A Numerically Efficient Valuation Method for American Currency Options with Stochastic Interest Rates

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ABSTRACT

This paper extends Ho, Stapleton, and Subrahmanyam's (1997a) generalized Geske-Johnson (1984) technique to price American currency options in a stochastic interest rate economy. We derive closed form solutions for European currency options and analytical form solutions for twice-exercisable currency options assuming that the volatility functions determining the term structure are deterministic. The two-point Geske and Johnson (1984) approximation formula is then applied to estimate American option prices. Numerical evaluations and comparative statics are presented to show the effect of stochastic interest rates. Although the model in this article is a special case of Amin and Bodurtha's (1995) general model, our numerical results are similar to theirs yet our method is numerically efficient.

JEL Classification: G13

Keywords: American option pricing, stochastic interest rates, domestic risk neutral valuation relationship, forward-risk-adjusted measure

I. Introduction

Option valuation theory in a stochastic interest rate economy has become an important issue for two reasons. First, as the time to maturity of many financial derivatives has lengthened significantly, the effects of stochastic interest rates can not be ignored. Moreover, some studies have shown that stochastic interest rate models can explain part of the undervaluation of options as compared to constant interest rate models.[1] Second, the payoff of many contingent claims, such as bond options and differential swaps, explicitly depends on bond prices or interest rates. Interest rate risk is therefore an important valuation factor.

Following Merton's (1973) approach, Grabbe (1983) explicitly combined stochastic interest rates with stochastic exchange rates to value European currency options.[2] Unfortunately, as pointed out by Amin and Jarrow (1991), this pricing approach does not integrate a full-fledged term structure model into the valuation framework, and thus can not be extended to price American type options. Amin and Jarrow (1991) provided an alternative class of option pricing models using the

[1] For example, Amin and Bodurtha (1995) show that large empirical biases in currency warrant premia can be significantly reduced by incorporating stochastic interest rates.

[2] Adams and Wyatt (1987) tested a modified version of Grabbe's model and concluded that the incorporation of the stochastic interest rate effect lead to nearly unbiased estimates of option prices.

Heath, Jarrow, and Morton (1992, hereafter HJM) term structure model to avoid the shortcomings of Merton's formulation. They obtained closed form solutions for European options on currencies and currency futures assuming that the volatility functions determining the term structure are deterministic.

Recently, Amin and Bodurtha (1995) developed a discrete time model, which converges to the continuous time models in Amin and Jarrow (1991, 1992), for valuing American currency options and warrants. Their model combined a discrete version of the HJM term structure model with the binomial model of Cox, Ross, and Rubinstein (1979). In their paper, they derived a path dependent model with arbitrary volatility functions and a path independent model with specific interest rate volatility functions. Although Amin and Bodurtha's (1995) method is computationally feasible, it may encounter the problem of computational inefficiency.[3] The computational inefficiency limits the potential use of their model in practice.

Ho, Stapleton, and Subrahmanyam (1997a, hereafter HSS) generalized the Geske and Johnson (1984, hereafter GJ) approach to price American options in a stochastic interest rate economy. In the GJ approach, an American option value is approximated, using Richardson extrapolation, by using a series of n-times exercisable option[4] values. The HSS method is attractive for two reasons. First, it is a computationally efficient method and gives quite accurate approximation for the American option price. Second, it has the potential to be extended to the multi-countries case since one only needs to model risk factors at few exercise dates.

This paper provides a numerically efficient valuation method for American currency options with stochastic interest rates using the GJ approach. In a similar manner to HSS, we first derive the risk neutral distributions for the spot exchange rate, domestic and foreign zero-coupon bond prices. After the risk neutral distributions for the asset prices are derived, one can build up the binomial tree, for instance HSS's (1995) multivariate binomial tree, to approximate the distributions for asset prices at two exercise dates and then price the option using this tree.[5] Alternatively, we derive two-dimensional analytic solutions for the European and twice-exercisable currency options. Comparing with HSS's (1995) and Chang's (1995) four-dimensional binomial method, our method is far more efficient and accurate. Moreover, a fast analytical solution has tremendous value not because it can compute numbers fast, but because it can compute hedges with high accuracy (numerical derivatives have been long known to be problematic, see for example Pelsser and Vorst (1994)).

Our solution provides a rather simple method to solve a problem of considerable complexity. For instance, the method presented in this paper may be applied to the valuation of American quanto options under stochastic interest rates,

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[3] For example, it takes about 15 minutes on a Pentium DOS personal computer to implement Amin and Bodurtha's path dependent model with time steps equal to twelve.

[4] An *n*-times exercisable option allows the holder to exercise the option at one of the *n* exercise dates. This option is also called an option with *n* exercise dates. In this paper, an *n*-times exercisable option and an option with *n* exercise dates are used interchangeably.

[5] This approach has been adopted by Chang (1995), HSS (1997a, 1997b) in pricing currency options, stock options, and bond options, respectively.

given that the distribution assumptions6 are satisfied. It is also consistent with approaches using a multifactor model of the term structure of interest rates because it involves the evaluations of options with only a small number of exercise dates. Furthermore, the model is arbitrage free while avoiding the complex problems involved in modelling the full evolution of the term structure.

The rest of this paper is organised as follows. Section 2 shows the valuation model of this paper. A general European option pricing formula is shown in Section 3. Section 4 prices European and twice-exercisable currency options and thus estimates the American currency option price. The simulation results are reported in Section 5. Section 6 concludes.

II. The Valuation Model

As in HSS (1997a), we use the European option price, P_1, and the twice-exercisable option price, P_2, to estimate the American option price, $\hat{P}_A$, that is,

$$\hat{P}_A = P_2 + \left(P_2 - P_1\right) \tag{1}$$

The above simple formula implies that P_1 and P_2 alone provide a large percentage of the information required for estimating P_{A}. Therefore, we will focus on pricing P_1 and P_2 hereafter.

The option valuation problem is considered from the domestic investor's viewpoint, and hence we need to establish the domestic risk neutral valuation relationship (RNVR) for valuing P_1 and P_2. As there will be considerable notation, for easy reference, we list it all in one place.

2.1 Notation and Assumptions

The following notation is employed:

$X(t)$ ≡ the spot exchange rate at time t (dollars per unit of foreign currency);

K_x ≡ the strike price of the currency option;

$F(t, T)$ ≡ the forward exchange rate at time t for settlement at T;

$B_d(t, T)$ ≡ the price in domestic currency at time t of a domestic zero-coupon bond with maturity T and unit face value;

$B_f(t, T)$ ≡ the price in foreign currency at time t of a foreign zero-coupon bond with maturity T and unit face value;

$r_d(t)$ ≡ the instantaneous domestic short rate of interest at time t;

$r_f(t)$ ≡ the instantaneous foreign short rate of interest at time t;

σ_Y ≡ the standard deviation of random variable ln$Y$;

$\sigma_{Y,Z}$ ≡ the covariance between random variables ln$Y$ and ln$Z$.

As in the HSS (1997a, 1997b), assume that the pricing kernel, the spot exchange rate, and the domestic and foreign zero-coupon bond prices, are joint lognormally distributed.[7] The following assumptions and relations are also used:[8]

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[6] The important restriction in this paper is that the asset prices must follow a multivariate lognormal distribution.

[7] As pointed out by HSS (1997a), under this assumption, there exist risk neutral measures for pricing options, that is the RNVR exists.

$$\frac{dX}{X} = \mu_x(.)\,dt + \sigma_x dW_x \text{; (spot exchange rate process)}$$

$$dr_d = \mu_d(.)\,dt + \sigma_d dW_d \text{; (domestic short rate process)}$$

$$dr_f = \mu_f(.)\,dt + \sigma_f dW_f \text{, (foreign short rate process)}$$

where $\mu_i(.), \sigma_i, \forall i = x, d, f$, are the instantaneous drift and volatility of each process, respectively. Notice that the exchange rate is lognormally distributed, and that the domestic and foreign short rate are normally distributed under the specified stochastic processes.[9] The specifications of the domestic and foreign short rate processes are consistent with Gaussian term structure models of interest rates in the literature such as Vasicek (1977), Ho and Lee (1986), and Hull and White (1990). Therefore the derivation of this paper can be consistent with the framework of no-arbitrage term structure models. The vectors of increments, $dW = (dW_x, dW_d, dW_f)$, are multivariate Wiener processes with the covariance matrix

$$Cov(dW, dW') = Corr(dW, dW')\,dt = \begin{vmatrix} 1 & \rho_{dx} & \rho_{fx} \\ \rho_{dx} & 1 & \rho_{df} \\ \rho_{fx} & \rho_{df} & 1 \end{vmatrix} dt,$$

where ρ_{ij} is the instantaneous correlation coefficient between *i* and *j*. These correlation coefficients are constant in this paper but can be generalized to be a time-varying function without any difficulty.

2.2 Domestic Risk Neutral Valuation Relationship

RNVR is one of the most important concepts in option pricing theory, but the concept of a RNVR has to be broadened within a stochastic interest rate economy. The main reason is because the effect of discounting an asset on a period by period basis is different from that of discounting an asset on a whole period basis within the stochastic interest rate economy. This actually reflects the difference between the futures price and the forward price in a stochastic interest rate economy, see for example, Cox, Ingersoll, and Ross (1981).

