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Oscillating–decaying solutions, Runge approximation property for the anisotropic elasticity system and their applications to inverse problems

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Abstract

We construct oscillating–decaying solutions for the general inhomogeneous anisotropic elasticity system. We also prove the Runge approximation property for the inhomogeneous transversely isotropic elasticity system. We apply the oscillating–decaying solutions and the Runge approximation property to the inverse problem of identifying an inclusion or a cavity embedded in a transversely isotropic elastic medium.

Résumé

Nous construisons des solutions oscillantes et décroissantes pour le système général de l'élasticité anisotrope inhomogène. Nous démontrons également la propriété d'approximation de Runge pour le système de l'élasticité inhomogène transversalement isotrope. Nous utilisons enfin ces résultats pour

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l'identification d'une inclusion ou d'une cavité située dans un domaine élastique transversalement isotrope.

Keywords: Anisotropic elasticity; Oscillating–decaying solutions; Runge approximation property

1. Introduction

Special type solutions for elliptic equation or systems have played an important role in inverse problems since the pioneering work of Calderón [1]. In 1987, Sylvester and Uhlmann [17] introduced complex geometrical optics solutions to solve the inverse boundary value problem for the conductivity equation. Recently, Ikehata used Calderón type solutions to the inverse problem of identifying inclusions [6]. We remark that the complex geometrical optics solutions considered in those papers are available only for operators and system of operators whose leading part is the Laplacian.

To consider inverse problems for general systems we look for a substitute of the complex geometrical optics solutions. In this paper we show that the *oscillating–decaying solutions* we construct can also be useful in inverse problems. Roughly speaking, given a hyperplane, an oscillating–decaying solution is oscillating very rapidly along this plane and decaying exponentially in the direction transversal to the same plane. They are also complex geometrical optics solutions but with the imaginary part of the phase function non-negative. From the point of view of operators these correspond to Fourier integral operators with complex phase. The first instance that we know of an application of this type of solutions to inverse problems was in the article [12] where it was shown that the Laplace–Beltrami operator can be factorized into forward and backward heat type equations. Solutions of the forward heat type operator are complex geometrical optics solutions whose phase has non-negative imaginary part. This type of solutions were also applied in inverse problems for scalar elliptic equations in [14]. In this paper we construct oscillating–decaying solutions for the general anisotropic elasticity system.

In Ikehata's work, the complex geometrical optics solutions are used to define the *indicator function* (see [6] for the definition). To implement the use of oscillating–decaying solutions to the inverse problem of identifying a cavity or an inclusion, we have to modify the definition of the indicator function using the Runge approximation property. It was first recognized by Lax [11] that the Runge approximation property is a consequence of the weak unique continuation property. The Runge approximation property with constraints on Dirichlet data for certain anisotropic elasticity whose elasticity tensor is either homogeneous or real-analytic was proved in [7,8]. The weak unique continuation property is an obvious fact in these two situations. The unique continuation property or the Runge approximation property for the general inhomogeneous anisotropic elasticity is not known at present. In this paper we consider a special case of anisotropy, namely transversely isotropic elasticity which models many physical situations in elasticity.

It is clear that if the domain is connected, then the weak unique continuation property is an easy consequence of the uniqueness of the Cauchy problem on any hypersurface. A general and powerful method of proving the uniqueness of the Cauchy problem was initiated by Calderón as an application of singular integral operators or pseudodifferential operators. His result has been improved by several people. To deal with our problem here, we extend a generalization of the Calderón uniqueness theorem due to Zuily [20] (or see the related work [19]) to the system of differential equations with decoupled principal part (see Theorem 3.1). A key ingredient of the proof is a Carleman estimate which is derived under some characteristic root conditions. For the case considered here, by imposing some restrictions on the coefficients of the system, the characteristic root conditions are satisfied for all but certain directions. Consequently, the uniqueness of the Cauchy problem for the transversely isotropic elasticity system holds except for some hypersurfaces. It turns out we can not conclude the weak unique continuation property for this system. However, for convex inner domains, we are still able to establish the classical Runge approximation property (without the restriction on the support of the Dirichlet data). To illustrate some applications of the oscillating–decaying solutions and this particular Runge approximation property, we will study the inverse problem of recovering the convex hull of an inclusion or a cavity embedded in an elastic body with transversely isotropic medium.

The rest of this paper is organized as follows. In Section 2, we will describe the construction of the oscillating–decaying solutions for the general anisotropic elasticity system. In Section 3, we establish the Runge approximation property for the transversely isotropic elasticity system when the inner domain is convex. Before doing that, we will first review a generalization of Calderón's theorem. Some applications of the oscillating–decaying solutions and the Runge approximation property obtained in the previous section to inverse problems are studied in Section 4.

2. Construction of oscillating–decaying solutions

This section is devoted to the construction of oscillating–decaying solutions for the general anisotropic elasticity system.

Assume that

$$C(x) = \big(C_{ijkl}(x)\big) \in B^{\infty}\big(\mathbb{R}^3\big) = \big\{ f \in C^{\infty}\big(\mathbb{R}^3\big)\colon\ \partial^{\alpha} f \in L^{\infty}\big(\mathbb{R}^3\big),\ \forall \alpha \in \mathbb{Z}^3_+ \big\}$$

is the elasticity tensor satisfying

$$C_{ijkl}(x) = C_{klij}(x) \quad \forall x \in \mathbb{R}^3,\ \forall i, j, k, l \text{ (hyperelasticity)}$$

and there exists $\delta > 0$ such that

$$\sum_{ijkl} C_{ijkl}(x) a_i b_j a_k b_l \geqslant \delta |a|^2 |b|^2 \quad \forall x \in \mathbb{R}^3 \text{ (strong ellipticity).} \tag{2.1}$$

Here and below, all Latin indices are set to be from 1 to 3 unless otherwise indicated. In what follows, we denote:

$$(\nabla u)_{kl} = \partial_l u_k$$

and

$$(\nabla \cdot G)_i = \sum_j \partial_j g_{ij} \text{ for any matrix function } G = (g_{ij}).$$

Before going to the main theme of the section, we want to define several notations. Assume that $\Omega \subset \mathbb{R}^3$ is an open set with smooth boundary and $\omega \in \mathbb{S}^2$ is given. Let $\eta \in \mathbb{S}^2$ and $\zeta \in \mathbb{S}^2$ be chosen so that $\{\eta, \zeta, \omega\}$ forms an orthonormal system of $\mathbb{R}^3$. We then denote $x' = (x \cdot \eta, x \cdot \zeta)$. Let $t \in \mathbb{R}$, $\Omega_t(\omega) = \Omega \cap \{x \cdot \omega > t\}$ and $\Sigma_t(\omega) = \{x \cdot \omega = t\} \cap \Omega$ be a non-empty open set. We consider a vector function $u_{\chi_t,b,t,N,\omega}(x,\tau) =: u(x,\tau) = [u_1(x,\tau), u_2(x,\tau), u_3(x,\tau)]^{\mathrm{T}} \in C^\infty(\overline{\Omega_t(\omega)} \setminus \partial\Sigma_t(\omega)) \cap C^0(\overline{\Omega_t(\omega)})$ with $\tau \gg 1$ satisfying:

$$\begin{cases} \mathcal{L}_C u = \nabla \cdot (C(x)\nabla u) = 0 \quad \text{in } \Omega_t(\omega), \\ u|_{\Sigma_t(\omega)} = \mathrm{e}^{\mathrm{i}\tau x\cdot\xi}\{\chi_t(x')Q_t(x')b + \beta_{\chi_t,b,t,N,\omega}(x',\tau)\}, \end{cases} \tag{2.2}$$

where $\xi \in \mathbb{S}^2$ lying in the span of η and ζ is chosen and fixed, $\chi_t(x') \in \mathbb{C}_0^\infty(\mathbb{R}^2)$ with $\operatorname{supp}(\chi_t) \subset \Sigma_t(\omega)$, $Q_t(x')$ is a nonsingular smooth matrix function and $0 \neq b \in \mathbb{C}^3$. Moreover, $\beta_{\chi_t,b,t,N,\omega}(x',\tau)$ is a smooth vector function supported in $\operatorname{supp}(\chi_t)$ satisfying:

$$\left\|\beta_{\chi_t,b,t,N,\omega}(\cdot\,,\tau)\right\|_{L^\infty(\mathbb{R}^2)} \leqslant \frac{c}{\tau}$$

for some constant $c > 0$. From now on, we use c to denote a general positive constant whose value may vary from line to line. Furthermore, $u_{\chi_t,b,t,N,\omega}$ can be written as

$$u_{\chi_t,b,t,N,\omega} = w_{\chi_t,b,t,N,\omega} + r_{\chi_t,b,t,N,\omega}$$

with

$$w_{\chi_t,b,t,N,\omega} = \chi_t(x')Q_t\mathrm{e}^{\mathrm{i}\tau x\cdot\xi}\mathrm{e}^{-\tau(x\cdot\omega-t)A_t(x')}b + \gamma_{\chi_t,b,t,N,\omega}(x,\tau) \tag{2.3}$$

and $r_{\chi_t,b,t,N,\omega}$ satisfying

$$\|r_{\chi_t,b,t,N,\omega}\|_{H^1(\Omega_t(\omega))} \leqslant c\tau^{-N-1/2}, \tag{2.4}$$

where $A_t(\cdot) \in B^\infty(\mathbb{R}^2)$ is a matrix function such that all eigenvalues of A_t, denoted by $\operatorname{spec}(A_t)$, satisfies $\operatorname{spec}(A_t) \subset \mathbb{C}_r = \{z \in \mathbb{C}\colon \operatorname{Re} z > 0\}$, and $\gamma_{\chi_t,b,t,N,\omega}$ is a smooth vector function supported in $\operatorname{supp}(\chi_t)$ satisfying

$$\left\|\partial_x^\alpha \gamma_{\chi_t,b,t,N,\omega}\right\|_{L^2(\Omega_s(\omega))} \leqslant c\tau^{|\alpha|-3/2}\mathrm{e}^{-\tau(s-t)\lambda} \tag{2.5}$$

for $|\alpha| \leqslant 1$ and $s \geqslant t$, where $\lambda > 0$ is some constant depending on $\operatorname{spec}(A_t)$.

Without loss of generality, we consider the special case where $t = 0$, $\omega = e_3 = (0,0,1)$ and choose $\eta = (1,0,0)$, $\zeta = (0,1,0)$. The general case can be easily obtained from this special case by obvious change of coordinates. Define $\mathcal{L} = \mathcal{L}_C$ and

$\widetilde{M}\cdot = \mathrm{e}^{-\imath\tau x'\cdot\xi'}\mathcal{L}(\mathrm{e}^{\imath\tau x'\cdot\xi'}\cdot)$, where $x' = (x_1, x_2)$ and $\xi' = (\xi_1, \xi_2)$ with $|\xi'| = 1$. Clearly, $\widetilde{M}$ is a matrix differential operator. To be precise, the component $\widetilde{M}_{ik}$ of $\widetilde{M}$ is given by:

$$\begin{aligned}\widetilde{M}_{ik} &= -\tau^2\sum_{jl} C_{ijkl}\xi_j\xi_l + \tau\sum_{jl} C_{ijkl}(\imath\xi_l)\partial_j + \tau\sum_{jl} C_{ijkl}(\imath\xi_j)\partial_l + \sum_{jl} C_{ijkl}\partial_j\partial_l \\ &\quad + \sum_{jl}(\partial_j C_{ijkl})(\mathrm{i}\tau\xi_l) + \sum_{jl}(\partial_j C_{ijkl})\partial_l \\ &= -\tau^2\sum_{jl} C_{ijkl}\xi_j\xi_l + \tau\sum_{l} C_{i3kl}(\imath\xi_l)\partial_3 + \tau\sum_{j} C_{ijk3}(\imath\xi_j)\partial_3 + C_{i3k3}\partial_3^2 \\ &\quad + \tau\sum_{j\neq 3,l} C_{ijkl}(\imath\xi_l)\partial_j + \tau\sum_{l\neq 3,j} C_{ijkl}(\imath\xi_j)\partial_l + {\sum_{jl}}' C_{ijkl}\partial_j\partial_l \\ &\quad + \sum_{jl}(\partial_j C_{ijkl})(\mathrm{i}\tau\xi_l) + \sum_{jl}(\partial_j C_{ijkl})\partial_l\end{aligned}$$

with $\xi_3 = 0$, where $\sum'_{jl} = \sum_{j,l\setminus\{3,3\}}$. Our task is now reduced to solve:

