1}^{-1} [y - (\mu + b_{m+1} \Sigma) C_{m+1}^*]^* / 2}}{(\sqrt{2\pi})^{m+1} |\det \Sigma_{m+1}|^{1/2}} dy \\ & - K \int_{A'_{m+1}} \frac{e^{-(y - \mu C_{m+1}^*) \Sigma_{m+1}^{-1} (y - \mu C_{m+1}^*)^* / 2}}{(\sqrt{2\pi})^{m+1} |\det \Sigma_{m+1}|^{1/2}} dy. \end{aligned}$$

The above analytic formula holds for the American-style reset call because such an option will not be exercised early if the underlying asset does not pay dividends. Here is the reasoning. It is well-known that an American-style vanilla call will not be exercised early if the underlying asset does not pay dividends. In fact, at any time t before maturity, the continuation value exceeds the exercise payoff $S(t) - K(t)$, where $K(t)$ denotes the prevailing strike price at time t . Because the strike price of an otherwise identical reset call could only be reset to a level lower than $K(t)$, its continuation value must be at least as high as that of the vanilla call, thus higher than the exercise payoff too. As a result, a reset call will not be exercised early either.

3.3 The Special Case of a Single Reset

We next consider the case of a single monitoring window as an illustration. The future value of this option is (4) with $m = 1$. The mean vector of X is

$$\mu = [(r - \frac{\sigma^2}{2})(t_1 - \frac{\ell}{2}), (r - \frac{\sigma^2}{2})T].$$

The covariance matrix of X is

$$\Sigma = \begin{bmatrix} \sigma^2(t_1 - \frac{2\ell}{3}) & \sigma^2(t_1 - \frac{\ell}{2}) \\ \sigma^2(t_1 - \frac{\ell}{2}) & \sigma^2 T \end{bmatrix}.$$

The desired transformation matrices C_1 and C_2 for integration areas A_1 and A_2 , respectively, are

$$C_1 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, C_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

By (3), the integration areas A'_1 and A'_2 are

$$\begin{aligned} A'_1 &= \{(Y_1, Y_2) | Y_1 \leq \ln(K/S(0)), Y_2 \leq 0\}, \\ A'_2 &= \{(Y_1, Y_2) | Y_1 \leq -\ln(K/S(0)), Y_2 \leq -\ln(K/S(0))\}. \end{aligned}$$

Let $b_1 = [1, 0]$ and $b_2 = [0, 1]$. The future value of the option is

$$\begin{aligned} & S(0) e^{rT} N(\nu_1 - (\mu + b_2 \Sigma) C_1^*, C_1 \Sigma C_1^*) \\ & - S(0) e^{r(t_1 - \ell/2) - \sigma^2 \ell / 12} N(\nu_1 - (\mu + b_1 \Sigma) C_1^*, C_1 \Sigma C_1^*) \\ & + S(0) e^{rT} N(\nu_2 - (\mu + b_2 \Sigma) C_2^*, C_2 \Sigma C_2^*) \\ & - KN(\nu_2 - \mu C_2^*, C_2 \Sigma C_2^*), \end{aligned} \quad (5)$$

where $\nu_1 \equiv (\ln(K/S(0)), 0)$, $\nu_2 \equiv (-\ln(K/S(0)), -\ln(K/S(0)))$, and $N(\cdot, \Sigma)$ is the cumulative normal distribution function with mean vector 0 and covariance matrix Σ .

3.4 Numerical Results

The analytic formula (5) is now compared against the formula in [3]. Suppose the initial underlying asset's price is 100, the initial strike price is 95, the interest rate is 5%, the volatility is 30%, the time to maturity for the option is one year, and the length of monitoring window is 0.06 year. The results are tabulated in Figure 2, where "CZ" denotes the Cheng-Zhang formula and "Exact" denotes (5). We use both the Monte Carlo simulation ("MC") based on 1,000,000 paths and the lattice algorithm ("Lattice"), that will be introduced later, as benchmarks. It is apparent that (5) produces values consistent with the simulation and lattice results, whereas Cheng and Zhang's formula does not. This confirms that their formula is incorrect.