Depending on the payoff functions of the options, there are different kinds of risk neutral measure for pricing options. To price options on foreign assets, it is very important to clarify which country the risk neutral distribution (measure) is based on. In a similar manner to HSS (1997b), we first define the domestic RNVR for the European option. For simplicity, except stated otherwise, the asset prices and payoffs are all in domestic currency under the domestic RNVR.

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[8] For simplicity of expression, we restrict interest rates in each country to the case of a one-factor model. However, the proposed method is consistent with approaches using a multifactor model of the term structure of interest rates because it involves the evaluations of options with only two exercise dates. For example, as shown in Proposition 1, only $B_d(t, T)$ and $B_f(t, T)$ are relevant for pricing a currency option with two exercise dates, *t* and *T*.

[9] The volatilities of these processes can be extended to be time-varying function (corresponding to Vasicek (1977) or Hull and White (1990) models), see, for example, HSS (1995) and Jamshidian (1993).

Definition 1. A domestic RNVR exists for a *T*-maturity European option if the option can be valued by taking the expected value of its payoff using a domestic *T*-maturity forward-risk-adjusted (FRA) measure, and discounting the expected value using the *T*-maturity domestic zero-coupon bond price. Under the domestic FRA measure, the distributions for the asset (even if it is a foreign asset) prices are identical to the true distributions except for a mean shift such that the asset prices relative to the *T*-maturity domestic zero-coupon bond price follow a martingale.

Definition 1 is close to the RNVR of Rubinstein (1976) and Brennan (1979) in the sense that both the true distributions and risk neutral distributions are lognormal. The definition of the domestic *T*-maturity FRA measure can be regarded as a generalization of Jamshidian (1991, 1993). In a similar manner, the foreign RNVR can also be defined. However, all the asset prices and payoffs should be expressed in terms of foreign currency in the foreign risk neutral measure.

We now give an example to show how one can apply Definition 1 to price a European option. The payoff of a European foreign equity/domestic strike call option is $\max\left(X(T)S_f(T)-K,0\right)$, where $S_f(T)$ is the foreign equity price in foreign currency at time *T*, and thus the option is valued as follows,

$$E_0^d\left[\max\left(X(T)S_f(T)-K,0\right)\right]B_d(0,T) \tag{2}$$

where $E_0^d(.)$ is the expectation under the domestic *T*-maturity FRA measure. Since $X(T)S_f(T)$ is the foreign equity price in domestic currency, from Definition 1 we obtain

$$E_0^d\left[X(T)S_f(T)\right]=E_0^d\left[\frac{X(T)S_f(T)}{B_d(T,T)}\right]=\frac{X(0)S_f(0)}{B_d(0,T)} \tag{3}$$

Substituting $E_0^d\left[X(T)S_f(T)\right]$ into the expectation of the payoff, one would then derive the pricing formula which is close to the Black-Scholes formula for the European foreign equity/domestic strike call option.

Similarly, we can define the domestic RNVR for the twice-exercisable option as follows.

Definition 2. A domestic RNVR exists for an option with two exercise dates, *t* and *T*, if it can be valued by taking the expected value of its payoff using a domestic two-period FRA measure, and discounting the expected value using the relevant domestic zero-coupon bond price. Under the domestic two-period FRA measure, the distributions for the asset prices are identical to the true distributions except for a mean shift such that the asset prices relative to the relevant domestic zero-coupon bond price in each period follows a martingale.[10]

Definition 2 can be applied to value the twice-exercisable option as follows. Consider a twice-exercisable option with payoff functions *g*(*Y*(*t*)) and *g*(*Y*(*T*)), if it is exercised at *t* and *T* respectively, then the option can be valued as

[10] Martingale theory implies that all asset prices, normalized by the same (random) variable (for instance a money market account), follow martingales if the economy is arbitrage free. Then, due to Girsanov's results, the normalizing variable can be replaced by another, such as a bond price of specific maturity, if also all drift terms are changed appropriately, giving a new probability measure. The two-period FRA measure uses two normalizing asset prices in the same framework, making appropriate changes to conditional expectations. The author thanks an anonymous referee for pointing out this explanation.

$$\tilde{E}_0^d\left[\max\left\{g\left(Y\left(t\right)\right),\tilde{E}_t^d\left[g\left(Y\left(T\right)\right)\right]B_d\left(t,T\right)\right\}\right]B_d\left(0,t\right) \tag{4}$$

where $\tilde{E}_\tau^d(.)$ is the expectation at time τ under the domestic two-period FRA measure. To get the option value, one needs to evaluate $\tilde{E}_0^d\left[B_d\left(t,T\right)\right]$, $\tilde{E}_0^d\left[Y\left(t\right)\right]$, and $\tilde{E}_0^d\left[Y\left(T\right)\right]$. These expectations can be obtained indirectly from the following martingale expectation:[11]

$$\tilde{E}_0^d\left[B_d\left(t,T\right)\right]=\frac{B_d\left(0,T\right)}{B_d\left(0,t\right)}, \tag{5}$$

$$\tilde{E}_0^d\left[Y\left(t\right)\right]=\frac{Y\left(0\right)}{B_d\left(0,t\right)}, \tag{6}$$

$$\tilde{E}_t^d\left[Y\left(T\right)\right]=\frac{Y\left(t\right)}{B_d\left(t,T\right)}. \tag{7}$$

The RNVR of HSS (1997a, 1997b) for the valuation of a twice-exercisable option is a special case of Definition 2. In a similar manner, one can define the domestic RNVR for a n-times exercisable option. In the limit case, one would obtained the domestic "risk-neutral" measure where the domestic money market account[12] is the denominator that transforms the asset price into a martingale.

If the exercise price at the first exercise date is ∞ (for call option) or 0 (for put option), then an option with two exercise dates will reduce to a European option. Therefore, a domestic two-period FRA measure can be applied to value the European currency and foreign market options as well. For example, a European

[11] The domestic T-maturity FRA measure can also be applied to value a twice-exercisable option as follows:

$$E_0^d\left[\max\left\{\frac{g\left(Y\left(t\right)\right)}{B_d\left(t,T\right)},E_t^d\left[g\left(Y\left(T\right)\right)\right]\right\}\right]B_d\left(0,T\right),$$

where the distributions for the asset prices under this measure are identical to the true distributions but with mean shifted to

$$E_0^d\left[B_d\left(t,T\right)\right]=\frac{B_d\left(0,T\right)\left[1-Cov\left(B_d\left(t,T\right),\frac{1}{B_d\left(t,T\right)}\right)\right]}{B_d\left(0,t\right)}$$

$$E_0^d\left[Y\left(t\right)\right]=\frac{Y\left(0\right)-B_d\left(0,T\right)Cov\left(Y\left(t\right),\frac{1}{B_d\left(t,T\right)}\right)}{B_d\left(0,t\right)}$$

$$E_t^d\left[Y\left(T\right)\right]=\frac{Y\left(t\right)}{B_d\left(t,T\right)}$$

Similarly, the domestic t-maturity FRA measure can also be applied to value a twice-exercisable option with appropriate changes to risk neutral expectations. The detailed proof of the risk neutral distributions can be obtained from the author on request.

[12] The domestic money market account (rolling over at the domestic short rate $r_d(t)$) is defined as $B(t)=\exp\left(\int_0^t r_d\left(s\right)ds\right)$.

option with payoff function in the domestic currency, $g(Y(T))$, can be valued using a domestic two-period FRA measure as the following,

$$E_0^{d}\left\{E_t^{d}\left[g\left(Y\left(T\right)\right)\right]B_d\left(t,T\right)\right\}B_d\left(0,t\right) \tag{8}$$

III. A General European Option Pricing Formula

The European option pricing model with stochastic interest rates is well developed, for example, see Merton (1973), Grabbe (1983), Hilliard, Madura, and Tucker (1991), and Amin and Jarrow (1992). Since for our valuation problem, the payoff usually has the same form as that of an option to exchange one asset for another, we show a slightly general European option pricing formula which is closer to that of Margrabe (1978). In Margrabe, both assets are traded in the same country, while we allow one asset to be domestic and the other foreign.

Lemma 1. Let $Y(T)$ and $Z(T)$ be random variables with bivariate lognormal distribution. If the domestic RNVR exists for a European option with payoff $\max\left(Y\left(T\right)-Z\left(T\right),0\right)$, then the option price at time 0 is given by

$$\left\{E_0^d\left[Y\left(T\right)\right]N\left(d_1\right)-E_0^d\left[Z\left(T\right)\right]N\left(d_2\right)\right\}B_d\left(0,T\right) \tag{9}$$

where $N(.)$ is the cumulative normal distribution function, $E_0^d\left(.\right)$ is the expectation under the domestic T-maturity FRA measure, and

$$\begin{aligned} d_1 &= \frac{\ln\dfrac{E_0^d\left[Y\left(T\right)\right]}{E_0^d\left[Z\left(T\right)\right]}+\dfrac{1}{2}\sigma^2}{\sigma}, \\ d_2 &= d_1-\sigma, \\ d_3 &= Var_0\left[\ln\frac{Y\left(T\right)}{Z\left(T\right)}\right]. \end{aligned}$$

Proof: See the Appendix.