$$\widetilde{M}v = 0. \tag{2.6}$$

An obvious strategy is to transform (2.6) into a first order system. First of all, we observe that (2.6) is equivalent to

$$Mv = 0, \tag{2.7}$$

where $M = C_3^{-1}\widetilde{M}$ and the (i,k) entry of C_3 is C_{i3k3}. Define $\langle a,b\rangle = (\langle a,b\rangle_{ik})$ for $a = (a_1,a_2,a_3)$ and $b = (b_1,b_2,b_3)$, where $\langle a,b\rangle_{ik} = \sum_{jl} C_{ijkl}a_j b_l$. Also, we denote $\langle a,b\rangle_0 = \langle a,b\rangle|_{x_3=0}$. Let P be a differential operator. We define the order of P, denoted by $\mathrm{ord}(P)$, in the following sense:

$$\big\| P\big(\mathrm{e}^{-\tau x_3 A(x')}\varphi(x')\big)\big\|_{L^2(\mathbb{R}^3_+)} \leqslant c\tau^{\mathrm{ord}(P)-1/2},$$

where $\mathbb{R}^3_+ = \{x_3 > 0\}$, $A(x')$ is a smooth matrix function of x' with $\mathrm{spec}(A) \subset \mathbb{C}_r$ and $\varphi(x') \in C_0^\infty(\mathbb{R}^2)$. In this sense, we can see that τ, ∂_3 are of order 1, ∂_1, ∂_2 are of order 0 and x_3 is of order -1. Note that the order of x_3 is simple consequence of the integration by parts. To verify the order of ∂_j, $j = 1$ or 2, we observe that

$$\partial_j\big(\mathrm{e}^{-\tau x_3 A(x')}\varphi(x')\big) = \mathrm{e}^{-\tau x_3 A(x')}\partial_j\varphi(x') - \tau\int_0^{x_3}\mathrm{e}^{-\tau(x_3-s)A(x')}\partial_j A(x')\mathrm{e}^{-\tau s A(x')}\varphi(x')\,\mathrm{d}s$$

and therefore

$$\big\|\partial_j\big(\mathrm{e}^{-\tau x_3 A(x')}\varphi(x')\big)\big\|_{L^2(\mathbb{R}^3_+)} \leqslant c\tau^{-1/2}.$$

Now according to this order, the principal part M_2 (order 2) of M is:

$$M_2 = -\big\{D_3^2 + \tau\langle e_3, e_3\rangle_0^{-1}\big(\langle e_3, \varrho\rangle_0 + \langle \varrho, e_3\rangle_0\big)D_3 + \tau^2\langle e_3, e_3\rangle_0^{-1}\langle \varrho, \varrho\rangle_0\big\} \tag{2.8}$$

with $D_3 = -\imath\partial_3$ and $\varrho = (\xi_1, \xi_2, 0)$. Notice that M_2 is obtained by the Taylor's expansion of M at $x_3 = 0$, i.e.,

$$\begin{aligned} M(x', x_3) &= M(x', 0) + x_3\partial_3 M(x', 0) + \cdots + \frac{x_3^{N-1}}{(N-1)!}\partial_3^{N-1}M(x', 0) + R \\ &= M_2 + M_1 + \cdots + M_{-N+1} + R, \end{aligned} \tag{2.9}$$

where $\operatorname{ord}(M_j) = j$ and $\operatorname{ord}(R) = -N$. By using (2.9), the system (2.7) is written as

$$M_2 v = -(M_1 + \cdots + M_{-N+1} + R)v := f. \tag{2.10}$$

Now we set $w_1 = v$ and $w_2 = -\tau^{-1}\langle e_3, e_3\rangle_0 D_3 v - \langle e_3, \varrho\rangle_0 v$. Then we can compute

$$D_3 w_1 = D_3 v = -\tau\langle e_3, e_3\rangle_0^{-1}\langle e_3, \varrho\rangle_0 w_1 - \tau\langle e_3, e_3\rangle_0^{-1} w_2 \tag{2.11}$$

and

$$\begin{aligned} D_3 w_2 &= -\tau^{-1}\langle e_3, e_3\rangle_0 D_3^2 v - \langle e_3, \varrho\rangle_0 D_3 v \\ &= \tau^{-1}\langle e_3, e_3\rangle_0\big\{\tau\langle e_3, e_3\rangle_0^{-1}\big(\langle e_3, \varrho\rangle_0 + \langle \varrho, e_3\rangle_0\big)D_3 v + \tau^2\langle e_3, e_3\rangle_0^{-1}\langle \varrho, \varrho\rangle_0 v + f\big\} \\ &\quad - \langle e_3, \varrho\rangle_0 D_3 v \\ &= \langle \varrho, e_3\rangle_0 D_3 v + \tau\langle \varrho, \varrho\rangle_0 w_1 + \tau^{-1}\langle e_3, e_3\rangle_0 f \\ &= \langle \varrho, e_3\rangle_0\big\{-\tau\langle e_3, e_3\rangle_0^{-1}\langle e_3, \varrho\rangle_0 w_1 - \tau\langle e_3, e_3\rangle_0^{-1} w_2\big\} + \tau\langle \varrho, \varrho\rangle_0 w_1 \\ &\quad + \tau^{-1}\langle e_3, e_3\rangle_0 f \\ &= -\tau\big\{\langle \varrho, e_3\rangle_0\langle e_3, e_3\rangle_0^{-1}\langle e_3, \varrho\rangle_0 - \langle \varrho, \varrho\rangle_0\big\}w_1 - \tau\langle \varrho, e_3\rangle_0\langle e_3, e_3\rangle_0^{-1} w_2 \\ &\quad + \tau^{-1}\langle e_3, e_3\rangle_0 f. \end{aligned} \tag{2.12}$$

Combining (2.11), (2.12) and setting $W = [w_1, w_2]^{\mathrm{T}}$, we get that

$$D_3 W = \tau K W + \begin{bmatrix} 0 \\ \tau^{-1}\langle e_3, e_3\rangle_0 f \end{bmatrix}, \tag{2.13}$$

where

$$K = \begin{bmatrix} \langle e_3, e_3\rangle_0^{-1}\langle e_3, \varrho\rangle_0 & \langle e_3, e_3\rangle_0^{-1} \\ \langle \varrho, e_3\rangle_0\langle e_3, e_3\rangle_0^{-1}\langle e_3, \varrho\rangle_0 - \langle \varrho, \varrho\rangle_0 & \langle \varrho, e_3\rangle_0\langle e_3, e_3\rangle_0^{-1} \end{bmatrix}.$$

Now we observe that f contains x_3 derivative of v up to order 1. By (2.10), we can express (2.13) as

$$D_3W = (\tau K + K_0 + \cdots + K_{-N} + S)W, \tag{2.14}$$

where $\operatorname{ord}(K_j) = j$ and $\operatorname{ord}(S) = -N - 1$. Notice that the differential operator K_j involves only x' derivatives. In addition, K is a matrix function independent of x_3 and its eigenvalues are determined from

$$\det(\lambda I - K) = 0$$

which is equivalent to

$$\det\big(\lambda^2\langle e_3, e_3\rangle_0 + \lambda\big\{\langle e_3, \varrho\rangle_0 + \langle \varrho, e_3\rangle_0\big\} + \langle \varrho, \varrho\rangle_0\big) = 0. \tag{2.15}$$

It follows from the strong ellipticity condition (2.1) that (2.15) has roots $\lambda_j^{\pm}$ with $\pm\operatorname{Im}\lambda_j^{\pm} > 0$, $1 \leqslant j \leqslant 3$. Let $\widetilde{Q} := [q_1^+, q_2^+, q_3^+, q_1^-, q_2^-, q_3^-]$ be a nonsingular matrix with linearly independent vectors $q_j^{\pm}$, $1 \leqslant j \leqslant 3$, chosen from the range $\operatorname{ran}(P_{\pm})$ of $P_{\pm}$, respectively, where

$$P_{\pm} = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_{\pm}} (\lambda I - K)^{-1}\, \mathrm{d}\lambda.$$

Here $\Gamma_{\pm}$ are closed C^1 curves in $\mathbb{C}_{\pm} := \{\pm\operatorname{Im}\lambda > 0\}$ enclosing $\lambda_j^{\pm}$, $1 \leqslant j \leqslant 3$. Although $\widetilde{Q}$ is defined only locally, it is not difficult to extend it globally. To do this, by considering the vector bundle over $\{x_3 = 0\}$ whose fiber at x is the linear span of $\{q_j^{\pm}\}_{j=1}^3 \subset \operatorname{ran}(P_{\pm})$ and noting that it is equivalent to $\{x_3 = 0\} \times \mathbb{C}^3$, we can find a globally defined smooth invertible matrix $\widetilde{Q} = [q_1^+, q_2^+, q_3^+, q_1^-, q_2^-, q_3^-]$ on $\{x_3 = 0\}$ with $\{q_j^{\pm}\}_{j=1}^3 \subset \operatorname{ran}(P_{\pm})$ (see for example [16]). Using this matrix function Q, we obtain that

$$\widetilde{K} = \widetilde{Q}^{-1} K \widetilde{Q} = \begin{bmatrix} \widetilde{K}_+ & 0 \\ 0 & \widetilde{K}_- \end{bmatrix}, \tag{2.16}$$

where $\operatorname{spec}(\widetilde{K}_{\pm}) \subset \mathbb{C}_{\pm}$, respectively. In fact, we can choose:

$$\widetilde{Q} = \begin{bmatrix} Q & \overline{Q} \\ Q' & \overline{Q'} \end{bmatrix}, \tag{2.17}$$

where

$$\begin{bmatrix} Q \\ Q' \end{bmatrix} = [q_1^+, q_2^+, q_3^+].$$

By virtue of the matrix $\widetilde{Q}$ in (2.17), we can see that $\widetilde{K}_- = \overline{\widetilde{K}_+}$. Furthermore, it is readily seen that Q is nonsingular. Now by setting $\widehat{W} = \widetilde{Q}^{-1}W$, we get from (2.14) that

$$D_3\widehat{W} = \big(\tau\widetilde{K} + \widehat{K}_0 + \cdots + \widehat{K}_{-N} + \widehat{S}\big)\widehat{W}, \tag{2.18}$$

where $\mathrm{ord}(\widehat{K}_j) = j$ and $\mathrm{ord}(\widehat{S}) = -N-1$. It should be pointed out that $\widehat{K}_j$ contains only x' derivatives. Additionally, $\widehat{K}_0$ can be divided into terms involving τx_3 and terms formed by the differential operator in $\partial_{x'}$ with coefficients independent of x_3. Likewise, $\widehat{K}_j$ can be grouped into terms containing τx_3^{-j+1}, $\tau^{-1}x_3^{-j-1}$ and x_3^{-j}, respectively, where $-N \leqslant j \leqslant -1$.