Reset Date	Lattice	MC	Exact	CZ
1.00	17.251	17.189	17.254	10.866
0.75	18.143	17.990	18.141	10.553
0.50	18.228	18.133	18.226	9.139
0.25	17.847	17.936	17.847	5.567

Figure 2: **Comparison with the Formula of [3].** The initial underlying asset's price is 100, the initial strike price is 95, the interest rate is 5%, the volatility is 30%, the time to maturity for the option is one year, and the length of monitoring window is 0.06 year. The Cheng-Zhang formula fails all four cases.

4 The Lattice Approach to Pricing

Although an analytic formula is very useful, it cannot be applied when the option is an American-style put or an American-style call on a dividend-paying stock. We next turn to lattice methods that can handle early exercise. We focus on calls as the extension to puts is straightforward.

4.1 The CRR Binomial Model

The standard CRR binomial lattice will be adopted for the underlying asset's price dynamics. The binomial model starts at (discrete) time 0 and ends at time n . Assume that the number of periods for a monitoring window is h . Let node (i, j) , $0 \leq j \leq i \leq n$, stand for the node at time i with j cumulative down moves and $S(i, j)$ denote the underlying asset's price at node (i, j) . See Figure 3 for illustration. Initially, $S(0, 0) = S(0)$. The price $S(i, j)$ at node (i, j) becomes $S(i, j)u$ at node $(i+1, j)$ —the up move—with probability p_u and $S(i, j)d$ at node $(i+1, j+1)$ —the down move—with probability $p_d \equiv 1 - p_u$. The identity $ud = 1$ of the CRR model will be crucial in the development of the algorithms. It is not hard to show that $S(i, j) = S(0)u^{i-2j}$ and node (i, j) is reached from the root node $(0, 0)$ with probability $\binom{i}{j}p_u^{i-j}p_d^j$. The underlying asset is assumed not to pay dividends for simplicity. For pricing purposes, the probability p_u is set to $(R-d)/(u-d)$, where R denotes the gross riskless return per period.

4.2 Basic Ideas

We first define a few key terms. Let (S_1, S_2, \dots, S_i) be a sequence of prices, where $1 \leq i \leq h+1$. Then $S_1 S_2 \dots S_i$ is the (partial) geometric price sum. When $i = h+1$, we call $(S_1 S_2 \dots S_h S_{h+1})^{1/(h+1)}$ a geometric price average. For a geometric price sum of form

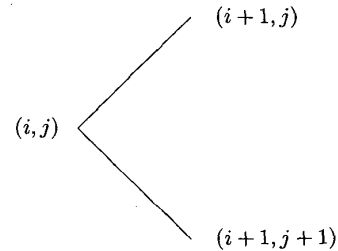


Figure 3: **The Binomial Model.** Node (i, j) is the result of j down moves and $i - j$ up moves, where $0 \leq j \leq i$.

$S(0)^i u^g$, we call g the price index.

To price a geometric average trigger reset option, the geometric price average in the current monitoring window is a key number because it might reset the strike price. The lattice algorithm keeps track of the prevailing strike price and the partial geometric price sums during a monitoring window at each node. These pairs of numbers constitute the states for a node.