Lemma 1 is a general result which nests with many papers in the literature. For example, if $B_d\left(0,T\right)=e^{-rT}$ (constant interest rates), $Y(T)$ is the stock price and $Z(T)=K$ (the constant strike price), then Lemma 1 gives Black-Scholes (1973) formula for a European call option. If $Y(T)$ and $Z(T)$ are two domestic asset prices, then Lemma 1 gives the value of an European option to exchange one asset for another (see Margrabe (1978)). Also note that $Y(T)$ and $Z(T)$ can be the multiplications of two or more random variables. For example, the payoff in domestic currency of a foreign equity/domestic strike call option is $\max\left(X\left(T\right)S_f\left(T\right)-K,0\right)$, and thus applying Lemma 1 will give the value of a European foreign equity/domestic strike call option.

IV. Pricing American Currency Options

4.1 Pricing European and Twice-exercisable Currency Options

A currency option is analogous to an option on a stock paying a known dividend yield in the non-stochastic interest rate economy, therefore one only needs to model the exchange rate. However, in the stochastic interest rate economy, there are three state variables relevant for the valuation of a currency option. They are the exchange rate, domestic and foreign interest rates. We first show the European currency option pricing formulae in a stochastic interest rate economy in Lemma 2. The approach adopted here is the risk neutral valuation technique developed by Cox and Ross (1976). Thus the proof is mainly to show the distribution for the spot exchange rate under the domestic *T*-maturity FRA measure.

Lemma 2. The price of a European call option on the spot exchange rate is

$$X(0)B_f(0,T)N(d_1)-KB_d(0,T)N(d_2) \tag{10}$$

and the price of a European put option on the spot exchange rate is

$$KB_d(0,T)N(-d_2)-X(0)B_f(0,T)N(-d_1) \tag{11}$$

where $d_1=\dfrac{\ln\dfrac{X(0)B_f(0,T)}{KB_d(0,T)}+\dfrac{1}{2}\sigma^2_{X(T)}}{\sigma_{X(T)}}$, $d_2=d_1-\sigma_{X(T)}$

Proof. See the Appendix.

Our pricing formulae are similar to the Black-Scholes formulae. In Lemma 2, we do not assume any processes for the domestic and foreign interest rates except that the volatility functions determining the term structure are deterministic. Thus it is valid for any Gaussian term structure models. Similar results to Lemma 2 are obtained independently by Hilliard, Madura, and Tucker (1991) using the Vasicek model, and Amin and Jarrow (1991) using the HJM model. Like Amin and Jarrow (1991), it is not necessary to assume that uncovered interest rate parity holds in this paper. We next show the implications of RNVR to the valuation of a twice-exercisable currency option.

Proposition 1. If a RNVR exists for the valuation of a currency option with two exercise dates, *t* and *T*, then this option can be valued by taking the expectation of its payoffs, $g(X(t))$ and $g(X(T))$ if exercised at *t* and *T* respectively, as follows,

$$E_0^{Q_d}\left[\max\left\{g(X(t)),E_t^{Q_d}\left[g(X(T))\right]B_d(t,T)\right\}\right]B_d(0,t) \tag{12}$$

where the distributions for asset prices under the domestic two-period FRA measure are identical to the true distributions but with mean shifted to

$$E_0^{Q_d}\left[B_d(t,T)\right]=\frac{B_d(0,T)}{B_d(0,t)}$$

$$E_0^{Q_d}\left[B_f(t,T)\right]=\frac{B_f(0,T)}{B_f(0,t)}\exp\left(-\sigma_{X(t),B_f(t,T)}\right)$$

$$E_0^{Q_d}\left[X(t)\right]=\frac{X(0)B_f(0,t)}{B_d(0,t)}$$

$$E_0^{Q_d}\left[X(T)\right]=\frac{X(0)B_f(0,T)}{B_d(0,T)}\exp\left(\sigma^2_{B_d(t,T)}-\sigma_{X(t),B_d(t,T)}-\sigma_{B_d(t,T),B_f(t,T)}\right)$$

$$E_t^{Q_d}\left[X(T)\right]=\frac{X(t)B_f(t,T)}{B_d(t,T)}$$

Proof. See the Appendix.

The expectation of the foreign zero-coupon bond price[13] is similar to that of the domestic zero-coupon bond price but is reduced by the covariance between the exchange rate and the foreign zero-coupon bond price. The finding is consistent with that of Amin and Bodurtha (1995). They find that the foreign rate drift is similar to the domestic rate drift but is decremented by the foreign-currency covariance.

Proposition 1 can be applied to show the following relationships between the domestic and foreign two-period FRA measures:[14]

$$E_0^{Q_f}\left[B_d(t,T)\right]=E_0^{Q_d}\left[B_d(t,T)\right]\exp\left(-\sigma_{X(t),B_d(t,T)}\right) \tag{13}$$

$$E_0^{Q_f}\left[B_f(t,T)\right]=E_0^{Q_d}\left[B_f(t,T)\right]\exp\left(\sigma_{X(t),B_f(t,T)}\right) \tag{14}$$

where $E_\tau^{Q_f}(.)$ is the expectation at τ under the foreign two-period FRA measure. In other words, the expectation of the foreign zero-coupon bond price under the domestic risk neutral measure is similar to that under the foreign risk neutral measure except adjusted by the covariance between the exchange rate and foreign bond price. Similar relationships have been given by Wei (1994) under the "risk neutral" equivalent martingale measure where the futures price follows a

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[13] Note that the foreign zero-coupon bond prices, such as $B_f(t,T)$, are expressed in the foreign currency. Therefore the expectation $E_0^{Q_d}\left[B_f(t,T)\right]$ actually means the expected value of the foreign bond price converted into the domestic currency using fixed exchange rate 1, that is $E_0^{Q_d}\left[1\times B_f(t,T)\right]$.

[14] It is straightforward to derive a symmetric result of Proposition 1 for the foreign two-period FRA measure as follows,

$$E_0^{Q_f}\left[B_d(t,T)\right]=\frac{B_d(0,T)}{B_d(0,t)}\exp\left(-\sigma_{X(t),B_d(t,T)}\right)$$

$$E_0^{Q_f}\left[B_f(t,T)\right]=\frac{B_f(0,T)}{B_f(0,t)}$$

These results together with Proposition 1 will provide the proof.

martingale.

It should be noted that the results of Lemma 2 and Proposition 1 are valid for any Gaussian term structure models. Later in the simulations and the sensitivity analysis, we will assume that the domestic and foreign short rate processes follow the Ho and Lee (1986) model to see the effects of the stochastic interest rate on the option prices and how the option prices vary with the interest rate parameters. Thus we will derive the variance-covariance terms under the Ho and Lee model in Corollary 1 for later use.

Corollary 1. Assume that the domestic and foreign interest rates follow Ho and Lee model, that is

$$dr_d = \mu_d(t)dt + \sigma_d dW_d$$
$$dr_f = \mu_f(t)dt + \sigma_f dW_f$$

For pricing a European currency option with maturity *T*, and a currency option with two exercise dates, *t* and *T*, the required variance-covariance terms[15] are

$$\sigma^2_{B_d(t,T)} = \sigma_d^2 (T-t)^2 t$$

$$\sigma^2_{B_f(t,T)} = \sigma_f^2 (T-t)^2 t$$

$$\sigma^2_{X(t)} = \sigma_x^2 t + \left(\rho_{dx}\sigma_d\sigma_x - \rho_{fx}\sigma_f\sigma_x\right)t^2 + \frac{1}{3}\left(\sigma_d^2 + \sigma_f^2 - 2\rho_{df}\sigma_d\sigma_f\right)t^3$$

$$\sigma^2_{X(T)} = \sigma_x^2 T + \left(\rho_{dx}\sigma_d\sigma_x - \rho_{fx}\sigma_f\sigma_x\right)T^2 + \frac{1}{3}\left(\sigma_d^2 + \sigma_f^2 - 2\rho_{df}\sigma_d\sigma_f\right)T^3$$

$$Var_t\left[\ln X(T)\right] = \sigma_x^2 (T-t) + \left(\rho_{dx}\sigma_d\sigma_x - \rho_{fx}\sigma_f\sigma_x\right)(T-t)^2 + \frac{1}{3}\left(\sigma_d^2 + \sigma_f^2 - 2\rho_{df}\sigma_d\sigma_f\right)(T-t)^3$$

$$\sigma_{B_d(t,T),B_f(t,T)} = \rho_{df}\sigma_d\sigma_f (T-t)^2 t$$

$$\sigma_{X(t),B_f(t,T)} = \frac{1}{2}\left[\sigma_f^2 - \rho_{df}\sigma_d\sigma_f\right](T-t)t^2 - \rho_{fx}\sigma_f\sigma_x (T-t)t$$

$$\sigma_{X(t),B_d(t,T)} = \frac{1}{2}\left[\rho_{df}\sigma_d\sigma_f - \sigma_d^2\right](T-t)t^2 - \rho_{dx}\sigma_d\sigma_x (T-t)t$$

Proof. See the Appendix.[16]

Corollary 1 shows that the total volatility of the exchange rate at *t* (or *T*) depends not only on the instantaneous volatility of the spot exchange rate, but also on the volatility of the domestic and foreign interest rates and the correlation terms.[17] Because the currency option value mainly depends on the total volatility of the exchange rate, it will be interesting to apply Corollary 1 in the sensitivity analysis to see how the interest rate parameters affect the option values.