We have already decoupled K by choosing a suitable matrix function $\widetilde{Q}$. Our next goal is to decouple $\widehat{K}_0, \ldots, \widehat{K}_{-N}$. We first show how to decouple $\widehat{K}_0$. Let $\widehat{W} = (I + x_3A^{(0)} + \tau^{-1}B^{(0)})\widetilde{W}^{(0)}$ with $A^{(0)}$ and $B^{(0)}$ being differential operators in $\partial_{x'}$ (with coefficients independent of x_3), then we have that

$$\begin{aligned}
D_3\widetilde{W}^{(0)} &= \bigl(I + x_3A^{(0)} + \tau^{-1}B^{(0)}\bigr)^{-1}\bigl(\tau\widetilde{K} + \widehat{K}_0 + \cdots + \widehat{S}\bigr)\bigl(I + x_3A^{(0)} + \tau^{-1}B^{(0)}\bigr)\widetilde{W}^{(0)} \\
&\quad + \bigl(I + x_3A^{(0)} + \tau^{-1}B^{(0)}\bigr)^{-1}\mathrm{i}A^{(0)}\widetilde{W}^{(0)} \\
&= \bigl(I - x_3A^{(0)} - \tau^{-1}B^{(0)} + \cdots\bigr)\bigl(\tau\widetilde{K} + \widehat{K}_0 + \cdots + \widehat{S}\bigr) \\
&\quad \times \bigl(I + x_3A^{(0)} + \tau^{-1}B^{(0)}\bigr)\widetilde{W}^{(0)} + \bigl(I - x_3A^{(0)} - \tau^{-1}B^{(0)} + \cdots\bigr)\mathrm{i}A^{(0)}\widetilde{W}^{(0)} \\
&= \bigl\{\tau\widetilde{K} + \bigl(\widehat{K}_0 - \tau x_3A^{(0)}\widetilde{K} + \tau x_3\widetilde{K}A^{(0)} - B^{(0)}\widetilde{K} + \widetilde{K}B^{(0)} + \mathrm{i}A^{(0)}\bigr) \\
&\quad + \widehat{K}'_{-1} + \cdots\bigr\}\widetilde{W}^{(0)},
\end{aligned}$$

where $\mathrm{ord}(\widehat{K}'_{-1}) = -1$ and the remainder contains terms of order at most -2. To look at $\widetilde{K}_0 := \widehat{K}_0 - \tau x_3A^{(0)}\widetilde{K} + \tau x_3\widetilde{K}A^{(0)} - B^{(0)}\widetilde{K} + \widetilde{K}B^{(0)} + \mathrm{i}A^{(0)}$ more carefully, we set $\widehat{K}_0 = \tau x_3\widehat{K}_{0,1} + \widehat{K}_{0,2}$ and express $\widehat{K}_{0,1}$, $\widehat{K}_{0,2}$, $A^{(0)}$ and $B^{(0)}$ in block forms, i.e.,

$$\widehat{K}_{0,\ell} = \begin{bmatrix} \widehat{K}_{0,\ell}(1,1) & \widehat{K}_{0,\ell}(1,2) \\ \widehat{K}_{0,\ell}(2,1) & \widehat{K}_{0,\ell}(2,2) \end{bmatrix}, \quad \ell = 1, 2,$$
$$A^{(0)} = \begin{bmatrix} A^{(0)}(1,1) & A^{(0)}(1,2) \\ A^{(0)}(2,1) & A^{(0)}(2,2) \end{bmatrix} \quad \text{and} \quad B^{(0)} = \begin{bmatrix} B^{(0)}(1,1) & B^{(0)}(1,2) \\ B^{(0)}(2,1) & B^{(0)}(2,2) \end{bmatrix}.$$

Then the off-diagonal blocks of $\widetilde{K}_0$ are given by:

$$\begin{aligned}
\widetilde{K}_0(1,2) &= \tau x_3\bigl\{\widehat{K}_{0,1}(1,2) - A^{(0)}(1,2)\widetilde{K}_- + \widetilde{K}_+A^{(0)}(1,2)\bigr\} \\
&\qquad + \bigl\{\widehat{K}_{0,2}(1,2) + \mathrm{i}A^{(0)}(1,2) - B^{(0)}(1,2)\widetilde{K}_- + \widetilde{K}_+B^{(0)}(1,2)\bigr\}, \\
\widetilde{K}_0(2,1) &= \tau x_3\bigl\{\widehat{K}_{0,1}(2,1) - A^{(0)}(2,1)\widetilde{K}_- + \widetilde{K}_+A^{(0)}(2,1)\bigr\} \\
&\qquad + \bigl\{\widehat{K}_{0,2}(2,1) + \mathrm{i}A^{(0)}(2,1) - B^{(0)}(2,1)\widetilde{K}_- + \widetilde{K}_+B^{(0)}(2,1)\bigr\}.
\end{aligned}$$

Since $\mathrm{spec}(\widetilde{K}_-) \cap \mathrm{spec}(\widetilde{K}_+) = \emptyset$, it is well-known that we can find suitable $A^{(0)}(1,2)$ and $A^{(0)}(2,1)$ such that

$$\begin{aligned}
\widehat{K}_{0,1}(1,2) - A^{(0)}(1,2)\widetilde{K}_- + \widetilde{K}_+A^{(0)}(1,2) &= \widehat{K}_{0,1}(2,1) - A^{(0)}(2,1)\widetilde{K}_- + \widetilde{K}_+A^{(0)}(2,1) \\
&= 0
\end{aligned}$$

(see similar arguments in [18]). Having found $A^{(0)}(1,2)$ and $A^{(0)}(2,1)$, we use the same argument and determine $B^{(0)}(1,2)$ and $B^{(0)}(2,1)$ so that

$$\begin{cases} \widehat{K}_{0,2}(1,2) + \mathrm{i}A^{(0)}(1,2) - B^{(0)}(1,2)\widetilde{K}_- + \widetilde{K}_+ B^{(0)}(1,2) = 0, \\ \widehat{K}_{0,2}(2,1) + \mathrm{i}A^{(0)}(2,1) - B^{(0)}(2,1)\widetilde{K}_- + \widetilde{K}_+ B^{(0)}(2,1) = 0. \end{cases} \tag{2.19}$$

It should be pointed out that Eqs. (2.19) are to be understood in the operator sense. More precisely, since $\widehat{K}_{0,2}(1,2)$ and $\widehat{K}_{0,2}(2,1)$ are differential operators in $\partial_{x'}$ (of order one) with coefficients independent of x_3. We will look for $B^{(0)}(1,2)$ and $B^{(0)}(2,1)$ as the same type of differential operator. By computing the related full symbols of Eqs. (2.19) and using the condition $\operatorname{spec}(\widetilde{K}_-) \cap \operatorname{spec}(\widetilde{K}_+) = \emptyset$, we can solve for $B^{(0)}(1,2)$ and $B^{(0)}(2,1)$. To find $A^{(0)}$ and $B^{(0)}$, we simply set diagonal blocks of them are zero, i.e.,

$$A^{(0)} = \begin{bmatrix} 0 & A^{(0)}(1,2) \\ A^{(0)}(2,1) & 0 \end{bmatrix} \quad \text{and} \quad B^{(0)} = \begin{bmatrix} 0 & B^{(0)}(1,2) \\ B^{(0)}(2,1) & 0 \end{bmatrix}.$$

With these matrices $A^{(0)}$ and $B^{(0)}$, we can see that

$$D_3\widetilde{W}^{(0)} = \{\tau\widetilde{K} + \widetilde{K}_0 + \widehat{K}'_{-1} + \cdots\}\widetilde{W}^{(0)} \tag{2.20}$$

with

$$\widetilde{K}_0 = \begin{bmatrix} \widetilde{K}_0(1,1) & 0 \\ 0 & \widetilde{K}_0(2,2) \end{bmatrix}.$$

To proceed further, we would like to decouple $\widehat{K}'_{-1}$. Notice that $\widehat{K}'_{-1}$ can be written as $\widehat{K}'_{-1} = \tau x_3^2\widehat{K}'_{-1,1} + x_3\widehat{K}'_{-1,2} + \tau^{-1}\widehat{K}'_{-1,3}$. Here, we can see that $\widehat{K}'_{-1,1}$, $\widehat{K}'_{-1,2}$ and $\widehat{K}'_{-1,3}$ are differential operators in $\partial_{x'}$ of order zero, one and two with coefficients independent of x_3, respectively. By the similar trick, we set $\widetilde{W}^{(0)} = (I + x_3^2A^{(1)} + \tau^{-1}x_3B^{(1)} + \tau^{-2}C^{(1)})\widetilde{W}^{(1)}$, where $A^{(1)}$, $B^{(1)}$ and $C^{(1)}$ are differential operators in $\partial_{x'}$. Now plugging $\widetilde{W}^{(0)}$ of above form into (2.20) yields:

$$\begin{aligned} D_3\widetilde{W}^{(1)} &= \{\tau\widetilde{K} + \widetilde{K}_0 + [\widehat{K}'_{-1} - \tau x_3^2(A^{(1)}\widetilde{K} - \widetilde{K}A^{(1)}) - x_3(B^{(1)}\widetilde{K} - \widetilde{K}B^{(1)}) \\ &\quad - \tau^{-1}(C^{(1)}\widetilde{K} - \widetilde{K}C^{(1)}) + 2x_3A^{(1)} + \mathrm{i}\tau^{-1}B^{(1)}] + \cdots\}\widetilde{W}^{(1)} \\ &= \{\tau\widetilde{K} + \widetilde{K}_0 + [\tau x_3^2(\widehat{K}'_{-1,1} - A^{(1)}\widetilde{K} + \widetilde{K}A^{(1)}) + x_3(\widehat{K}'_{-1,2} - B^{(1)}\widetilde{K} + \widetilde{K}B^{(1)} \\ &\quad + 2A^{(1)}) + \tau^{-1}(\widehat{K}'_{-1,3} - C^{(1)}\widetilde{K} + \widetilde{K}C^{(1)} + \mathrm{i}B^{(1)})] + \cdots\}\widetilde{W}^{(1)} \end{aligned} \tag{2.21}$$

where the remainder consists of terms with order at most -2. Therefore, by the same argument, we can find suitable $A^{(1)}$, $B^{(1)}$ and $C^{(1)}$ such that the off-diagonal blocks of the order -1 term on the right side of (2.21) are zero. That is, we get that

$$D_3\widetilde{W}^{(1)} = \{\tau\widetilde{K} + \widetilde{K}_0 + \widetilde{K}_{-1} + \cdots\}\widetilde{W}^{(1)}$$

with

$$\widetilde{K}_{-1} = \begin{bmatrix} \widetilde{K}_{-1}(1,1) & 0 \\ 0 & \widetilde{K}_{-1}(2,2) \end{bmatrix}.$$

Recursively, by defining

$$\widehat{W} = \left(I + x_3 A^{(0)} + \tau^{-1} B^{(0)}\right)\left(I + x_3^2 A^{(1)} + \tau^{-1} x_3 B^{(1)} + \tau^{-2} C^{(1)}\right)\cdots \left(I + x_3^{N+1} A^{(N)} + \tau^{-1} x_3^N B^{(N)} + \tau^{-2} x_3^{N-1} C^{(N)}\right)\widetilde{W}^{(N)}$$

with suitable $A^{(j)}$, $B^{(j)}$ and $C^{(j)}$ for $0 \leqslant j \leqslant N$ ($C^{(0)} = 0$), we can transform (2.18) into

$$D_3 \widetilde{W}^{(N)} = \left\{\tau \widetilde{K} + \widetilde{K}_0 + \cdots + \widetilde{K}_{-N} + \widetilde{S}\right\} \widetilde{W}^{(N)}, \tag{2.22}$$

where $\widetilde{K}_{-j}$ for all $0 \leqslant j \leqslant N$ are decoupled and $\operatorname{ord}(\widetilde{S}) = -N-1$. Notice that all diagonal blocks of $A^{(j)}$ and $B^{(j)}$ for $1 \leqslant j \leqslant N$ are zero.

Now in view of (2.22), we consider the equation,

$$D_3 \hat{v}^{(N)} = \left\{\tau \widetilde{K}_+ + \widetilde{K}_0(1,1) + \cdots + \widetilde{K}_{-N}(1,1)\right\} \hat{v}^{(N)},$$

with an approximated solution of the form:

$$\hat{v}^{(N)} = \sum_{j=0}^{N+1} \hat{v}_{-j}^{(N)},$$

where $\hat{v}_{-j}^{(N)}$ for $0 \leqslant j \leqslant N+1$ satisfy:

$$\begin{cases} D_3 \hat{v}_0^{(N)} = \tau \widetilde{K}_+ \hat{v}_0^{(N)}, \quad \hat{v}_0^{(N)}|_{x_3=0} = \chi_t(x')b, \\ D_3 \hat{v}_{-1}^{(N)} = \tau \widetilde{K}_+ \hat{v}_{-1}^{(N)} + \widetilde{K}_0(1,1)\hat{v}_0^{(N)}, \quad \hat{v}_{-1}^{(N)}|_{x_3=0} = 0, \\ \vdots \\ D_3 \hat{v}_{-N-1}^{(N)} = \tau \widetilde{K}_+ \hat{v}_{-N-1}^{(N)} + \sum_{j=0}^{N} \widetilde{K}_{-j}(1,1)\hat{v}_{j-N}^{(N)}, \quad \hat{v}_{-N-1}^{(N)}|_{x_3=0} = 0, \end{cases} \tag{2.23}$$

where $\chi_t(x') \in C_0^\infty(\mathbb{R}^2)$ and $b \in \mathbb{C}^3$. Clearly, we have that $\hat{v}_0^{(N)} = \exp(\mathrm{i}\tau x_3 \widetilde{K}_+)\chi_t(x')b$ and $\hat{v}_{-1}^{(N)} = \exp(\mathrm{i}\tau x_3 \widetilde{K}_+)\int_0^{x_3} \exp(-\mathrm{i}\tau s \widetilde{K}_+)\widetilde{K}_0(1,1)\hat{v}_0^{(N)}\,\mathrm{d}s$. Furthermore, we can derive that