The underlying asset's price for any node on an n -period lattice has form $S(0)u^x$ where $-n \leq x \leq n$. The maximum geometric price sum for any node at the i th period of a monitoring window is there-

fore smaller than $\overbrace{S(0)u^n \times \dots \times S(0)u^n}^i = S(0)^i u^{ni}$. Similarly, the minimum geometric price sum is larger than $S(0)^i u^{-ni}$. Thus, the set of possible partial geometric price sums for this node is some subset of

$$\{S(0)^i u^{-ni}, S(0)^i u^{-ni+1}, \dots, S(0)^i u^{ni-1}, S(0)^i u^{ni}\},$$

which has $2ni + 1 \leq 2n(h+1) + 1$ elements. The number of possible geometric price sums for a node in a monitoring window is hence bounded by $O(nh)$. The number of possible prevailing strike prices is also bounded by $O(nh)$ because a strike price must be a prevailing geometric price average or K . As a result, the maximum number of states for each node is bounded by $O(n^2 h^2)$. This estimate is loose as some nodes need much fewer states. We use $M_{i,j}(\hat{K}, x)$ to denote the option value at node (i, j) whose prevailing strike price and the price index of the partial geometric price sum are \hat{K} and x , respectively.

Assume node (i, j) is at the a th period in a monitoring window. A price path with a geometric price sum $S(0)^a u^b$ at node (i, j) has a geometric price sum $S(0)^{a+1} u^{b+(i+1-2j)}$ at node $(i+1, j)$ and a geometric price sum $S(0)^{a+1} u^{b+(i+1)-2(j+1)}$ at node $(i+1, j+1)$.

If time $i + 1$ is not a reset date, $M_{i,j}(\hat{K}, b)$ equals

$$\frac{p_u M_{i+1,j}(\hat{K}, b + (i + 1) - 2j)}{R} + \frac{p_d M_{i+1,j+1}(\hat{K}, b + (i + 1) - 2(j + 1))}{R}.$$

On the other hand, if time $i + 1$ is a reset date, the strike price can be reset to the prevailing geometric price average if it is lower than \hat{K} . Therefore, the prevailing strike price at node $(i + 1, j)$ is

$$K' = \min(\hat{K}, S(0) u^{\frac{b+(i+1)-2j}{h+1}}).$$

Similarly, the prevailing strike price at node $(i + 1, j + 1)$ is

$$K'' = \min(\hat{K}, S(0) u^{\frac{b+(i+1)-2(j+1)}{h+1}}).$$

Thus $M_{i,j}(\hat{K}, b)$ equals

$$\frac{p_u M_{i+1,j}(K', b + (i + 1) - 2j)}{R} + \frac{p_d M_{i+1,j+1}(K'', b + (i + 1) - 2(j + 1))}{R}.$$

The backward induction can be simplified because many nodes on the lattice need much fewer states. For example, nodes that do not fall in any monitoring windows do not need to keep track of geometric price sums. Assume node (i, j) does not fall in any monitoring windows. Use $M_{i,j}(\hat{K}, 0)$ to denote the option value whose prevailing strike price is \hat{K} . If time $i + 1$ is not in a monitoring window, then the backward induction for node (i, j) is the simplified

$$M_{i,j}(\hat{K}, 0) = \frac{p_u M_{i+1,j}(\hat{K}, 0) + p_d M_{i+1,j+1}(\hat{K}, 0)}{R}.$$

If, on the other hand, time $i + 1$ is in a monitoring window I , the underlying asset's price for a node at time $i + 1$ will influence the geometric price average over I . The backward induction for node (i, j) sets $M_{i,j}(\hat{K}, 0)$ to be

$$\frac{p_u M_{i+1,j}(\hat{K}, (i + 1) - 2j) + p_d M_{i+1,j+1}(\hat{K}, (i + 1) - 2(j + 1))}{R}.$$

The early-exercise property should be considered when pricing American-style options. Assume $V_{i,j}(\hat{K})$ denotes the payoff of exercising the option at node (i, j) when the prevailing strike price is \hat{K} . Then $M_{i,j}(\hat{K}, b)$ in the algorithm keeps the maximum of the early exercise value (i.e., $V_{i,j}(\hat{K})$) and the continuation value as computed by the backward-induction formula.

Because the maximum number of states for each node is bounded by $O(n^2 h^2)$ and there are $O(n^2)$ nodes in the lattice, the running time is $O(n^4 h^2)$.