In a similar way, one can also derive the distributions for the asset prices under

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[15] Some covariance terms, such as $\sigma_{X(T),B_f(t,T)}$, are not shown here because they are redundant for pricing options. The reason for this is clearer later when we implement these terms in Proposition 2.

[16] Actually Hilliard, Madura, and Tucker (1991) have derived the bond and exchange rate variances under Vasicek's model which can be directly used in our model. When the mean reversion tends to zero, one would obtain the same results as in our Corollary 1.

[17] The result is as expected since future exchange rates generally depend on the domestic and foreign interest rates.

the domestic n-period FRA measure which can be applied to price a n-times exercisable currency option. However, for practical purposes and computational feasibility, it is recommended to use at most the three-times exercisable option price to forecast the American option price. The required derivations for the n-times exercisable currency options may be tremendous because one needs to show the conditional and unconditional means and variance-covariance terms for $3n-2$ variables (exchange rates at n exercise dates, and domestic and foreign bond prices at $n-1$ exercise dates).

4.2 Critical Exercise Price and the Option Value

This section shows how one can apply the risk neutral distributions from Proposition 1 to value the European and twice-exercisable currency options. We derive a two-dimension analytic solution for pricing the European and twice-exercisable currency options.[18] The idea is to find out the minimum set of asset price(s) at the first exercise date such that the critical exercise price and the European and twice-exercisable option values at the first exercise date are explicit functions of this set of price(s). The twice-exercisable option value at time 0 is then the discounted value of the integral of the option value at t using the density function(s) for this set of price(s). For example, Proposition 2 shows that the critical exercise price and the European and twice-exercisable option prices are explicit functions of the domestic zero-coupon bond price and the forward (for delivery at T) price of the exchange rate.

Proposition 2. If a RNVR exists for currency options with two exercise dates, t and T, then the critical exercise prices at time t for twice-exercisable call and put options are explicit functions of $B_d(t, T)$ and $F(t, T)$ respectively as the following,

$$\bar{X}_c(t) = K_x - K_x B_d(t,T) N(h) + B_d(t,T) F(t,T) N(h+\sigma) \tag{15}$$

$$\bar{X}_p(t) = K_x - K_x B_d(t,T) N(-h) + B_d(t,T) F(t,T) N(-h-\sigma) \tag{16}$$

where $h = \dfrac{\ln \dfrac{F(t,T)}{K_x} - \dfrac{1}{2}\sigma^2}{\sigma}$, and $\sigma^2 = Var_t\left[\ln X(T)\right]$. The values of the European and twice-exercisable call and put options at time 0 are respectively given as the following,

[18] As in Chang (1995) and HSS (1997a, 1997b), the alternative approach is to apply HSS's (1995) multivariate binomial tree method to approximate the distributions of $X(t)$, $X(T)$, $B_d(t, T)$, and $B_f(t, T)$, and then use this tree to price the options. However, this method is a four-dimension solution, thus is not very efficient compared with the proposed two-dimension method. The numerical results indicate that it takes less than a second to price an option running on the UNIX system in Lancaster University.

$$C_1(0) = B_d(0,t)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_1\big(B_d(t,T), F(t,T)\big)\, f_1(.)\, d\ln B_d(t,T)\, d\ln F(t,T)$$

$$C_2(0) = B_d(0,t)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_2\big(B_d(t,T), F(t,T)\big)\, f_1(.)\, d\ln B_d(t,T)\, d\ln F(t,T)$$

$$P_1(0) = B_d(0,t)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_1\big(B_d(t,T), F(t,T)\big)\, f_1(.)\, d\ln B_d(t,T)\, d\ln F(t,T)$$

$$P_2(0) = B_d(0,t)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_2\big(B_d(t,T), F(t,T)\big)\, f_1(.)\, d\ln B_d(t,T)\, d\ln F(t,T)$$

where $f_1(.)$ is the probability density functions (p.d.f.) of $\ln B_d(t, T)$ and $\ln F(t, T)$ under the domestic two-period FRA measure and

$$C_1\big(B_d(t,T), F(t,T)\big) = B_d(t,T)F(t,T)N(h+\sigma) - K_x B_d(t,T)N(h)$$

$$\begin{aligned} C_2\big(B_d(t,T), F(t,T)\big) &= \exp\left(\mu_1 + \frac{1}{2}\sigma_1^2\right)N(d_{1c}) - K_x N(d_{2c}) \\ &\quad + \big[B_d(t,T)F(t,T)N(h+\sigma) - K_x B_d(t,T)N(h)\big]N(-d_{2c}) \end{aligned}$$

$$P_1\big(B_d(t,T), F(t,T)\big) = K_x B_d(t,T)N(-h) - B_d(t,T)F(t,T)N(-h-\sigma)$$

$$\begin{aligned} P_2\big(B_d(t,T), F(t,T)\big) &= K_x N(-d_{2p}) - \exp\left(\mu_1 + \frac{1}{2}\sigma_1^2\right)N(-d_{1p}) \\ &\quad + \big[K_x B_d(t,T)N(-h) - B_d(t,T)F(t,T)N(-h-\sigma)\big]N(d_{2p}) \end{aligned}$$

where

$$\mu_1 = E^{\mathcal{Q}_d}\big[\ln X(t) \big| B_d(t,T), F(t,T)\big]$$

$$\sigma_1^2 = Var\big[\ln X(t) \big| B_d(t,T), F(t,T)\big]$$

$$d_{2c} = \frac{\mu_1 - \ln \bar{X}_c(t)}{\sigma_1}, d_{1c} = d_{2c} + \sigma_1$$

$$d_{2p} = \frac{\mu_1 - \ln \bar{X}_p(t)}{\sigma_1}, d_{1p} = d_{2p} + \sigma_1$$

Proof. See the Appendix.

Proposition 2 is an analytic solution and thus is very easy to implement. Comparing with the HSS (1997a, 1997b), their solution is four-dimension while ours is two-dimension. Therefore, the proposed approach is far more efficient and accurate than the HSS method.

To implement Proposition 2, one first plugs in the risk neutral distributions from Proposition 1 (and maybe variance-covariance terms from Corollary 1) to get the p.d.f. of $\ln B_d(t, T)$ and $\ln F(t, T)$, the conditional mean, μ_1, and the conditional variance, σ_1^2. The double integration can then be calculated either using the numerical integration method or using the binomial method to approximate the distributions of $B_d(t, T)$ and $F(t,T)$.

V. Numerical Evaluation

We use the proposed approach to value a range of American currency options. The numerical evaluation includes pricing call and put options on premium, parity and discount currencies with varying interest rates, time to maturities, and depth-in-the- money. The benchmark parameters, adopted from Chang (1995), are as follows: $\sigma_x = 0.1$, $\sigma_d = \sigma_f = 0.02$, $\rho_{dx} = 0.1$, $\rho_{df} = \rho_{fx} = 0.05$, and the spot exchange rate, $X(0)$, is 150 cents. We also assume that the initial term structure of interest rates is flat in each country.

To implement the analytic solution in Proposition 2, we will adapt the numerical integration method. In the numerical integration method, the range is from the mean minus five standard deviations (lower limit) to the mean plus five standard deviations (upper limit), rather than from $-\infty$ to ∞.[19] The range is divided into equally spaced sections where the length of each section equals one tenth of the standard deviation. The numerical integration method used here is identical to the extended trapezoidal rule (see Press *et al*. (1994)) except that the weights on the lower and upper limits are one.

Tables 1 and 2 report the magnitude of stochastic interest rate effect on the American currency options' values. Columns (1) to (3) are from Chang's (1995) Table 6.1. Columns (4) and (5) are the true values of the European currency options in the static and stochastic interest rate models, respectively. Columns (6) and (7) ((8) and (9)) indicate the European and American currency option values with non-stochastic (stochastic) interest rates using the proposed method.[20] The results suggest that the effect of stochastic interest rates is small in general, especially for options on premium currencies and options on discount currencies. However, the effect significantly increases as the time to maturity of the options increases.

Tables 1 and 2 also confirm the well-known results that there is no early exercise premium for American put options on discount currencies and American call options on premium currencies for constant interest rate models. However, the early exercise premium of these two kinds of options are not zero under stochastic interest rate models. The reason is because, for example, a put option on discount currencies today may become a put option on premium currencies one month later due to the increase of the domestic interest rates. Therefore the early exercise premium of an American put option on a discount currency is not zero.[21]

A comparison of the estimates in columns (4), (5), (6), (8) shows that the pricing errors using the proposed method for valuing European currency options are very small.[22] The pricing errors are smaller than 0.1 cent and 0.01% of the true value

[19] The reason for this choice is because the probability density function outside the lower and upper limits is small and thus negligible.

[20] Note that the proposed method can also be applied to a non-stochastic interest rate model which is a limiting case of stochastic interest rate model as the variances and covariances of interest rates tend to zero. We let $\sigma_d = \sigma_f = 0.000002$, and $\rho_{dx} = \rho_{df} = \rho_{fx} = 0.0$ to represent a non-stochastic interest rate economy in this paper.