$$\left\| x_3^\beta \partial_{x'}^\alpha \hat{v}_0^{(N)} \right\|_{L^2(\mathbb{R}_+^3)} \leqslant c\tau^{-\beta-1/2}$$

for and $\beta \in \mathbb{Z}_+$ and multi-index α. Likewise, we can compute:

$$
\begin{aligned}
\|\hat{v}_{-1}^{(N)}\|_{L^2(\mathbb{R}^3_+)}^2 &= \int_{\mathbb{R}^2} \mathrm{d}x' \int_0^\infty \left| \exp\big(\mathrm{i}\tau x_3 \widetilde{K}_+\big) \int_0^{x_3} \exp\big(-\mathrm{i}\tau s \widetilde{K}_+\big) \widetilde{K}_0(1,1) \hat{v}_0^{(N)}\, \mathrm{d}s \right|^2 \mathrm{d}x_3 \\
&\leqslant \int_{\mathbb{R}^2} \mathrm{d}x' \int_0^\infty \mathrm{e}^{-2\tau x_3 \lambda} \left(\int_0^{x_3} \mathrm{e}^{\tau s \lambda} \big| \widetilde{K}_0(1,1) \hat{v}_0^{(N)} \big|\, \mathrm{d}s \right)^2 \mathrm{d}x_3 \\
&\leqslant \int_{\mathbb{R}^2} \mathrm{d}x' \int_0^\infty \mathrm{e}^{-2\tau x_3 \lambda} x_3 \left(\int_0^{x_3} \mathrm{e}^{2\tau s \lambda} \big| \widetilde{K}_0(1,1) \hat{v}_0^{(N)} \big|^2\, \mathrm{d}s \right) \mathrm{d}x_3 \\
&= (2\tau\lambda)^{-1} \int_{\mathbb{R}^2} \mathrm{d}x' \int_0^\infty \mathrm{e}^{-2\tau x_3 \lambda} \left\{ \int_0^{x_3} \mathrm{e}^{2\tau s \lambda} \big| \widetilde{K}_0(1,1) \hat{v}_0^{(N)} \big|^2\, \mathrm{d}s \right. \\
&\qquad \left. + x_3 \mathrm{e}^{2\tau x_3 \lambda} \big| \widetilde{K}_0(1,1) \hat{v}_0^{(N)} \big|^2 \right\} \mathrm{d}x_3 \\
&= (2\tau\lambda)^{-2} \int_{\mathbb{R}^2} \mathrm{d}x' \int_0^\infty \big| \widetilde{K}_0(1,1) \hat{v}_0^{(N)} \big|^2\, \mathrm{d}x_3 \\
&\qquad + (2\tau\lambda)^{-1} \int_{\mathbb{R}^2} \mathrm{d}x' \int_0^\infty x_3 \big| \widetilde{K}_0(1,1) \hat{v}_0^{(N)} \big|^2\, \mathrm{d}x_3 \\
&\leqslant c\tau^{-3}.
\end{aligned} \tag{2.24}
$$

Notice that the last two equalities of (2.24) are obtained from the integration by parts. To get the last inequality of (2.24), we make use of $\mathrm{ord}(\widetilde{K}_0(1,1)) = 0$ and $\mathrm{ord}(x_3^{1/2}) = -1/2$. Furthermore, by similar computations in (2.24), one can show that

$$
\big\| x_3^\beta \partial_{x'}^\alpha \big(\hat{v}_{-1}^{(N)}\big) \big\|_{L^2(\mathbb{R}^3_+)} \leqslant c\tau^{-\beta-3/2}.
$$

Inductively, similar estimates can be obtained for $\hat{v}_{-j}^{(N)}$, $j = 2, \ldots, N+1$, i.e.,

$$
\big\| x_3^\beta \partial_{x'}^\alpha \big(\hat{v}_{-j}^{(N)}\big) \big\|_{L^2(\mathbb{R}^3_+)} \leqslant c\tau^{-\beta-j-1/2}
$$

for $2 \leqslant j \leqslant N+1$.

Therefore, if we set $V^{(N)} = \begin{bmatrix} \hat{v}^{(N)} \\ 0 \end{bmatrix}$, then

$$
\begin{cases}
D_3 V^{(N)} - \{\tau \widetilde{K} + \widetilde{K}_0 + \cdots + \widetilde{K}_{-N}\} V^{(N)} = \widetilde{R}, \\
V^{(N)}|_{x_3=0} = \begin{bmatrix} \chi_t(x') b \\ 0 \end{bmatrix},
\end{cases}
$$

where

$$\|\widetilde{R}\|_{L^2(\mathbb{R}^3_+)} \leqslant c\tau^{-N-3/2}.$$

Defining the vector function:

$$\tilde{v} = [\tilde{v}_1, \tilde{v}_2, \tilde{v}_3]^{\mathrm{T}}$$

with $\tilde{v}_j$ being the jth component of the vector $\widetilde{Q}(I + x_3A^{(0)} + \tau^{-1}B^{(0)})(I + x_3^2A^{(1)} + \tau^{-1}x_3B^{(1)} + \tau^{-2}C^{(1)}) \cdots (I + x_3^{N+1}A^{(N)} + \tau^{-1}x_3^NB^{(N)} + \tau^{-2}x_3^{N-1}C^{(N)})V^{(N)}$ and setting $w = \exp(\mathrm{i}\tau x' \cdot \xi')\tilde{v}$, we can see that

$$\begin{aligned} w &= Q\exp(\mathrm{i}\tau x' \cdot \xi')\exp\bigl(\mathrm{i}\tau x_3\widetilde{K}_+(x')\bigr)\chi_t(x')b + \exp(\mathrm{i}\tau x' \cdot \xi')\tilde{\gamma}(x,\tau) \\ &= Q\exp(\mathrm{i}\tau x' \cdot \xi')\exp\bigl(-\tau x_3\bigl(-\mathrm{i}\widetilde{K}_+(x')\bigr)\bigr)\chi_t(x')b + \gamma(x,\tau) \end{aligned}$$

and

$$w|_{x_3=0} = \exp(\mathrm{i}\tau x' \cdot \xi')\bigl\{\chi_t(x')Qb + \beta_0(x',\tau)\bigr\},$$

where γ satisfies the estimate (2.5) on $\Omega_s := \{x_3 > s\} \cap \Omega$ for $s \geqslant 0$ and $\beta_0(x',\tau) = \tilde{\gamma}(x',0,\tau)$ is supported in $\operatorname{supp}(\chi_t)$ with $\|\beta_0(\cdot\,,\tau)\|_{L^\infty} \leqslant c\tau^{-1}$. Also, we have that

$$\|M\tilde{v}\|_{L^2(\Omega_0)} \leqslant c\tau^{-N-1/2}.$$

Notice that Q as the $(1,1)$ block of $\widetilde{Q}$ in (2.17) is invertible.

Now let $u = w + r = \mathrm{e}^{\mathrm{i}x'\cdot\xi'}\tilde{v} + r$ and r be the solution to the boundary value problem

$$\begin{cases} \mathcal{L}r = -\mathrm{e}^{\mathrm{i}x'\cdot\xi'}\widetilde{M}\tilde{v} & \text{in } \Omega_0, \\ r|_{\partial\Omega_0} = 0. \end{cases} \tag{2.25}$$

The existence of r solving (2.25) is guaranteed by the Lax–Milgram theorem. Moreover, we have the following estimate

$$\|r\|_{H^1(\Omega_0)} \leqslant c\tau^{-N-1/2},$$

which is the estimate (2.4) on Ω_0. We now complete the construction of the oscillating–decaying solution for the case $t = 0$ and $\omega = (0,0,1)$. The oscillating–decaying solution in the general case can be obtained using an easy change of coordinates. On the other hand, since the construction of the oscillating–decaying solution is local near any point on the hyperplane $\Sigma_t(\omega)$ and the strong convexity condition is invariant under change of coordinates, we can construct the oscillating–decaying solution with respect to any curved hypersurface as well. The only extra work needed to do is to flatten the boundary.

3. Runge approximation property for the transversely isotropic system

3.1. Generalization of Calderón's theorem

To begin, we first review Zuily's results in [20, Chapter 2]. The purpose is to make this paper as self-contained as possible. Let V be an open neighborhood of $x_0 \in \mathbb{R}^n$. In this section we do not specify the dimension $n \in \mathbb{N}$. In the neighborhood of V we define a C^∞ hypersurface:

$$S = \{x \in V\colon\ \psi(x) = \psi(x_0)\}. \tag{3.1}$$

Let

$$\mathbb{P}(x, D) = P(x, D) + Q(x, D) \tag{3.2}$$

be a system of differential operators, where

$$P(x, D) = \operatorname{diag}\big[P_1(x, D), \ldots, P_k(x, D)\big] \tag{3.3}$$

is a decoupled system of differential operators with $P_j(x, D) = \sum_{|\alpha|=m} a^j_\alpha(x) D^\alpha$ being an mth order differential operator with C^∞ coefficients for every $1 \leqslant j \leqslant k$. Denote $p_j(x, \xi) = \sum_{|\alpha|=m} a^j_\alpha(x)\xi^\alpha$ the full symbol of $P_j(x, D)$ for $1 \leqslant j \leqslant k$. The lower order terms $Q(x, D)$ contain bounded coefficients. As usual, the hypersurface S is assumed to be non-characteristic for $\mathbb{P}$ at x_0, i.e., $\prod_{j=1}^k p_j(x_0, N_0) \neq 0$, where $N_0 = d\psi(x_0)$. Before stating the main theorem of this section, we want to clearly describe the assumptions on the characteristic roots. For each $p_j(x, \xi)$, $1 \leqslant j \leqslant k$, we assume that

(C.1) there exist a conic neighborhood Γ_{N_0} and m functions $\{\lambda^j_\ell(x, \xi, \mathcal{N})\}_{\ell=1}^m$ which are C^∞ in $(x, \xi, \mathcal{N}) \in V \times (\mathbb{R}^n \setminus 0) \times \Gamma_{N_0}$ with $\xi \nparallel \mathcal{N}$ such that for every $\xi \nparallel \mathcal{N}$, $p_j(x, \xi + \tau\mathcal{N})$ is written as

$$p_j(x, \xi + \tau\mathcal{N}) = p_j(x, \mathcal{N}) \prod_{\ell=1}^m \big(\tau - \lambda^j_\ell(x, \xi, \mathcal{N})\big)$$

in $V \times (\mathbb{R}^n \setminus 0) \times \Gamma_{N_0}$;

(C.2) for any ℓ, $1 \leqslant \ell \leqslant m$, if $\lambda^j_\ell(x, \xi, \mathcal{N})$ is real (or complex) at one point, then it remains real (or complex) at every point;

(C.3) the real roots are simple and the multiplicity of the complex roots is not more than two.

With the conditions (C.1)–(C.3), we are able to prove the following uniqueness result:

Theorem 3.1. *Let* $\mathbb{P}(x, D)$ *be a system of differential operators defined as in* (3.2), *where* $P(x, D)$ *is of the form* (3.3) *with* C^∞ *coefficients and the lower order terms* $Q(x, D)$

contain only bounded coefficients. Assume that the characteristic roots conditions (C.1), (C.2), *and* (C.3) *hold. Then there exists a neighborhood V_0 of x_0 such that if $U \in C^\infty(V)$ is a k dimensional vector-valued function satisfying*:

$$\begin{cases} |\mathbb{P}U| \leqslant c\sum_{|\alpha|\leqslant m-1}|D^\alpha U|, & x \in V, \text{ for some constant } C > 0, \\ \partial^\alpha U|_S = 0, & |\alpha| \leqslant m-1, \end{cases}$$

then U vanishes identically in V_0.