5 The Combinatorial Approach

In the former algorithm, every node contains $O(n^2 h^2)$ states. This results in an $O(n^4 h^2)$ algorithm as there are $O(n^2)$ nodes. A faster $O(n^3 h m)$ -time algorithm using combinatorics will be described below, where m denotes the number of monitoring windows. This approach, based on the CRR lattice, prices only European-style reset options. Its speed comes from that at most $O(nh)$ states are required at each node.

The goal is to calculate the probability for each state in a forward fashion. The option value is then computed by taking the discounted expected value of the terminal payoff. Start with a probability of 1 for the root node. We propagate that probability forward in time. A few terms are defined for convenience. Let z be the unique integer such that

$$S(0) u^{\frac{z-1}{h+1}-n} < K \leq S(0) u^{\frac{z}{h+1}-n}. \quad (6)$$

For ease of presentation, we replace K with $S(0) u^{\frac{z}{h+1}-n}$. $P_{i,j}(x)$ denotes the probability to reach node (i, j) with prevailing strike price $S(0) u^{\frac{x}{h+1}-n}$ from the root node, where $x \leq z$.

Outside of monitoring windows, the strike price will not be reset. Inside of monitoring windows, however, the resetting of the strike price will affect how probabilities are propagated forward. The details are as follows.

5.1 Outside of a Monitoring Window

Before starting the probability propagation, set $P_{0,0}(z) = 1$ and $P_{0,0}(x) = 0$ for all $x < z$. Let p_{abcd} denote the probability to reach node (c, d) from node (a, b) . Then

$$p_{abcd} = \begin{cases} \binom{c-b}{d-b} p_u^{c-a-d+b} p_d^{d-b}, & \text{if } c-a-d+b \geq 0 \text{ and } d-b \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Now let node (c, d) be at the end of the nonmonitoring window which begins at time a . Then

$$P_{c,d}(x) = \sum_{b=0}^a p_{abcd} P_{a,b}(x),$$

for all $x \leq z$.

5.2 Inside of a Monitoring Window

Since the strike price will be reset at a reset date if the geometric price average over the monitoring window is lower than the prevailing strike price, an efficient method is needed to calculate the probability distribution of the geometric price averages in a monitoring window.

Probability Distribution for Geometric Price Averages

Assume that node $(i-1, j)$ has price $\hat{S} = S(0)u^{(i-1)-2j}$ and time i is the beginning of monitoring window I . The reset date for I is at time $i+h$. Define $N_{\alpha, \beta}$ as the number of price paths that start from node $(i-1, j)$ and go through node $(i+h, j+\beta)$ with geometric price sum $\hat{S}^{h+1}u^\alpha$. The probability that this happens is clearly $N_{\alpha, \beta}P_u^{h+1-\beta}P_d^\beta$.

One key observation of the algorithm is that the integer $N_{\alpha, \beta}$ is the coefficient of $x^\alpha y^\beta$ in the following generating function:

$$(x^{-h-1}y + x^{h+1})(x^{-h}y + x^h) \cdots (x^{-2}y + x^2)(x^{-1}y + x). \quad (7)$$

Here is the intuitive justification. Assume that the next $h+1$ prices from node $(i-1, j)$ are $\hat{S}a_1, \hat{S}a_1a_2, \dots, \hat{S}a_1 \cdots a_{h+1}$, where each a_i can be either u or u^{-1} . Its geometric price sum equals

$$(\hat{S}a_1)(\hat{S}a_1a_2) \cdots (\hat{S}a_1a_2 \cdots a_{h+1}) = \hat{S}^{h+1}a_1^{h+1}a_2^h \cdots a_n^2 a_{h+1}.$$

Observe that a_i contributes either u^{h+2-i} or u^{-h-2+i} to the geometric price sum. Express this fact by $x^{-h-2+i}y + x^{h+2-i}$, where x^{h+2-i} denotes the geometric price sum contribution when $a_i = u^1$ and $x^{-h-2+i}y$ denotes the geometric price sum contribution (through x) and the downward movement contribution (through y) when $a_i = -u$. The exponents of x and y in (7) thus keep track of geometric price sum contributions and the number of down movements in monitoring window I , respectively. Similar generating functions can be found in [12] for efficient pricing of geometric average Asian options.