[21] Because the chance that options on premium currencies becomes options on discount currencies is small for short term options, the early exercise premium of these options are very small in general.

[22] These results also confirm our argument that one can apply a domestic two-period FRA

in all cases. We would expect that the pricing errors of the proposed method for valuing twice-exercisable option are also small. Therefore, the main pricing errors of valuing American option come from using two-point GJ approximation to estimate the American option price.[23]

Table 1
The Prices of American Put Options:
Stochastic and Non-Stochastic Interest Rates

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
exact solution			Chung					
r_d	r_f	T	P_E^{NSR}	P_E^{SR}	P_E^{NSR}	P_A^{NSR}	P_E^{SR}	P_A^{SR}
Panel A:	In-The-Money Options (K_x = 155)							
0.06	0.04	0.25	5.5484	5.5536	5.5484	5.7825	5.5536	5.7919
0.06	0.04	0.50	6.1051	6.1276	6.1051	6.5062	6.1276	6.5452
0.06	0.04	1.00	6.7588	6.8592	6.7588	7.5003	6.8592	7.6480
0.06	0.06	0.25	6.0811	6.086	6.0811	6.1063	6.086	6.1367
0.06	0.06	0.50	7.0407	7.0619	7.0407	7.0996	7.0619	7.1896
0.06	0.06	1.00	8.3865	8.4815	8.3865	8.5435	8.4815	8.8246
0.04	0.06	0.25	6.6685	6.6731	6.6685	6.6685	6.6731	6.6743
0.04	0.06	0.50	8.1223	8.1421	8.1223	8.1223	8.1421	8.1536
0.04	0.06	1.00	10.4019	10.4902	10.4019	10.4019	10.4902	10.5653
Panel B:	At-The-Money Options (K_x = 150)							
0.06	0.04	0.25	2.599	2.605	2.599	2.6545	2.605	2.6625
0.06	0.04	0.50	3.4358	3.4589	3.4358	3.5787	3.4589	3.6116
0.06	0.04	1.00	4.3767	4.4743	4.3766	4.7349	4.4743	4.8626
0.06	0.06	0.25	2.9472	2.9532	2.9472	2.9509	2.9532	2.9633
0.06	0.06	0.50	4.1055	4.1287	4.1055	4.1213	4.1287	4.1716
0.06	0.06	1.00	5.6333	5.7319	5.6333	5.6975	5.7319	5.8965
0.04	0.06	0.25	3.3397	3.3457	3.3397	3.3397	3.3457	3.3459
0.04	0.06	0.50	4.8988	4.9219	4.8988	4.8988	4.9219	4.9254
0.04	0.06	1.00	7.2304	7.328	7.2304	7.2304	7.328	7.3629
Panel C:	Out-The-Money Options (K_x = 145)							
0.06	0.04	0.25	0.9096	0.914	0.9096	0.9157	0.914	0.9205
0.06	0.04	0.50	1.6575	1.6764	1.6575	1.692	1.6764	1.7146
0.06	0.04	1.00	2.608	2.6928	2.608	2.7503	2.6928	2.8506
0.06	0.06	0.25	1.0769	1.0816	1.0769	1.0771	1.0816	1.0825
0.06	0.06	0.50	2.066	2.0864	2.066	2.0688	2.0864	2.0962
0.06	0.06	1.00	3.5	3.5916	3.5	3.5213	3.5916	3.6569
0.04	0.06	0.25	1.2726	1.2776	1.2726	1.2726	1.2776	1.2776
0.04	0.06	0.50	2.5675	2.5891	2.5675	2.5675	2.5891	.5899
0.04	0.06	1.00	4.6732	4.7702	4.6732	4.6732	4.7702	4.7836

This table shows the price difference of American put options between stochastic and non-stochastic interest rate models. Columns (1) to (3) are from Chang's (1995) Table 6.1. T is the time to maturity of the option. Columns (4) and (5) show the exact (closed form) solution of the European options without, and with stochastic interest rates, respectively. Columns (6) to (9) indicate values of European

measure to price the European currency options.

[23] We expect that the pricing errors should be small since HSS (1994) has shown that two-point GJ approximation is quite accurate for currency options.

option without stochastic interest rates, American option without stochastic interest rates, European option with stochastic interest rates, American option with stochastic interest rates, respectively, using our analytical solution implementation with the numerical integration.

Table 2
The Prices of American Call Options
Stochastic and Non-Stochastic Interest Rate

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
			exact solution		Chung			
r_d	r_f	T	P_E^{NSR}	P_E^{SR}	P_E^{NSR}	P_A^{NSR}	P_E^{SR}	P_A^{SR}
Panel A: In-The-Money Options (K_x = 155)								
0.06	0.04	0.25	1.3635	1.3688	1.3635	1.3635	1.3688	1.3688
0.06	0.04	0.50	2.7158	2.7383	2.7158	2.7158	2.7383	2.7392
0.06	0.04	1.00	4.9037	5.0041	4.9037	4.9037	5.0041	5.0190
0.06	0.06	0.25	1.1555	1.1605	1.1555	1.1558	1.1605	1.1615
0.06	0.06	0.50	2.1885	2.2097	2.1885	2.1916	2.2097	2.2207
0.06	0.06	1.00	3.6776	3.7727	3.6776	3.7007	3.7727	3.8440
0.04	0.06	0.25	0.9776	0.9822	0.9776	0.9844	0.9822	0.9895
0.04	0.06	0.50	1.7583	1.7781	1.7583	1.7961	1.7781	1.8201
0.04	0.06	1.00	2.7443	2.8325	2.7442	2.8955	2.8325	3.0030
Panel B: At-The-Money Options (K_x = 150)								
0.06	0.04	0.25	3.3397	3.3457	3.3397	3.3397	3.3457	3.3459
0.06	0.04	0.50	4.8988	4.9219	4.8988	4.8988	4.9219	4.9255
0.06	0.04	1.00	7.2304	7.328	7.2304	7.2304	7.328	7.3641
0.06	0.06	0.25	2.9472	2.9532	2.9472	2.9509	2.9532	2.9635
0.06	0.06	0.50	4.1055	4.1287	4.1055	4.1212	4.1287	4.1724
0.06	0.06	1.00	5.6333	5.7319	5.6333	5.6975	5.7319	5.8995
0.04	0.06	0.25	2.599	2.605	2.599	2.6542	2.605	2.6628
0.04	0.06	0.50	3.4358	3.4589	3.4358	3.5791	3.4589	3.6127
0.04	0.06	1.00	4.3767	4.4743	4.3766	4.7354	4.4743	4.8663
Panel C: Out-The-Money Options (K_x = 145)								
0.06	0.04	0.25	6.5759	6.5803	6.5759	6.5759	6.5802	6.5815
0.06	0.04	0.50	7.9727	7.9916	7.9727	7.9727	7.9916	8.0036
0.06	0.04	1.00	10.1705	10.2554	10.1705	10.1705	10.2554	10.3318
0.06	0.06	0.25	6.0024	6.0071	6.0024	6.0281	6.0071	6.0588
0.06	0.06	0.50	6.9183	6.9386	6.9182	6.9776	6.9386	7.0675
0.06	0.06	1.00	8.2088	8.3004	8.2088	8.3642	8.3004	8.6443
0.04	0.06	0.25	5.4822	5.4872	5.4821	5.7172	5.4872	5.7268
0.04	0.06	0.50	6.0055	6.0272	6.0055	6.4041	6.0272	6.4445
0.04	0.06	1.00	6.6234	6.7204	6.6234	7.3574	6.7204	7.5046

This table shows the price difference of American call options between stochastic and non-stochastic interest rate models. Columns (1) to (3) are from Chang's (1995) Table 6.1. T is the time to maturity of the option. Columns (4) and (5) show the exact (closed form) solution of the European options without, and with stochastic interest rates, respectively. Columns (6) to (9) indicate values of European option without stochastic interest rates, American option without stochastic interest rates, European option with stochastic interest rates, American option with stochastic interest rates, respectively, using our analytical solution implementation with the numerical integration.

5.1 Sensitivity Analysis of Model Parameters

Given our pricing model, we conduct comparative static, pricing analysis over the model parameters to see the stochastic interest rate effects on the American currency options. The effects of the stochastic interest rates on the prices of American currency options can be decomposed into two parts. First, stochastic interest rates influence the volatility of the spot exchange rate (see Corollary 1), and thus change the holding value of the option. This effect is reflected in the price difference of European option between stochastic and non-stochastic interest rate models. Second, the early exercise decision for such options is affected by the term structure of interest rates on future dates, since the live value of the claim on each future date depends on the discount rates on that date. In other words, the early exercise premium may differ under two models. Therefore, our analysis has two parts: European option values and early exercise values.[24] The benchmark parameters and the spot exchange rate are the same as those for Tables 1 and 2. The time to maturity of the option, *T*, is one year, and the strike price is 150 cents.