We sketch the main ideas of the proof of Theorem 3.1 here. The interested reader is referred to [20, Chapter 2] for further details in which the scalar case was studied. As in [20], assuming $x_0 = 0$ and using the Holmgren transform:

$$x_i = x_i, \quad 1 \leqslant i \leqslant n-1, \quad t = \langle x, N_0\rangle + \delta|x|^2, \tag{3.4}$$

with a suitable constant $\delta > 0$, the principal part of $\mathbb{P}$ becomes:

$$P(x,t;D_x,D_t) = \operatorname{diag}\big[P_1(x,t,D_x,D_t),\ldots,P_k(x,t,D_x,D_t)\big].$$

For each $p_j(x,t,\xi,\tau)$, the full symbol of $P_j(x,t,D_x,D_t)$, $1 \leqslant j \leqslant k$, there exist functions $c^j(x,t)$ and $\{\lambda_\ell^j(x,t,\xi)\}_{\ell=1}^m$, such that

$$p_j(x,t,\xi,\tau) = c^j(x,t)\prod_{\ell=1}^m\big(\tau - \lambda_\ell^j(x,t,\xi)\big)$$

in $\widetilde{V} \times (\mathbb{R}^n \setminus 0)$, where $\widetilde{V}$ is a small neighborhood of $(0,0)$ and $c^j(x,t)$ is a C^∞ function with $c^j(0,0) \neq 0$ and $\lambda_\ell^j(x,t,\xi)$ is C^∞ in $\widetilde{V} \times (\mathbb{R}^n \setminus 0)$ with homogeneous of degree one in ξ, $1 \leqslant \ell \leqslant m$. Moreover, for any j, $\{\lambda_\ell^j(x,t,\xi)\}_{\ell=1}^m$ satisfy conditions (C.2) and (C.3). Since the result is local near $(0,0)$, it suffices to assume that the characteristic roots $\{\lambda_\ell^j\}_{\ell=1,\ j=1}^{m,k}$ vanish outside of a small neighborhood of $(0,0)$. Furthermore, it is readily seen that the new solution $\widetilde{U}$ under the transform (3.4) satisfies

$$\operatorname{supp}\widetilde{U} \subset \big\{(x,t) \in \mathbb{R}^n\colon\ t \geqslant \tilde{c}\|x\|^2\big\}$$

for some constant $\tilde{c}$.

Now the proof of Theorem 3.1 relies on the following Carleman estimate, which was proved in [20].

Lemma 3.1. [20] *For all j, $1 \leqslant j \leqslant k$, there exist positive constants c, T_0, η_0 and r such that for $T \leqslant T_0$ and $\eta \geqslant \eta_0$ we have that*

$$\sum_{|\alpha|\leqslant m-1}\int_0^T e^{\eta(t-T)^2}\|D^\alpha v\|^2_{L^2(\mathbb{R}^{n-1})}\,dt \leqslant c\left(T^2 + \frac{1}{\eta}\right)\int_0^T e^{\eta(t-T)^2}\|P_j v\|^2_{L^2(\mathbb{R}^{n-1})}\,dt$$

for any $v \in C^\infty(\mathbb{R}^n)$ *with* $\operatorname{supp} v \subset \{(x,t)\colon\ 0 \leqslant t \leqslant T,\ |x| \leqslant r\}$.

In view of Lemma 3.1, by taking T and η^{-1} sufficiently small, if necessary, we can easily show that

$$\sum_{|\alpha|\leqslant m-1} \int_0^T \mathrm{e}^{\eta(t-T)^2} \|D^\alpha W\|^2_{L^2(\mathbb{R}^{n-1})}\,\mathrm{d}t \leqslant c\left(T^2 + \frac{1}{\eta}\right) \int_0^T \mathrm{e}^{\eta(t-T)^2} \|\mathbb{P}W\|^2_{L^2(\mathbb{R}^{n-1})}\,\mathrm{d}t \tag{3.5}$$

for any k-dimensional vector-valued function $W \in C^\infty(\mathbb{R}^n)$ with $\operatorname{supp} W \subset \{(x,t)\colon\ 0 \leqslant t \leqslant T,\ |x| \leqslant r\}$. Now Theorem 3.1 follows from the Carleman estimate (3.5) by the standard arguments.

3.2. Transversely isotropic elasticity

Let the body $\mathcal{B}$ with reference configuration Ω be occupied by a transversely isotropic medium. More precisely, let the axis of rotational symmetry coincide with the x_3 axis, then the non-zero components of the elasticity tensor $C_{ijk\ell}(x)$ are,

$$C_{1111},\ C_{2222},\ C_{3333},\ C_{1122},\ C_{1133},\ C_{2233},\ C_{2323},\ C_{1313},\ C_{1212},$$

and they satisfy:

$$C_{1111} = C_{2222}, \quad C_{1133} = C_{2233}, \quad C_{2323} = C_{1313}, \quad C_{1212} = (C_{1111} - C_{1122})/2.$$

For notational simplicity, we set:

$$C_{1111} = A, \quad C_{1122} = N, \quad C_{1133} = F, \quad C_{3333} = C, \quad C_{2323} = L. \tag{3.6}$$

Notice that here the elasticity tensor $C(x)$ satisfies the full symmetry properties:

$$C_{ijkl}(x) = C_{klij}(x) = C_{jikl}(x) \quad \forall x \in \mathbb{R}^3,\ \forall i, j, k, l. \tag{3.7}$$

Instead of the strong ellipticity condition (2.1), we assume that the elasticity tensor satisfies the strong convexity condition, i.e., there exists $\delta > 0$ such that for any real symmetric matrix E

$$C(x)E \cdot E \geqslant \delta |E|^2 \quad \text{for all } x \in \overline{\Omega}. \tag{3.8}$$

In terms of (3.8), we obtain that

$$A > \tilde{\delta}, \quad C > \tilde{\delta}, \quad L > \tilde{\delta}, \quad \frac{1}{2}(A - N) > \tilde{\delta}, \quad (A + N)C - 2F^2 > \tilde{\delta} \tag{3.9}$$

in $x \in \overline{\Omega}$ for some $\tilde{\delta} > 0$. Now let $u(x)$ be the displacement vector, then the stationary elastic equation is given by:

$$(\mathcal{L}u)_i = \sum_{j=1}^{3} \partial_{x_j}\sigma_{ij} = 0 \quad \text{in } \Omega,\ 1 \leqslant i \leqslant 3, \tag{3.10}$$

where the stress-strain relation is:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} A & N & F & 0 & 0 & 0 \\ N & A & F & 0 & 0 & 0 \\ F & F & C & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & (A-N)/2 \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{pmatrix}.$$

It should be noted that the strong convexity condition implies the strong ellipticity condition for the elasticity tensor, which ensures that the system of Eqs. (3.10) is strongly elliptic.

In this section, we will study the weak unique continuation property and the Runge approximation property for (3.10). As before, the uniqueness of the Cauchy problem for (3.10) is key to our investigation. We aim to apply Theorem 3.1 to the case here. Of course, we first need to diagonalize the principal part of (3.10). A direct way is to use the cofactor of the principal part. The question is now whether the characteristic roots conditions (C.1), (C.2) and (C.3) are satisfied? By assuming the elasticity tensor $C_{ijk\ell}(x) \in C^{\infty}(\overline{\Omega})$ and in view of (3.8) (or (3.9)), we only have to check the smoothness of the characteristic roots in (C.1) and the multiplicity condition (C.3). It should be noted that when the characteristic roots are not smooth, Plis [15] constructed a fourth order elliptic differential operator for which the solution of the Cauchy problem is not unique. We will first discuss the multiplicity condition (C.3). It turns out we need to exclude certain directions and put some extra conditions on A, C, F, L, N in order to guarantee (C.3). The details are contained in the following lemma.

Lemma 3.2. *Let $\{\vec{\mathbf{m}}, \vec{\mathbf{n}}\}$ be a pair of orthogonal vectors in $\mathbb{R}^3$. Consider the characteristic equation*

$$\det\left(\sum_{j,\ell=1}^{3} C_{ijk\ell}\xi_j\xi_\ell;\ i\downarrow,\ k\to 1,2,3\right) = 0 \quad \textit{in } p, \tag{3.11}$$

where $\xi = (\xi_1, \xi_2, \xi_3) = \vec{\mathbf{m}} + p\vec{\mathbf{n}}$. Let $\vec{\ell} := \vec{\mathbf{m}} \times \vec{\mathbf{n}}$ and ϕ be the angle between $\vec{\ell}$ and the x_3 axis. Then the necessary and sufficient conditions for the characteristic roots of (3.11) *to be at most double are $\phi \neq 0$, $\phi \neq \pi$ and*

$$\sqrt{AC} - F - 2L \neq 0 \quad \textit{or} \quad \frac{2L}{A-N} - \sqrt{\frac{C}{A}} \neq 0. \tag{3.12}$$

Proof. The main idea of the proof is taken from Tanuma's paper [18] (see the Remark after Lemma in [18]). Let

$$\mathbf{Q} = (Q_{ik}) = \sum_{j,\ell=1}^{3} C_{ijk\ell} m_j m_\ell, \qquad \mathbf{R} = (R_{ik}) = \sum_{j,\ell=1}^{3} C_{ijk\ell} m_j n_\ell,$$
$$\mathbf{T} = (T_{ik}) = \sum_{j,\ell=1}^{3} C_{ijk\ell} n_j n_\ell,$$

where $i, k = 1, 2, 3$, and $\vec{\mathbf{m}} = (m_1, m_2, m_3)$, $\vec{\mathbf{n}} = (n_1, n_2, n_3)$, then the characteristic equation (3.11) is equivalent to

$$\det\big[\mathbf{Q} + (\mathbf{R} + \mathbf{R}^t) p + \mathbf{T} p^2\big] = 0. \tag{3.13}$$

From (3.9), we see that (3.13) contains only complex roots and they form conjugate pairs. Since the axis of rotational symmetry coincides with the x_3 axis, the elasticity tensor $C_{ijk\ell}$ is invariant under the orthogonal transform $\mathbf{O}$ rotating around the x_3 axis, i.e.,

$$\mathbf{O} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, the multiplicities of the characteristic roots for (3.13) are invariant under the same transform $\mathbf{O}$ on $\vec{\mathbf{m}}$ and $\vec{\mathbf{n}}$. Moreover, the multiplicities are also invariant to the rotation of the vectors $\vec{\mathbf{m}}$ and $\vec{\mathbf{n}}$ on the plane spanned by $\{\vec{\mathbf{m}}, \vec{\mathbf{n}}\}$. Thus, it suffices to prove this proposition for

$$\vec{\mathbf{m}} = (\cos\phi, 0, -\sin\phi), \qquad \vec{\mathbf{n}} = (0, 1, 0),$$

where $\vec{\ell} = (\sin\phi, 0, \cos\phi)$ (see [18]). Let $\xi_i = m_i + p n_i$, $i = 1, 2, 3$, then

$$\xi_1 = \cos\phi, \qquad \xi_2 = p, \qquad \xi_3 = -\sin\phi. \tag{3.14}$$

Now we have that

$$\begin{aligned} &\det\big[\mathbf{Q} + (\mathbf{R} + \mathbf{R}^t) p + \mathbf{T} p^2\big] \\ &\quad = \det\Bigg(\sum_{j,\ell=1}^{3} C_{ijk\ell} \xi_j \xi_\ell;\ i \downarrow,\ k \to 1, 2, 3\Bigg) = \big[(1/2)(A - N)\big(\xi_1^2 + \xi_2^2\big) + L\xi_3^2\big] \\ &\qquad \times \big[AL\big(\xi_1^2 + \xi_2^2\big)^2 + \big(AC - F^2 - 2FL\big)\big(\xi_1^2 + \xi_2^2\big)\xi_3^2 + CL\xi_3^4\big] \\ &\quad = 0. \end{aligned} \tag{3.15}$$

We take p_1, p_2, p_3 to be three roots of (3.15) with positive imaginary part. Let p_1 satisfies:

$$\frac{1}{2}(A - N)\big(\xi_1^2 + \xi_2^2\big) + L\xi_3^2 = \frac{1}{2}(A - N)\big(p^2 + \cos^2\phi\big) + L\sin^2\phi = 0 \tag{3.16}$$

and p_2, p_3 satisfy:

$$\begin{aligned} &AL\big(\xi_1^2+\xi_2^2\big)^2+\big(AC-F^2-2FL\big)\big(\xi_1^2+\xi_2^2\big)\xi_3^2+CL\xi_3^4 \\ &\quad =\big(\xi_1^2+\xi_2^2+G\xi_3^2\big)^2+J\xi_3^4=\big(p^2+\cos^2\phi+G\sin^2\phi\big)^2+J\sin^4\phi=0, \end{aligned} \tag{3.17}$$

where

$$G=\frac{AC-F^2-2FL}{2AL} \tag{3.18}$$

and

$$J=\frac{-(AC-F^2)(\sqrt{AC}+F+2L)(\sqrt{AC}-F-2L)}{4A^2L^2}. \tag{3.19}$$

We first prove the necessity. If $\phi=0$ or $\phi=\pi$, then from (3.16) and (3.17) we have that $p_1=p_2=p_3=i$, i.e., a triple root. Now assume that

$$\sqrt{AC}-F-2L=0 \quad \text{and} \quad \frac{2L}{A-N}-\sqrt{\frac{C}{A}}=0.$$

It is clear that $J=0$ and

$$p_1=p_2=p_3=\mathrm{i}\sqrt{\cos^2\phi+\sqrt{\frac{C}{A}}\sin^2\phi}$$

is also a triple root.