Because (7) can be computed in $O(h^4)$ time, the probability distribution for geometric price sums can be obtained in $O(h^4)$ time as well. This calculation needs only be done once in the algorithm.

Probabilities at a Reset Date

Let a_l , $0 \leq l \leq z$, denote the probability of reaching the state at node $(i-1, j)$ whose prevailing strike price is $S(0)u^{\frac{l}{h+1}-n}$. The number b_l denotes the probability to start from node $(i-1, j)$ and end at reset-date node $(i+h, j+\beta)$ with geometric price average $S(0)u^{\frac{l}{h+1}-n}$. Thus

$$b_l = N_{l+(2j-i-n+1)(h+1), \beta} P_u^{h+1-\beta} P_d^\beta.$$

The number $a_k b_l$ is the probability of reaching node $(i-1, h)$ with a prevailing strike price $S(0)u^{\frac{k}{h+1}-n}$ and going on to reach node $(i+h, j+\beta)$ with geometric price average $S(0)u^{\frac{l}{h+1}-n}$. Our goal is to evaluate the probabilities for all the states $0 \leq l \leq z$ at node $(i+h, j+\beta)$ efficiently.

Recall that the prevailing strike price will be reset to the new geometric price average if the geometric price average is lower than the prevailing strike price. Thus probability $a_k b_l$ contributes to state $\min(k, l)$ of node $(i+h, j+\beta)$. The probability of reaching state l of node $(i+h, j+\beta)$ hence equals

$$\left(\sum_{k=l}^z a_k \right) b_l + a_l \left(\sum_{k=l+1}^{2n(h+1)} b_k \right).$$

Because it costs at most $O(nh)$ time to evaluate all $\sum_{k=l}^z a_k$ and $\sum_{k=l+1}^{2n(h+1)} b_k$ for $l = 0, 1, \dots, z$, the probabilities for the $z+1$ states at node $(i+h, j+\beta)$ can be evaluated in $O(nh)$ time. Since there are $O(n)$ nodes at times $i-1$ and $i+h$ each, it takes $O(nnh) = O(n^3h)$ time to propagate the probabilities through a monitoring window. Given m monitoring windows in the lattice, the algorithm takes $O(n^3hm)$ time. On the other hand, it takes $O(n^3hm)$ time to propagate the probabilities outside monitoring windows. After accounting for the $O(h^4)$ time in calculating the coefficients of (7), we conclude that the total running time is $O(n^3hm + h^4 + n^3hm) = O(n^3hm)$ because $h = O(n)$.

5.3 Numerical Results

The lattice approach, the analytic approach, and the Monte Carlo simulations give consistent results as shown in Figure 2. The data in Figure 4 show how the number of monitoring windows influences the option value. The settings are the same as in Figure 2 except that it is a put option and the length of a monitoring window is 0.1 year. We use a 50-period lattice in this experiment. Pricing results for both European-style and American-style options are illustrated in the same table. Since the strike price is more likely to be reset as the number of monitoring windows increases, the option value increases with the number of reset dates. Monte Carlo simulation results are listed in the same table to show the accuracy of our algorithms.

Figure 5 compares the running times of the lattice approach and the combinatorial approach. The results show that the combinatorial approach is significantly more efficient. This conclusion is consistent with the theoretical analysis.

6 Conclusions

The geometric average trigger reset option resets the strike price based on the geometric average of the underlying asset's prices over monitoring windows. Similar contracts have been traded on exchanges in Asia.