5.1.1 European option values

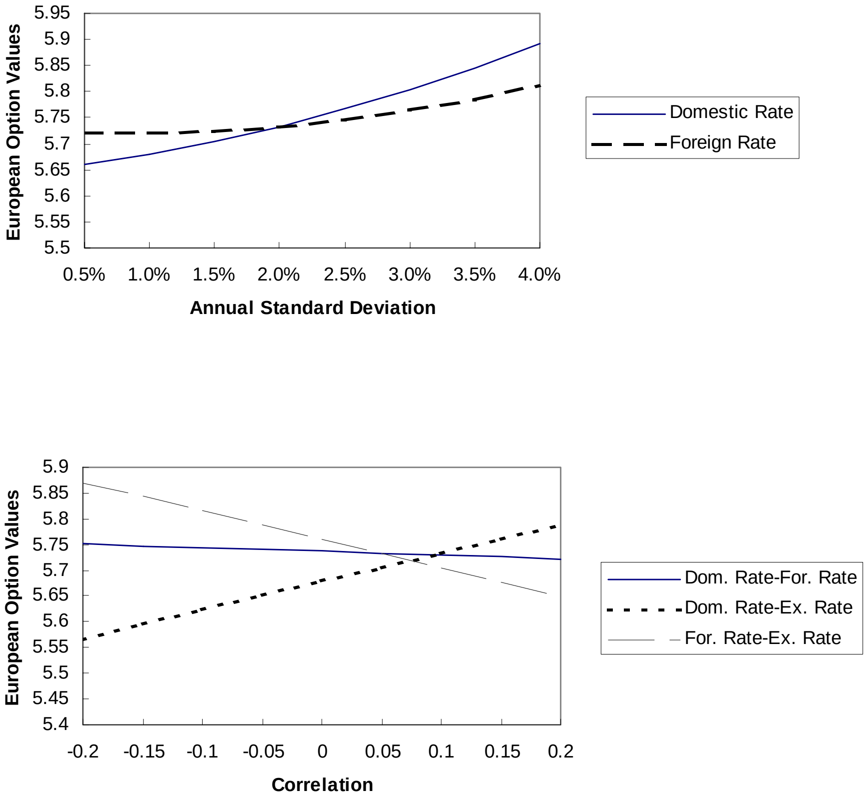
Figure 1 shows the comparative statics results for the counterpart European option value of the American put option on parity currency. The pricing effect of changing standard deviations of interest rate and that of changing correlations are presented in the first and second panels respectively. We vary σ_d and σ_f from 0.5 to 4 percent. Increasing the standard deviation of the interest rate will increase the European option value and the effect of the domestic interest rate is more significant. The explanation is straightforward if we consider the influence of parameter changes on the total volatility of the exchange rate. The total volatility of the exchange rate, $\sigma^2_{X(T)}$, equals $\sigma_x^2 T + \left(\rho_{dx}\sigma_d\sigma_x - \rho_{fx}\sigma_f\sigma_x\right) T^2 + \frac{1}{3}\left(\sigma_d^2 + \sigma_f^2 - 2\rho_{df}\sigma_{d\sigma_f}\right)T^3$ within stochastic interest rate economy (see Corollary 1), and equals $\sigma_x^2 T$ within non-stochastic interest rate economy. Within the chosen parameters, the effects of stochastic interest rate are positive and the effect of σ_d is more significant than that of σ_f.

Similar observations can be applied to the analysis when changing correlation from -0.2 to +0.2. Varying the interest rate and exchange rate correlations gives price changes of the same magnitude but opposite sign. Increasing the domestic rate and exchange rate correlation or decreasing the foreign rate and exchange rate correlation will increase the European option value. Raising the domestic rate and foreign rate correlation has a smaller negative effects. When the correlation between the domestic rate and exchange rate is very negative (for example, $\sigma_{dx} = -0.2$), the effect of stochastic interest rates is negative. These results are not surprising if we investigate the total volatility of exchange rates. It is obvious that the sign of ρ_{dx} is positive and the sign of ρ_{fx} and ρ_{df} is negative in the total volatility. For the

[24] HSS(1997a) only analyse the effect on the early exercise value because the volatility of the underlying asset remains unchanged in their stochastic interest rate economy.

chosen parameters σ_d and σ_f , varying the interest rate and exchange rate correlations gives the same magnitude (but different sign) of price change of the European option.

Figure 1. Values of European Put Options on Parity Currency



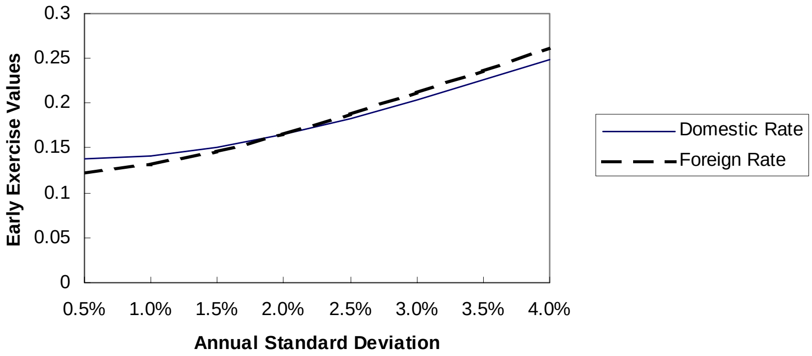
This figure shows the comparative statics analysis of European put options on parity currency. We vary one parameter, while holding all others at the following benchmark values: $X(0) = K_x = 150$, $r_d = r_f = 0.06$, $\sigma_x = 0.1$, $\sigma_d = \sigma_f = 0.02$, $\rho_{dx} = 0.1$, and $\rho_{df} = \rho_{fx} = 0.05$. The European option value under constant interest rates is 5.6333.

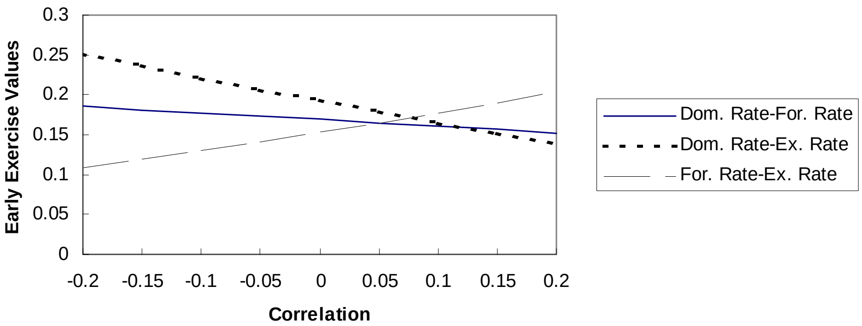
5.1.2 Early exercise values

Figure 2 plots the early exercise value of American options on the parity currency as the model parameters are varied. The first panel shows that the early exercise value may be still sensitive to changes in interest rate volatility, which is different from the findings of Amin and Bodurtha (1995). The second panel indicates that the effect of currency-related correlation changes on early exercise values are opposite to those of European option values. For example, increasing the domestic-currency correlation (ρ_{df}) will decrease the early exercise value while at the same time increasing the European option value. Taken together, these results indicate that the composite value of the American currency option is not significantly affected by currency-related correlation changes.

A change in the correlation between domestic and foreign interest rates (ρ_{df}) affects the early exercise values by a smaller amount than changes in the other correlations. Raising ρ_{df} decreases the early exercise value and also decrease the European option value. Our results concerning the effect of correlation changes are of the same sign as those of Amin of Bodurtha (1995). The intuition of the early exercise values can also be found in their paper.

Figure 2. Early Exercise Premium of American Put Options on Parity Currency





This figure shows the comparative statics analysis of the early exercise premium for American put options on parity currency. We vary one parameter, while holding all others at the following benchmark values: $X(0) = K_x = 150$, $r_d = r_f = 0.06$, $\sigma_x = 0.1$, $\sigma_d = \sigma_f = 0.02$, $\rho_{dx} = 0.1$, and $\rho_{df} = \rho_{fx} = 0.05$. The early exercise premium under constant interest rates is 0.0642.

VI. Conclusion

This paper extended HSS's (1997a) generalized GJ technique to price

American currency options for which domestic interest rate risk, foreign interest rate risk and currency risk are important. We derived the domestic risk neutral distributions for the spot exchange rate, domestic and foreign zero-coupon bond prices and showed analytic solutions for the European and twice-exercisable currency options. The analytic solutions are at most two-dimensional even though the valuation problem involves three state variables at two exercise dates. Thus our method is numerically efficient at pricing the options as well as calculating option hedge ratios and other risk management parameters.

The findings of simulations are summarised as follows: (1) We find that the early exercise premium for American put option on discount currency and American call option on premium currency are not zero under stochastic interest rates models. These results are different from those reported in the literature using constant interest rate models which found no early exercise premium for these two options. (2) The magnitude of stochastic interest rate effects depends on both the volatilities of interest rates and the correlations between interest rates and exchange rate. The numerical results are similar to those of Amin and Bodurtha (1995).

There are many potential applications of our method which we have not pursued in this article. For example, Proposition 1 can be extended to a domestic multi-period FRA measure and thus be applied to value cross-currency derivatives, such as differential swaps and options on differential swaps, where risk arises from the exchange rates and the domestic and foreign interest rates at multiple dates.[25] The other possible application is to generalise the proposed approach to value quanto options whose payoff depends on foreign equity risk, exchange rate risk and interest rate risk. We leave the above mentioned potential applications to further research.