Next we want to prove the sufficiency. Suppose that $\phi\neq 0,\pi$. If $\sqrt{AC}-F-2L\neq 0$ (note that $L>0$), then $J\neq 0$. Thus, from (3.17), we must have that $p_2\neq p_3$ and the multiplicity is at most two. The last case we need to discuss is $\phi\neq 0,\pi$ and $\sqrt{AC}-F-2L=0$, but $2L/(A-N)\neq\sqrt{C/A}$. It immediately follows from (3.16) and (3.17) that $p_1\neq p_2=p_3$. Hence, the multiplicity is at most two. The proof of the lemma is now complete. □

In the next lemma, we will give some sufficient conditions on which the smoothness of the characteristic roots is guaranteed.

Lemma 3.3. *Let V be an open neighborhood of $x_0\in\Omega$ and Γ_{N_0} be any conic neighborhood of $N_0\in\mathbb{R}^3$. Assume that*

$$\sqrt{AC}-F-2L=0 \tag{3.20}$$

or

$$\sqrt{AC}-F-2L>0 \quad \textit{and} \quad F+L>0 \tag{3.21}$$

for all $x \in \overline{V}$. *Then the roots satisfying the characteristic equation* (3.11) *as defined in* (C.1) *are* C^∞ *in* $(x, \xi, \mathcal{N}) \in V \times (\mathbb{R}^3 \setminus 0) \times \Gamma_{N_0}$ *with* $\xi \nparallel \mathcal{N}$.

Proof. To prove this lemma, we only need to check the expression of the characteristic equation (3.11). From (3.15) in the proof of Lemma 3.2, we have that

$$\begin{aligned}
&\det\left(\sum_{j,\ell=1}^{3} C_{ijk\ell}\xi_j\xi_\ell;\ i\downarrow,\ k\to 1,2,3\right)\\
&\quad= \left[(1/2)(A-N)\left(\xi_1^2+\xi_2^2\right)+L\xi_3^2\right]\\
&\qquad\times\left[AL\left(\xi_1^2+\xi_2^2\right)^2+\left(AC-F^2-2FL\right)\left(\xi_1^2+\xi_2^2\right)\xi_3^2+CL\xi_3^4\right]\\
&\quad= AL\left[(1/2)(A-N)\left(\xi_1^2+\xi_2^2\right)+L\xi_3^2\right]\left[\left(\xi_1^2+\xi_2^2+G\xi_3^2\right)^2+J\xi_3^4\right],
\end{aligned}\tag{3.22}$$

where G and J are given in (3.18) and (3.19), respectively. In view of (3.9), we see that the first factor in (3.22),

$$(1/2)(A-N)\left(\xi_1^2+\xi_2^2\right)+L\xi_3^2,$$

represents a second order strongly elliptic operator in V. Now if the condition (3.20) holds, then $G=\sqrt{C/A}>0$ and $J=0$. We immediately obtain that the second factor in (3.22) is $(\xi_1^2+\xi_2^2+G\xi_3^2)^2$, which represents the principal symbol of the square of a second order strongly elliptic operator. Even though we may have a triple root in this case, the smoothness of the characteristic roots is obvious.

Next, we discuss the condition (3.21). It follows from (3.9) and (3.21) that

$$\begin{aligned}
AC-F^2-2FL &> 2FL+4L^2 = 2L(F+2L)>0,\\
AC-F^2 &> 4FL+4L^2 = 4L(F+L)>0
\end{aligned}$$

and therefore, $G>0$, $-J>0$. Furthermore, through straightforward computations, we can see that

$$\begin{aligned}
G^2-(-J) &= \left(1/4A^2L^2\right)\left\{\left(AC-F^2-2FL\right)^2\right.\\
&\qquad\left.-\left(AC-F^2\right)\left(\sqrt{AC}+F+2L\right)\left(\sqrt{AC}-F-2L\right)\right\}\\
&= \left(1/4A^2L^2\right)\left(4ACL^2\right)=C/A>0,\quad \forall x\in\overline{V}.
\end{aligned}\tag{3.23}$$

Therefore, the second factor in (3.22) becomes:

$$\begin{aligned}
&\left(\xi_1^2+\xi_2^2+G\xi_3^2\right)^2+J\xi_3^4\\
&\quad=\left(\xi_1^2+\xi_2^2+\left(G+\sqrt{(-J)}\right)\xi_3^2\right)\left(\xi_1^2+\xi_2^2+\left(G-\sqrt{(-J)}\right)\xi_3^2\right).
\end{aligned}\tag{3.24}$$

From (3.23) we obtain that $G \pm \sqrt{(-J)} > 0$ for $x \in \overline{V}$. Therefore, (3.24) represents the principal symbol of the composition of two second order strongly elliptic operators in V. In other words, the characteristic equation (3.11) can be treated as the principal symbol of the composition of three second order strongly elliptic operators in V. Hence, the smoothness of the characteristic roots can be easily verified. □

With the help of Lemmas 3.2 and 3.3, we are at the position of proving the uniqueness theorem of the Cauchy problem for the system of Eqs. (3.10).

Theorem 3.2. *Let $x_0 \in \mathbb{R}^3$ and V be a neighborhood of x_0. Assume that $S = \{x\colon \psi(x) = \psi(x_0)\}$ is a C^∞ surface with $N_0 = d\psi(x_0)$ satisfying*

$$N_0 \text{ is not perpendicular to the } x_3 \text{ axis}. \tag{3.25}$$

Let the elasticity tensor $C_{ijk\ell}$ with non-zero components defined by (3.6) *be C^∞ satisfying* (3.9) *and* (3.21) *or* (3.9) *and*

$$\sqrt{AC} - F - 2L = 0 \quad \text{and} \quad \frac{2L}{A-N} - \sqrt{\frac{C}{A}} \neq 0, \tag{3.26}$$

in V. Then if u is a C^∞ function in V solving the following Cauchy problem:

$$\begin{cases} \mathcal{L}u = 0 & \text{in } V, \\ \partial^\alpha u|_S = 0 & |\alpha| \leqslant 1, \end{cases} \tag{3.27}$$

then u vanishes identically in some neighborhood V_0 of x_0.

Proof. Let $P(x, D)$ be the principal part of $\mathcal{L}$ with symbol $p(x,\xi) = (p_{ik}(x,\xi))$, where

$$p_{ik}(x,\xi) = \sum_{j,\ell=1}^{3} C_{ijk\ell}(x)\xi_j\xi_\ell.$$

Let $q(x,\xi)$ be the cofactor of $p(x,\xi)$ with the corresponding operator $Q(x, D)$, which is a fourth order strongly elliptic system. It is not hard to see that

$$R(x,D) := Q\mathcal{L} = r(x,D)I + \text{l.o.t.},$$

with principal symbol

$$r(x,\xi) = \det\left(\sum_{j,\ell=1}^{3} C_{ijk\ell}(x)\xi_j\xi_\ell;\ i\downarrow,\ k \to 1,2,3\right).$$

Now one can see that u is also a solution to the following new Cauchy problem:

$$\begin{cases} R(x,D)u = 0 & \text{in } V, \\ \partial^\alpha u|_S = 0 & |\alpha| \leqslant 5. \end{cases} \tag{3.28}$$

Note that the Cauchy data in (3.28) is easily obtained from (3.27) since S is noncharacteristic near x_0. In view of (3.25) and (3.21) (or (3.26)), it follows from Lemma 3.2 and Lemma 3.3 that the characteristic roots conditions (C.1), (C.2) and (C.3) are satisfied for $R(x,D)$ near N_0. By using Theorem 3.1, we obtain that u vanishes in some neighborhood of x_0. □

In Theorem 3.2, we see that the uniqueness of the Cauchy problem for (3.10) near directions perpendicular to the x_3 axis can not be proved by the techniques used in Section 2. Neither can the weak unique continuation property for (3.10) be inferred from Theorem 3.2. In spite of this fact, for convex inner domains, we are still able to prove the classical Runge approximation property for (3.10) without the restriction on the support of the Dirichlet data. Similar results for some elliptic operators can be found in [11,13] (or in [10] where the application of the classical Runge approximation property to the inverse conductivity problem was discussed). It should be noted that the classical Runge approximation property for (3.10) does not follow directly from the results in [11] or [13] because we do not have the uniqueness of the Cauchy problem for (3.10) near every direction.

Theorem 3.3. *Let O and Ω be two open bounded domains with C^∞ boundary in $\mathbb{R}^3$ such that O is convex and $\overline{O} \subset \Omega$. Assume that the coefficients A, C, F, N, L of the transversely isotropic elasticity tensor are $C^\infty(\overline{\Omega})$ and satisfy* (3.9) *and* (3.21) (*or* (3.26)) *for all $x \in \overline{\Omega}$. Let $u \in H^1(O)$ satisfy*

$$\mathcal{L}u = 0 \quad \text{in } O.$$

Then for any compact subset $K \subset O$ and any $\varepsilon > 0$ there exists $U \in H^1(\Omega)$ solving,

$$\mathcal{L}U = 0 \quad \text{in } \Omega,$$

such that

$$\|U - u\|_{H^1(K)} < \varepsilon.$$

Proof. Let $\widetilde{K}$ be a compact subset in O such that $K \subset \operatorname{int}(\widetilde{K})$ and $\Omega \setminus \widetilde{K}$ is connected. Applying the interior estimate to $U - u$ satisfying $\mathcal{L}(U-u) = 0$ in $\operatorname{int}(\widetilde{K})$, we have that

$$\|U - u\|_{H^1(K)} \leqslant \kappa \|U - u\|_{L^2(\widetilde{K})}$$

for some constant $\kappa > 0$ (see for example [2]). Therefore, proving this theorem is equivalent to showing that

$$X = \left\{w:\ w = v|_{\widetilde{K}},\ v \in H^1(\Omega),\ \mathcal{L}v = 0 \text{ in } \Omega\right\}$$

is dense in

$$Y = \{w\colon\ w = v|_{\widetilde{K}},\ v \in H^1(O),\ \mathcal{L}v = 0 \text{ in } O\}$$

in terms of $L^2(\widetilde{K})$. By the Hahn–Banach theorem, this is equivalent to the following statement. If $f \in L^2(\widetilde{K})$ such that

$$(f, w)_{L^2(\widetilde{K})} = 0 \quad \text{for all } w \text{ in } X, \tag{3.29}$$

then

$$(f, w)_{L^2(\widetilde{K})} = 0 \quad \text{for all } w \text{ in } Y. \tag{3.30}$$

Now let $f \in L^2(\widetilde{K})$ satisfy (3.29) and define:

$$\tilde{f} = \begin{cases} f & \text{in } \widetilde{K}, \\ 0 & \text{in } \Omega \setminus \widetilde{K}. \end{cases}$$

Let $\tilde{u}$ be the solution of:

$$\begin{cases} \mathcal{L}\tilde{u} = \tilde{f} & \text{in } \Omega, \\ \tilde{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.31}$$

then for any $v \in H^1(\Omega)$ with $\mathcal{L}v = 0$ in Ω we have that

$$0 = \int_{\widetilde{K}} f \cdot v \,\mathrm{d}x = \int_{\Omega} \tilde{f} \cdot v \,\mathrm{d}x = \int_{\Omega} \mathcal{L}\tilde{u} \cdot v \,\mathrm{d}x = \int_{\partial\Omega} \sigma(\tilde{u})\nu \cdot v \,\mathrm{d}s, \tag{3.32}$$

where

$$\sigma(\tilde{u})\nu = (C\nabla\tilde{u})\nu,$$

which is known as the surface traction. It follows from (3.32) that $\sigma(\tilde{u})\nu = 0$ on $\partial\Omega$. Combining this result with (3.31), we obtain that $\tilde{u}$ is the solution to:

$$\begin{cases} \mathcal{L}\tilde{u} = \tilde{f} & \text{in } \Omega, \\ \tilde{u} = 0, \sigma(\tilde{u})\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

In other words, on $\Omega \setminus \widetilde{K}$, we have that

$$\begin{cases} \mathcal{L}\tilde{u} = 0 & \text{in } \Omega \setminus \widetilde{K}, \\ \tilde{u} = 0,\ \sigma(\tilde{u})\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.33}$$

Now we are going to show that

$$\partial^\alpha \tilde{u} = 0 \quad \text{on } \partial O \text{ for } |\alpha| \leqslant 1. \tag{3.34}$$

If (3.34) is true, then

$$\int_{\widetilde{K}} f \cdot v \,\mathrm{d}x = \int_{O} \mathcal{L}\tilde{u} \cdot v \,\mathrm{d}x = 0$$

for any $v \in H^1(O)$ with $\mathcal{L}v = 0$ in O; therefore, (3.30) holds.