Reset Dates	L (A)	L(E)	MC
1	8.322	8.302	8.302
1, 0.8	10.854	10.451	10.368
1, 0.8, 0.6	12.452	11.982	11.905
1, 0.8, 0.6, 0.4	13.732	13.188	13.104
1, 0.8, 0.6, 0.4, 0.2	14.735	14.117	14.032

Figure 4: **Option Values with Respect to Different Reset Dates.** The option value increases as the number of monitoring windows increases. The Monte Carlo simulation, denoted as “MC”, is based on 1,000,000 sample paths and 1,000 time periods. “L” denotes the lattice approach in our paper. The term (E)/(A) denotes that the option is an European-style/American-style one.

	$n = 200$	$n = 400$
Lattice	1.24 s	29.12 s
Combinatorial	0.21 s	0.58 s

Figure 5: **Running-Time Comparison.** The running times are measured in seconds. The parameters are identical to those in Figure 2 with n denoting the number of periods. The experiments are performed on a PC with an Intel Pentium III 866 MHz processor and 1 GB DRAM.

This paper derives an analytic formula for such options. It is proved that the reset call will not be exercised early if the underlying asset does not pay dividends; hence the same formula applies to American-style reset calls in this case. The formula is in fact a corollary of a much more general formula that is of independent interest as it is applicable to a large class of path-dependent options. An $O(n^4h^2)$ -time algorithm for general American-style reset options is presented. A much more efficient $O(n^3hm)$ -time algorithm exists for pricing European-style options. The correctness of these three approaches are verified by numerical experiments. Finally, numerical evidence suggests that our pricing formula and algorithms give very tight upper (lower) bounds on arithmetic average trigger reset calls (puts, respectively).

Acknowledgements

We thank Chien-Chou Lee for discussions.

References

[1] Mark Broadie, P. Glasserman, and S. Kou. A Continuity Correction for Discrete Barrier Op-

tions. *Mathematical Finance*, 7:325–349, 1997.

- [2] Chuang-Chang Chang, S.-L. Chung, and M. Shackleton. Pricing Options with American Style Average Reset Features. unpublished, National Central University, 2002.
- [3] Wai-Yan Cheng, and S. Zhang. The Analytics of Reset Options. *Journal of Derivatives*, 8(1):59–71, Fall 2000.
- [4] T.-S. Dai, Y.-Y. Fang. Pricing Options With Average Reset Feature. *Proceedings of the 11th Conference on the Theories and Practices of Securities and Financial Markets*, Kaohsiung, Taiwan, Dec. 2002.
- [5] Yu-Yuan Fang, *A General Analytic Formula for Path-Dependent Options: Theory and Applications*. Master’s Thesis, Department of Finance, National Taiwan University, 2002.
- [6] P.A. Forsyth, K. R. Vetzal, and R. Zvan. The Use of Interpolation in Pricing Path-Dependent Options: A Detailed Examination of Convergence for the Case of Asian Options. unpublished, University of Waterloo, 2001.
- [7] Stephen F. Gray, and R. Whaley. Valuing S&P 500 Bear Market Warrants with a Periodic Reset. *Journal of Derivatives*, 5:99–106, 1997.
- [8] Stephen F. Gray, and R. Whaley. Reset Put Options: Valuation, Risk Characteristics, and an Application. *Australian Journal of Management*, 24:1–20, 1999.
- [9] R. C. Heynen, and H. M. Kat. Lookback Options with Discrete and Partial Monitoring of the Underlying Price. *Applied Mathematical Finance*, 2:273–283, 1995.
- [10] Paul Hsueh, and Y. Angela Liu. Step-Reset Options: Design and Valuation. *Journal of Futures Markets*, 22(2):155–171, 2002.
- [11] John Hull, and A. White. Efficient Procedures for Valuing European and American Path-Dependent Options. *Journal of Derivatives*, 1:21–31, 1993.
- [12] Yuh-Dauh Lyuu, *Financial Engineering and Computation: Principles, Mathematics, Algorithms*. Cambridge, U.K.: Cambridge University Press, 2002.