[25] The domestic FRA measure can be applied to value each foreign interest rate payment in the differential swaps (see Wei's (1994) eq. (7) and (9) and Proposition for details). It can also be applied to value a general differential swap where both payments at each swap date are driven by two different foreign interest rates but denominated in domestic currency.

Appendix: Mathematical Proofs

We apply a lemma of Dravid, Richardson, and Sun (1993) to establish another lemma for later reference. We first quote their results as the following:

Lemma A1. Let x_1 and x_2 be standard normal random variables with correlation coefficient ρ. Then, for arbitrary constants *a*, *b*, *c*, *d*, and *k*,

$$E\left[e^{cx_1+dx_2} I\left(ax_1+bx_2 \geq k\right)\right] = e^{\frac{c^2+d^2+2\rho cd}{2}} N\left(\frac{ac+bd+\rho\left(ad+bc\right)-k}{\sqrt{a^2+b^2+2\rho ab}}\right)$$

where *I*(.) denotes the indicator function that takes the value 1 if the expression in parentheses is true and 0 otherwise.

Lemma A2. Let *y* and *z* be normal random variables with means μ_y, and μ_z, and variances σ_y^2, and σ_z^2, and covariance σ_{yz}. Then

(a). $$\int_{-\infty}^{\infty}\int_{h}^{\infty} e^{y+z} f\left(y,z\right) dydz = e^{\left(\mu_y+\mu_z+\frac{1}{2}\sigma_y^2+\frac{1}{2}\sigma_z^2+\sigma_{yz}\right)} N\left(\frac{\mu_y+\sigma_y^2+\sigma_{yz}-h}{\sigma_y}\right)$$

where *h* is a constant, and *f*(.) is the p.d.f. of *y* and *z*.

(b).

$$\int_{-\infty}^{\infty}\int_{z}^{\infty} e^{y} f\left(y,z\right) dydz = e^{\left(\mu_y+\frac{1}{2}\sigma_y^2\right)} N\left(\frac{\mu_y+\sigma_y^2-\mu_z-\sigma_{yz}}{\sqrt{\sigma_y^2+\sigma_z^2-2\sigma_{yz}}}\right)$$

$$\int_{-\infty}^{\infty}\int_{z}^{\infty} e^{z} f\left(y,z\right) dydz = e^{\left(\mu_z+\frac{1}{2}\sigma_z^2\right)} N\left(\frac{\mu_y+\sigma_{yz}-\mu_z-\sigma_z^2}{\sqrt{\sigma_y^2+\sigma_z^2-2\sigma_{yz}}}\right)$$

(c).

$$\int_{-\infty}^{\infty}\int_{h}^{\infty} e^{z} f\left(y,z\right) dydz = e^{\left(\mu_y+\frac{1}{2}\sigma_y^2\right)} N\left(-\frac{\mu_y+\sigma_y^2-h}{\sigma_y}\right)$$

$$\int_{-\infty}^{\infty}\int_{h}^{\infty} e^{z} f\left(y,z\right) dydz = e^{\left(\mu_z+\frac{1}{2}\sigma_z^2\right)} N\left(-\frac{\mu_y+\sigma_{yz}-h}{\sigma_y}\right)$$

Proof. Let $x_1 = \frac{y-\mu_y}{\sigma_y}$, and $x_2 = \frac{z-\mu_z}{\sigma_z}$, then it is not difficult to show that x_1 and x_2 are two standard normal random variables with correlation coefficient $\frac{\sigma_{yz}}{\sigma_y\sigma_z}$. Since $y=\sigma_y x_1+\mu_y$, and $z=\sigma_z x_2+\mu_z$, it is straightforward that (*a*) is a special case of lemma A1. That is

$$(a) = e^{\mu_y+\mu_z} E\left[e^{\sigma_y x_1 + \sigma_z x_2} I\left(x_1 \ge \frac{h-\mu_y}{\sigma_y} \right) \right]$$
$$= e^{\left(\mu_y+\mu_z+\frac{1}{2}\sigma_y^2+\frac{1}{2}\sigma_z^2+\sigma_{yz}\right)} N\left(\frac{\mu_y+\sigma_y^2+\sigma_{yz}-h}{\sigma_y} \right)$$

Similarly, (*b*) are two special cases of Lemma A1 with

$$a=\sigma_y, b=-\sigma_z, c=\sigma_y, d=0, k=\mu_z-\mu_y$$
$$a=\sigma_y, b=-\sigma_z, c=0, d=\sigma_z, k=\mu_z-\mu_y$$

respectively, and (*c*) are two special cases of Lemma A1 with

$$a=1, b=0, c=\sigma_y, d=0, k=\frac{h-\mu_y}{\sigma_y}$$
$$a=1, b=0, c=0, d=\sigma_z, k=\frac{h-\mu_y}{\sigma_y} \quad \square$$

Proof of Lemma 1. We now apply the RNVR to finish the proof. Since the payoff and the price are expressed in terms of the domestic currency (i.e. from the point of view of domestic investors), the relevant probability density function is the domestic risk neutral density function. Assume that the domestic risk neutral density function of $y=\ln Y_T$ and $z=\ln Z_T$ is $f(y, z)$. Under the RNVR, the European option price will be equal to the expected value, using the domestic T-maturity FRA measure, of the payoff at time T multiplied by the discount factor $B_d(0, T)$. The expected value of the payoff at time T can be shown as the following:

$$\int_{-\infty}^{\infty}\int_{z}^{\infty}\left(e^y-e^z\right) f(y,z)\,dydz$$
$$= e^{\left(\mu_y+\frac{1}{2}\sigma_y^2\right)} N\left(\frac{\mu_y+\sigma_y^2-\mu_z-\sigma_{yz}}{\sqrt{\sigma_y^2+\sigma_z^2-2\sigma_{yz}}} \right) - e^{\left(\mu_z+\frac{1}{2}\sigma_z^2\right)} N\left(\frac{\mu_y+\sigma_{yz}-\mu_z-\sigma_z^2}{\sqrt{\sigma_y^2+\sigma_z^2-2\sigma_{yz}}} \right)$$

(17)

where the equality comes from part (b) of Lemma A2. Noting that $e^{\left(\mu_y+\frac{1}{2}\sigma_y^2\right)} = E_0^d\left[Y_T\right]$, $e^{\left(\mu_z+\frac{1}{2}\sigma_z^2\right)} = E_0^d\left[Z_T\right]$, and $\sigma_y^2+\sigma_z^2-2\sigma_{yz} = Var_0\left[\ln\frac{Y_T}{Z_T}\right]$, and substituting these results into equation (17), one can verify that the expected value of the payoff at time T is equal to $E_0^d\left[Y_T\right]N\left(d_1\right)-E_0^d\left[Z_T\right]N\left(d_2\right)$. $\square$

Proof of Lemma 2. The payoff of a European currency option at maturity is

$$\max\left[\pm\left(K_x - X(T)\right), 0\right]$$

where "+" is for put and "−" for call. If the risk neutral distribution of $X(T)$ is derived, the option pricing formulae can be obtained from Lemma 1. Thus one only needs to derive $E_0^d\left[X(T)\right]$[26] here.

[26] Because $X(T)$ is not actually a price, the meaning of $E_0^d\left[X(T)\right]$ should be interpreted as

The prices in domestic currency of a T-maturity foreign zero-coupon bond are $X(0)B_f(0, T)$ at time 0 and $X(T)B_f(T, T)$ at time T. By the definition of the domestic T-maturity FRA measure, one can show that

$$E_0^d\left[\frac{X(T)B_f(T,T)}{B_d(T,T)}\right]=\frac{X(0)B_f(0,T)}{B_d(0,T)} \tag{18}$$

Since $B_f(T,T)=B_d(T,T)=1$, $E_0^d\left[X(T)\right]=\dfrac{X(0)B_f(0,T)}{B_d(0,T)}$. □

Proof of Proposition 1. The mean $\tilde{E}_0^d\left[B_d(t,T)\right]$ is implied by the definition of the domestic two-period FRA measure. The means $\tilde{E}_0^d\left[X(t)\right]$ and $\tilde{E}_t^d\left[X(T)\right]$ can be shown in the same way as that of $\tilde{E}_0^d\left[X(T)\right]$ in Lemma 2. The mean $\tilde{E}_0^d\left[B_f(t,T)\right]$ is not straightforward and is reasoned as the following. Considering a T-maturity foreign zero-coupon bond, the price in domestic currency of this bond are $X(0)B_f(0, T)$ at time 0 and $X(t)B_f(t, T)$ at time t. By the definition of the domestic two-period FRA measure, one can derive that

$$\tilde{E}_0^d\left[\frac{X(t)B_f(t,T)}{B_d(t,t)}\right]=\frac{X(0)B_f(0,T)}{B_d(0,t)}$$

Since $B_d(t,t)=1$,

$$\tilde{E}_0^d\left[X(t)B_f(t,T)\right]=\frac{X(0)B_f(0,T)}{B_d(0,t)} \tag{19}$$