To prove the statement (3.34) by Theorem 3.2, it is easier and clearer to work on a simple domain such as a ball rather than the general domain Ω. Therefore, we pick a domain $\widetilde{\Omega}$ with Lipschitz boundary such that $\Omega \subset \overline{B} \subset \widetilde{\Omega}$, where B is an open ball. Suppose that the coefficients A, C, F, N, L are extended to $\widetilde{\Omega}$, still denoted by the same notations, such that they remain C^∞ in $\widetilde{\Omega}$ and satisfy (3.9), (3.21) or (3.9), (3.26) for all $x \in \widetilde{\Omega}$. Let Φ be defined by:

$$\Phi(x) = \begin{cases} \tilde{u}(x) & \text{in } \Omega, \\ 0 & \text{in } \widetilde{\Omega} \setminus \Omega, \end{cases}$$

then

$$\int_{\widetilde{\Omega}\setminus\widetilde{K}} \Phi \cdot \mathcal{L}\varphi \,\mathrm{d}x = \int_{\Omega\setminus\widetilde{K}} \tilde{u} \cdot \mathcal{L}\varphi \,\mathrm{d}x = \int_{\Omega\setminus\widetilde{K}} \mathcal{L}\tilde{u} \cdot \varphi + \int_{\partial(\Omega\setminus\widetilde{K})} \big(\sigma(\tilde{u})\nu \cdot \varphi - \sigma(\varphi)\nu \cdot \tilde{u}\big) \,\mathrm{d}s = 0$$

for any test function $\varphi \in C_0^\infty(\widetilde{\Omega} \setminus \widetilde{K})$. Therefore Φ satisfies:

$$\mathcal{L}\Phi = 0 \quad \text{in } \widetilde{\Omega} \setminus \widetilde{K} \text{ in the sense of distribution.}$$

In view of $\tilde{u}|_{\partial\Omega} = 0$ and the definition of Φ, we obtain $\Phi \in H^1(\widetilde{\Omega} \setminus \widetilde{K})$. By the elliptic regularity theorem [2], we get that $\Phi \in C^\infty(\widetilde{\Omega} \setminus \widetilde{K})$. Therefore, solving (3.33) is equivalent to solving the Cauchy problem:

$$\begin{cases} \mathcal{L}\Phi = 0 & \text{in } B \setminus \widetilde{K}, \\ \partial^\alpha \Phi = 0 & \text{on } \partial B \text{ for } |\alpha| \leqslant 1. \end{cases} \tag{3.35}$$

Now we denote $\mathrm{co}(\widetilde{K})$ the closed convex hull of $\widetilde{K}$. It is obvious that $\mathrm{co}(\widetilde{K}) \subset O$. Let Γ_τ, $\tau \in [0, 1]$, be a continuous deformation of surfaces such that

(i) for each $\tau \in [0, 1]$, Γ_τ is a C^∞ surface contained in $B \setminus \mathrm{co}(\widetilde{K})$ with boundary $\partial\Gamma_\tau$ lies in $\partial B \setminus \omega$, where ω is the equator on ∂B, i.e., the normal vector to ∂B at $x \in \omega$ is perpendicular to the x_3 axis;
(ii) for each $\tau \in [0, 1]$, the surface Γ_τ contains no normal vector perpendicular to the x_3 axis;
(iii) $\Gamma_0 \subset \partial B \setminus \omega$.

With the definition of Γ_τ, combining Theorem 3.2 and Fritz John's arguments [9], we can show that Φ vanishes on any set swept out by Γ_τ for $\tau \in [0,1]$. It is readily seen that each point $x \in B \setminus \mathrm{co}(\widetilde{K})$ can be covered by suitably chosen Γ_τ satisfying (i)–(iii). Therefore, we conclude that $\Phi = 0$ in $B \setminus \mathrm{co}(\widetilde{K})$, in particular, (3.34) holds. The proof is now complete. □

4. Detecting the convex hull of inclusions or cavities in a transversely isotropic body

As an application of what we have established in the previous sections, we study the inverse problem of determining the convex hull of an inclusion or cavity embedded in a transversely isotropic body. We first consider the inclusion case.

4.1. Reconstruction of inclusions

As before, let $\Omega \in \mathbb{R}^3$ be a bounded domain with smooth boundary. Assume that D is a subset of Ω. We called D an inclusion embedded in Ω. Here we do not assume $\overline{D} \subset \Omega$ or D is open. Suppose that the elasticity tensor $C(x)$ takes the form:

$$C(x) = C_0(x) + \chi_D H(x),$$

where $C_0(x) \in B^\infty(\mathbb{R}^3)$ is a transversely isotropic elasticity tensor satisfying (3.7), (3.8) and the assumptions in Theorem 3.3, χ_D is the characteristic function D and $H(x) \in L^\infty(D)$ is a fourth-rank tensor so that the whole $C(x)$ satisfies (3.7) and (3.8) with possibly different constant δ. Then, for any $f \in H^{1/2}(\partial\Omega)$, there exists a unique weak solution to the boundary value problem:

$$\begin{cases} \mathcal{L}_C w = \nabla \cdot (C(x)\nabla w) = 0 & \text{in } \Omega, \\ w|_{\partial\Omega} = f. \end{cases}$$

We now define the Dirichlet-to-Neumann map $\Lambda_D : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ by the formula

$$\langle \Lambda_D(f), g \rangle = \int_\Omega C\nabla w \cdot \nabla v \, \mathrm{d}x,$$

where $g \in H^{1/2}(\partial\Omega)$ and v is any function in $H^1(\Omega)$ with $v|_{\partial\Omega} = g$. Here we assume that both D and H are unknown. We are interested in the inverse problem of determining the information of the inclusion D by the knowledge of $\Lambda_D(f)$ for infinitely many f. We want to remark that the same inverse problem associated with the anisotropic elasticity system has been studied in [7] where the reference elasticity tensor C_0 is assumed to be homogeneous. In which case, the Runge approximation property is trivial. For the case considered here, we have a special type of Runge approximation property whose inner domain is required to be convex (see Theorem 3.3). This Runge approximation property is not enough to solve the inverse problem of determining the full information of D. Nevertheless, with the help the oscillating–decaying solutions, we can prove that one can determine the convex hull of D by $\Lambda_D(f)$ for infinitely many f.

To begin, we define B an open ball in $\mathbb{R}^3$ such that $\overline{\Omega} \subset B$. Assume that $\widetilde{\Omega} \subset \mathbb{R}^3$ is an open set with Lipschitz boundary satisfying $\overline{B} \subset \widetilde{\Omega}$. Defined as before, $\omega \in \mathbb{S}^2$ and $\{\eta, \zeta, \omega\}$ forms an orthonormal basis of $\mathbb{R}^3$. Suppose $t_0 = h_D(\omega) = \inf_{x\in D} x \cdot \omega = x_0 \cdot \omega$, where $x_0 = x_0(\omega) \in \partial D$. For any $t \leqslant t_0$ and $\varepsilon > 0$ sufficiently small, let

$$\begin{aligned} u_{\chi_{t-\varepsilon},b,t-\varepsilon,N,\omega}(x,\tau) &= \chi_{t-\varepsilon}(x') Q_{t-\varepsilon} \mathrm{e}^{\mathrm{i}\tau x\cdot\xi} \mathrm{e}^{-\tau(x\cdot\omega-(t-\varepsilon))A_{t-\varepsilon}(x')} b + \gamma_{\chi_{t-\varepsilon},b,t-\varepsilon,N,\omega} \\ &\quad + r_{\chi_{t-\varepsilon},b,t-\varepsilon,N,\omega} \end{aligned}$$

be the oscillating–decaying solution for $\mathcal{L}_{C_0}$ in $B_{t-\varepsilon}(\omega) := B \cap \{x \cdot \omega > t - \varepsilon\}$ as constructed in Section 2, where $\chi_{t-\varepsilon}(x') \in C_0^\infty(\mathbb{R}^2)$ and $b \in \mathbb{C}^3$. Here $A_{t-\varepsilon}(x')$ is in fact given by $-\mathrm{i}\widetilde{K}_+(x)$ with $x \in \Sigma_{t-\varepsilon}(\omega) = B \cap \{x \cdot \omega = t - \varepsilon\}$ (see (2.16)). Related to $u_{\chi_{t-\varepsilon},b,t-\varepsilon,N,\omega}(x,\tau)$, we can establish,

$$u_{\chi_t,b,t,N,\omega}(x,\tau) = \chi_t(x') Q_t \mathrm{e}^{\mathrm{i}\tau x\cdot\xi} \mathrm{e}^{-\tau(x\cdot\omega-t)A_t(x')} b + \gamma_{\chi_t,b,t,N,\omega} + r_{\chi_t,b,t,N,\omega}$$

which is an oscillating–decaying solution for $\mathcal{L}_{C_0}$ in $B_t(\omega)$. Let $\varepsilon_0 > 0$ be a sufficiently small given number. We take $\chi_{t-\varepsilon}(x') = \chi_t(x') = \tilde{\chi}(x')$ for all $0 < \varepsilon < \varepsilon_0$, where $\tilde{\chi} \in C_0^\infty(\mathbb{R}^2)$ with $\operatorname{supp}(\tilde{\chi}) \subset \bigcap_{0\leqslant\varepsilon<\varepsilon_0} \Pi_t \Sigma_{t-\varepsilon}(\omega)$. Here $\Pi_t \Sigma_{t-\varepsilon}(\omega)$ denotes the projection of $\Sigma_{t-\varepsilon}(\omega)$ onto the $\{x \cdot \omega = t\}$ plane. Before going further, we want to show that for any τ, $u_{\chi_{t-\varepsilon},b,t-\varepsilon,N,\omega}(x,\tau)$ converges to $u_{\chi_t,b,t,N,\omega}(x,\tau)$ in an appropriate sense as $\varepsilon \to 0$. Indeed, since $C_0(x) \in B^\infty(\mathbb{R}^3)$, we can see that for any τ:

$$\begin{aligned} &\chi_{t-\varepsilon}(x') Q_{t-\varepsilon} \mathrm{e}^{\mathrm{i}\tau x\cdot\xi} \mathrm{e}^{-\tau(x\cdot\omega-(t-\varepsilon))A_{t-\varepsilon}(x')} b \xrightarrow{\varepsilon} \chi_t(x') Q_t \mathrm{e}^{\mathrm{i}\tau x\cdot\xi} \mathrm{e}^{-\tau(x\cdot\omega-t)A_t(x')} b \\ &\quad \text{in } H^2\big(B_t(\omega)\big). \end{aligned} \tag{4.1}$$

Likewise, $\gamma_{\chi_{t-\varepsilon},b,t-\varepsilon,N,\omega}$ is obtained by solving the system (2.23) with coefficients depending smoothly on ε. Using the property of continuous dependence on parameters in ordinary differential equations, one can easily deduce that for any τ:

$$\gamma_{\chi_{t-\varepsilon},b,t-\varepsilon,N,\omega} \xrightarrow{\varepsilon} \gamma_{\chi_t,b,t,N,\omega} \quad \text{in } H^2\big(B_t(\omega)\big). \tag{4.2}$$

On the other hand, we recall that $r_{\chi_{t-\varepsilon},b,t-\varepsilon,N,\omega} \in H_0^1(B_{t-\varepsilon}(\omega))$ satisfies:

$$\begin{aligned} \mathcal{L}_{C_0}(r_{\chi_{t-\varepsilon},b,t-\varepsilon,N,\omega}) &= -\mathrm{e}^{\mathrm{i}\tau x'\cdot\xi'} \mathcal{L}_{C_0}\big(\chi_{t-\varepsilon}(x') Q_{t-\varepsilon} \mathrm{e}^{\mathrm{i}\tau x\cdot\xi} \mathrm{e}^{-\tau(x\cdot\omega-(t-\varepsilon))A_{t-\varepsilon}(x')} b \\ &\quad + \gamma_{\chi_{t-\varepsilon},b,t-\varepsilon,N,\omega}\big) \quad \text{in } B_{t-\varepsilon}(\omega). \end{aligned} \tag{4.3}$$