Using equation (19), $\tilde{E}_0^d\left[X(t)\right]$, and the log-normality that $E(XY)=E(X)E(Y)\ \exp(\sigma_{XY})$ if X and Y are lognormally distributed, one can derive that

$$\begin{aligned}&\tilde{E}_0^d\left[B_f(t,T)\right]\\&=\frac{\tilde{E}_0^d\left[X(t)B_f(t,T)\right]}{\tilde{E}_0^d\left[X(t)\right]}\exp\left(-\sigma_{X(t),B_f(t,T)}\right)\\&=\frac{B_f(0,T)}{B_f(0,t)}\exp\left(-\sigma_{X(t),B_f(t,T)}\right)\end{aligned} \tag{20}$$

Finally, using the law of iterated expectations, one can show that

$$\begin{aligned}\tilde{E}_0^d\left[X(T)\right]&=\tilde{E}_0^d\left[\tilde{E}_t^d\left[X(T)\right]\right]=\tilde{E}_0^d\left[\frac{X(t)B_f(t,T)}{B_d(t,T)}\right]\\&=\frac{X(0)B_f(0,T)}{B_d(0,T)}\exp\left(\sigma^2_{B_d(t,T)}-\sigma_{X(t),B_d(t,T)}-\sigma_{B_d(t,T),B_f(t,T)}\right)\end{aligned}$$

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the expected value of one unit foreign currency converted to domestic currency, that is $E_0^d\left[1\times X(T)\right]$.

where the third equality comes from substituting $E_0^{\mathcal{Q}_d}\left[B_f\left(t,T\right)\right]$(equation (20)), $E_0^{\mathcal{Q}_d}\left[X\left(t\right)\right]$, and $E_0^{\mathcal{Q}_d}\left[B_d\left(t,T\right)\right]$ into the formula, $E\left(\frac{XY}{Z}\right)=\frac{E\left(X\right)E\left(Y\right)}{E\left(Z\right)}\exp\left(\sigma_Z^2+\sigma_{XY}-\sigma_{XZ}-\sigma_{YZ}\right)$, which is true if *X*, *Y*, and *Z* are lognormally distributed.

Proof of Corollary 1. Because the domestic and foreign short rate processes follow the Ho and Lee model, the domestic and foreign bond price processes are (for example, see Hull and White (1994))

$$\frac{dB_d\left(t,T\right)}{B_d\left(t,T\right)}=\mu_1\left(.\right)dt-\sigma_d\left(T-t\right)dZ_d$$

$$\frac{dB_f\left(t,T\right)}{B_f\left(t,T\right)}=\mu_2\left(.\right)dt-\sigma_f\left(T-t\right)dZ_f$$

Therefore the instantaneous variance of the log of forward exchange rate, *F*(*t*, *T*), is $\sigma_x^2+\left(\sigma_d^2+\sigma_f^2-2\rho_{df}\sigma_d\sigma_f\right)\left(T-t\right)^2+2\left(\rho_{dx}\sigma_d\sigma_x-\rho_{fx}\sigma_f\sigma_x\right)\left(T-t\right)$. Following Merton's (1973) approach, for example, the variance of the log of spot exchange rate at *T* in a stochastic interest rate economy is

$$\sigma_{X(T)}^2=\int_0^T Var\left(d\ln F\left(m,T\right)\right)=\int_0^T Var\left(d\ln\frac{X\left(m\right)B_f\left(m,T\right)}{B_d\left(m,T\right)}\right)$$

$$=\int_0^T\left[\sigma_x^2+\left(\sigma_d^2+\sigma_f^2-2\rho_{df}\sigma_d\sigma_f\right)\left(T-m\right)^2+2\left(\rho_{dx}\sigma_d\sigma_x-\rho_{fx}\sigma_f\sigma_x\right)\left(T-m\right)\right]dm$$

$$=\sigma_x^2T+\left(\rho_{dx}\sigma_d\sigma_x-\rho_{fx}\sigma_f\sigma_x\right)T^2+\frac{1}{3}\left(\sigma_d^2+\sigma_f^2-2\rho_{df}\sigma_d\sigma_f\right)T^3$$

the covariance between the spot exchange rate and the domestic *T*-maturity zero-coupon bond price at *t* is

$$\sigma x(t),\ B_d(t,T)=\int_0^t Cov\left(d\ln\frac{X(m)B_f(m,t)}{B_d(m,t)},d\ln\frac{B_d(m,T)}{B_d(m,t)}\right)$$

$$=\int_0^t\Big[Cov(d\ln X(m),d\ln B_d(m,T))+Cov(d\ln B_f(m,T),d\ln B_d(m,T))$$

$$+Var(d\ln B_d(m,t)-Cov(d\ln X(m),d\ln B_d(m,t))$$

$$-Cov(d\ln B_f(m,T),d\ln B_d(m,t))-Cov(d\ln B_d(m,t),d\ln B_d(m,T))\Big]$$

$$=\int_0^t\left[(\rho_{df}\sigma_d\sigma_f-\sigma_d^2)(T-t)(t-m)-\rho_{dx}\sigma_d\sigma_x(T-t)\right]dm$$

$$=\frac{1}{2}\left[\rho_{df}\sigma_d\sigma_f-\sigma_d^2\right](T-t)t^2-\rho_{dx}\sigma_d\sigma_x(T-t)t.$$

The other variance-covariance terms can be derived in the same way and thus the detailed proof is omitted.

Proof of Proposition 2. The critical exercise price is the price where the holding value equals to the early exercise value. In the case of put option, for example, the critical exercise price at time *t* is the solution of the following equation,

$$K_x - X(t) = \left[K_x N(-h) - \frac{X(t)B_f(t,T)}{B_d(t,T)} N(-h-\sigma) \right] B_d(t,T)$$
$$= K_x B_d(t,T) N(-h) - B_d(t,T) F(t,T) N(-h-\sigma)$$

where the left hand side and the right hand side of the first equality are the early exercise value and the holding (the counterpart European option) value respectively. Given $\ln B_d\left(t,T\right)$ and $\ln F\left(t,T\right)$, the European option values at the first exercise date (time *t*) are straightforward. For example, the European put option price in domestic currency is

$$P_1(B_d(t,T), F(t,T)) = \left[K_x N(-h) - \frac{X(t)B_f(t,T)}{B_d(t,T)} N(-h-\sigma) \right] B_d(t,T)$$
$$= K_x B_d(t,T) N(-h) - B_d(t,T) F(t,T) N(-h-\sigma)$$

The value of the twice-exercisable option at *t* is equal to the holding value plus the early exercise value. In the case of put option, for example, the holding value is

$$\int_{\ln \bar{X}_p(t)}^{\infty} \left[K_x B_d(t,T) N(-h) - B_d(t,T) F(t,T) N(-h-\sigma) \right] f_2(.) d\ln X(t)$$
$$= \left[K_x B_d(t,T) N(-h) - B_d(t,T) F(t,T) N(-h-\sigma) \right] N(d_{2p}),$$

where $f_2(.)$ is the conditional p.d.f. of $\ln X\left(t\right)$ given $\ln B_d\left(t,T\right)$ and $\ln F\left(t,T\right)$ underr the domestic two-period FRA measure, and the early exercise value is

$$\int_{-\infty}^{\ln \bar{X}_p(t)} \left[K_x - X(t) \right] f_2(\ln X(t) \mid B_d(t,T), F(t,T)) d\ln X(t)$$
$$= K_x N(-d_{2p}) - \exp(\mu_1 + \frac{1}{2}\sigma_1^2) N(-d_{1p}).$$

Therefore the twice-exercisable put option price at *t*, given $\ln B_d\left(t,T\right)$ and $\ln F\left(t,T\right)$, is

$$P_2\left(B_d\left(t,T\right), F\left(t,T\right)\right) = K_x N\left(-d_{2p}\right) - \exp\left(\mu_1 + \frac{1}{2}\sigma_1^2\right) N\left(-d_{1p}\right)$$
$$+ \left[K_x B_d\left(t,T\right) N\left(-h\right) - B_d\left(t,T\right) F\left(t,T\right) N\left(-h-\sigma\right) \right] N\left(d_{2p}\right)$$

The option value at 0 is then equal to $B_d(0, t)$ multiplied by the integration of the option value at *t*, given $\ln B_d\left(t,T\right)$ and $\ln F\left(t,T\right)$, along the p.d.f of $\ln B_d\left(t,T\right)$ and $\ln F\left(t,T\right)$ under the domestic two-period FRA measure.

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隨機利率下美式外匯選擇權之數值效率評價法

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摘要

本文擴展Ho, Stapleton, and Subrahmanyam (1997a)的一般化Geske-Johnson (1984)法，以評價隨機利率下的美式外匯選擇權，在假設利率的波動函數非隨機(deterministic)的情況下，我們推導出歐式外匯選擇權價格的封閉式解，以及可在兩個時點履約的外匯選擇權價格的分析解，再利用兩點外插的Geske-Johnson(1984)的近似公式來計算美式外匯選擇權的價格。我們也進行數值及比較靜態分析，以瞭解隨機利率對於美式外匯選擇權價格的影響。雖然本文的模型只是Amin and Bodurtha (1995)一般化模型的特例，但是我們的數值結果和他們類似但卻更加具有數值效率性。

關鍵字：美式選擇權定價、隨機利率，本國風險中立評價關係，遠期風險中立

測度