Since C_0 is infinitely smooth and so is the right hand term of (4.3), by the elliptic regularity theorem, $r_{\chi_{t-\varepsilon},b,t-\varepsilon,N,\omega}$ is infinitely smooth on $B_{t-\varepsilon}(\omega) \cup (\partial B_{t-\varepsilon}(\omega) \setminus \partial\Sigma_{t-\varepsilon}(\omega))$. So it is clear that

$$r_{\chi_{t-\varepsilon},b,t-\varepsilon,N,\omega}|_{\Sigma_t(\omega)} \xrightarrow{\varepsilon} 0 \quad \text{in } H^{1/2}\big(\Sigma_t(\omega)\big).$$

Denote $z_\varepsilon = r_{\chi_{t-\varepsilon},b,t-\varepsilon,N,\omega} - r_{\chi_t,b,t,N,\omega}$. Then we have that

$$\begin{cases} \mathcal{L}_{C_0} z_\varepsilon = f_\varepsilon & \text{in } B_t(\omega), \\ z_\varepsilon|_{\partial B_t(\omega)} \stackrel{\varepsilon}{\to} 0 & \text{in } H^{1/2}(\partial B_t(\omega)), \end{cases}$$

where $\|f_\varepsilon\|_{L^2(B_t(\omega))} \stackrel{\varepsilon}{\to} 0$ due to estimates (4.1) and (4.2). By the elliptic regularity theorem, we therefore deduce that

$$z_\varepsilon \stackrel{\varepsilon}{\to} 0 \quad \text{in } H^1\big(B_t(\omega)\big),$$

i.e.,

$$r_{\chi_{t-\varepsilon},b,t-\varepsilon,N,\omega} \stackrel{\varepsilon}{\to} r_{\chi_t,b,t,N,\omega} \quad \text{in } H^1\big(B_t(\omega)\big).$$

In summary, we have proved that, for any τ,

$$u_{\chi_{t-\varepsilon},b,t-\varepsilon,N,\omega}(x,\tau) \stackrel{\varepsilon}{\to} u_{\chi_t,b,t,N,\omega}(x,\tau) \quad \text{in } H^1\big(B_t(\omega)\big). \tag{4.4}$$

Obviously, $B_{t-\varepsilon}(\omega)$ is a convex set and $\overline{\Omega}_t(\omega) \subset B_{t-\varepsilon}$ for all $t \leqslant t_0$. Thus, by Theorem 3.3, we can see that there exist a sequence of functions $\tilde{u}_{\varepsilon,j}$, $j = 1, 2, \ldots,$ such that

$$\tilde{u}_{\varepsilon,j} \stackrel{j}{\to} u_{\chi_{t-\varepsilon},b,t-\varepsilon,N,\omega} \quad \text{in } H^1\big(\Omega_t(\omega)\big), \tag{4.5}$$

where $\tilde{u}_{\varepsilon,j} \in H^1(\widetilde{\Omega})$ satisfy $\mathcal{L}_{C_0}\tilde{u}_{\varepsilon,j} = 0$ in $\widetilde{\Omega}$ for all ε, j. Define the indicator function $I(\tau, \chi_t, b, t, \omega)$ by the formula:

$$I(\tau, \chi_t, b, t, \omega) = \lim_{\varepsilon\to 0} \lim_{j\to\infty} \big\langle (\Lambda_D - \Lambda_0)\tilde{u}_{\varepsilon,j}|_{\partial\Omega}, \overline{\tilde{u}_{\varepsilon,j}|_{\partial\Omega}} \big\rangle. \tag{4.6}$$

Here Λ_0 is the Dirichlet-to-Neumann map related to $\mathcal{L}_{C_0}$ in Ω, i.e.,

$$\big\langle \Lambda_0(f), g \big\rangle = \int_\Omega C_0 \nabla u \cdot \nabla v \, \mathrm{d}x,$$

where u is the solution to:

$$\begin{cases} \mathcal{L}_{C_0} u = \nabla \cdot (C_0 \nabla u) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = f \end{cases}$$

and $g \in H^{1/2}(\partial\Omega)$ and v is any function in $H^1(\Omega)$ with $v|_{\partial\Omega} = g$. We assume that H satisfies the jump condition:

$$H(x)E \cdot E \geqslant C_\omega |E|^2 \text{ for almost all } x \in D_\omega(\delta_\omega) \text{ and for all real symmetric matrix } E \tag{4.7}$$

or

$$H(x)E \cdot E \leqslant -C_\omega |E|^2 \text{ for almost all } x \in D_\omega(\delta_\omega) \text{ and for all real symmetric matrix } E, \tag{4.8}$$

where

$$D_\omega(\delta_\omega) = \{x \in D\colon\ h_D(\omega) \leqslant x \cdot \omega < h_D(\omega) + \delta_\omega\} \quad \text{with } 0 < \delta_\omega \ll 1$$

and C_ω is a positive constant. Additionally, in dealing with the inverse problem, we assume that D satisfies the following condition: for each $\omega \in \mathbb{S}^2$, there exist $c_\omega > 0$, $\varepsilon_\omega > 0$ and $p_\omega \in [0, 1]$ such that

$$\frac{1}{c_\omega} s^{p_\omega} \leqslant \mu(\{x \in D\colon\ x \cdot \omega = t_0 + s\}) \leqslant c_\omega s^{p_\omega} \quad \text{for all } s \in (0, \varepsilon_\omega), \tag{4.9}$$

where μ is the surface measure. Note that the assumption of D is quite similar to the one used by Ikehata in [5]. It is not hard to check that if $\partial D \in C^2$ and has nonzero curvature everywhere, then D satisfies (4.9) with $p_\omega = 1$. In addition to (4.9), we assume that $D_\omega(\delta_\omega)$ satisfies the *cone property* for all sufficiently small δ_ω. The purpose of imposing the cone property is to validate Korn's inequality in δ_ω. Now the characterization of the convex hull of D is based on the following theorem:

Theorem 4.1. (i) *If $t < t_0$, then for any $\chi_t \in C_0^\infty(\mathbb{R}^2)$ and $b \in \mathbb{C}^3$, we have that $I(\tau, \chi_t, b, t, \omega) \to 0$ as $\tau \to \infty$;*

(ii) *If $t = t_0$, then for any $\chi_{t_0} \in C_0^\infty(\mathbb{R}^2)$ with $x_0' = (x_0 \cdot \eta, x_0 \cdot \zeta)$ being an interior point of* $\operatorname{supp}(\chi_{t_0})$ *and $0 \neq b \in \mathbb{C}^3$, we get that $|I(\tau, \chi_{t_0}, b, t_0, \omega)| \geqslant c\tau^{1-p_\omega}$ (as $\tau \to \infty$) for some positive constant c.*

Proof. We first recall a formula established in [4]:

$$\begin{aligned} \int_D \{C_0^{-1} - (C_0 + H)^{-1}\} C_0 \nabla u \cdot C_0 \overline{\nabla u} \, \mathrm{d}x &\leqslant \langle (\Lambda_D - \Lambda_0) u|_{\partial\Omega}, \overline{u|_{\partial\Omega}} \rangle \\ &\leqslant \int_D H \nabla u \cdot \overline{\nabla u} \, \mathrm{d}x, \end{aligned} \tag{4.10}$$

where u satisfies $\mathcal{L}_{C_0} u = 0$ in Ω. Here C_0^{-1} and $(C_0 + H)^{-1}$ are called the compliance tensors (see [3]). Notice that C_0^{-1} (also $(C_0 + H)^{-1}$) satisfies the estimate (2.1) with possibly different constant δ. Using (4.4), (4.5) and (4.10), we can obtain that for $t \leqslant t_0$ and any $\chi_t \in C_0^\infty(\mathbb{R}^2)$, $b \in \mathbb{C}^3$,

$$\int_D \left\{C_0^{-1} - (C_0+H)^{-1}\right\} C_0 \nabla u_{\chi_t,b,t,N,\omega}(x,\tau) \cdot C_0 \overline{\nabla u_{\chi_t,b,t,N,\omega}(x,\tau)}\, \mathrm{d}x$$
$$\leqslant I(\tau,\chi_t,b,t,\omega) \leqslant \int_D H \nabla u_{\chi_t,b,t,N,\omega}(x,\tau) \cdot \overline{\nabla u_{\chi_t,b,t,N,\omega}(x,\tau)}\, \mathrm{d}x. \tag{4.11}$$

It is readily seen that

$$\left| \int_D \left\{C_0^{-1} - (C_0+H)^{-1}\right\} C_0 \nabla u_{\chi_t,b,t,N,\omega}(x,\tau) \cdot C_0 \overline{\nabla u_{\chi_t,b,t,N,\omega}(x,\tau)}\, \mathrm{d}x \right|$$
$$\leqslant c \int_{\Omega_{t_0}(\omega)} \left|\nabla u_{\chi_t,b,t,N,\omega}(x,\tau)\right|^2 \mathrm{d}x \tag{4.12}$$

and

$$\left| \int_D H \nabla u_{\chi_t,b,t,N,\omega}(x,\tau) \cdot \overline{\nabla u_{\chi_t,b,t,N,\omega}(x,\tau)}\, \mathrm{d}x \right|$$
$$\leqslant c \int_{\Omega_{t_0}(\omega)} \left|\nabla u_{\chi_t,b,t,N,\omega}(x,\tau)\right|^2 \mathrm{d}x. \tag{4.13}$$

Now if $t < t_0$, we substitute $u_{\chi_t,b,t,N,\omega} = w_{\chi_t,b,t,N,\omega} + r_{\chi_t,b,t,N,\omega}$ with $w_{\chi_t,b,t,N,\omega}$ being described by (2.3) into (4.12), (4.13) and use estimates (2.4), (2.5) to obtain that

$$\left|I(\tau,\chi_t,b,t,\omega)\right| \leqslant c\tau^{-2N-1}$$

for $\tau \gg 1$, which immediately implies (i). To prove (ii), we first consider the jump condition (4.7) of H. Using the formula,

$$\begin{aligned}
&\left(C_0^{-1} - (C_0+H)^{-1}\right) C_0 \nabla u \cdot C_0 \nabla u \\
&\quad = H(C_0+H)^{-1} C_0 \nabla u \cdot (C_0+H)^{-1} C_0 \nabla u \\
&\qquad + C_0^{-1} H (C_0+H)^{-1} C_0 \nabla u \cdot H (C_0+H)^{-1} C_0 \nabla u
\end{aligned} \tag{4.14}$$

(see [7] for the derivation of (4.14)), the jump condition (4.7) and the strong convexity conditions for corresponding elasticity and compliance tensors in (4.14), we get that

$$\left(C_0^{-1} - (C_0+H)^{-1}\right) C_0 \nabla u \cdot C_0 \nabla u \geqslant c |\operatorname{Sym} \nabla u|^2 \quad \text{for all } x \in D_\omega(\delta_\omega), \tag{4.15}$$

where $\operatorname{Sym} A$ is the symmetric part of the matrix A. Combining the first half of (4.11) and (4.15) yields:

$$
\begin{aligned}
&I(\tau,\chi_{t_0},b,t_0,\omega)\\
&\quad\geqslant \int\limits_{D\setminus D_\omega(\delta_\omega)} \bigl\{C_0^{-1}-(C_0+H)^{-1}\bigr\}C_0\nabla u_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)\cdot C_0\overline{\nabla u_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)}\,\mathrm{d}x\\
&\qquad+\int\limits_{D_\omega(\delta_\omega)} \bigl\{C_0^{-1}-(C_0+H)^{-1}\bigr\}C_0\nabla u_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)\cdot C_0\overline{\nabla u_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)}\,\mathrm{d}x\\
&\quad\geqslant \int\limits_{D\setminus D_\omega(\delta_\omega)} \bigl\{C_0^{-1}-(C_0+H)^{-1}\bigr\}C_0\nabla u_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)\cdot C_0\overline{\nabla u_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)}\,\mathrm{d}x\\
&\qquad+c\int\limits_{D_\omega(\delta_\omega)} \bigl|\mathrm{Sym}\,\nabla u_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)\bigr|^2\,\mathrm{d}x,
\end{aligned}\tag{4.16}
$$

where $u_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)$ is the associated oscillating–decaying solution of $\mathcal{L}_{C_0}$ in $B_{t_0}(\omega)$. Argued as above, we can show that for $\tau\gg 1$,

$$
\begin{aligned}
&\left|\int\limits_{D\setminus D_\omega(\delta_\omega)} \bigl\{C_0^{-1}-(C_0+H)^{-1}\bigr\}C_0\nabla u_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)\cdot C_0\overline{\nabla u_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)}\,\mathrm{d}x\right|\\
&\quad\leqslant c\tau^{-2N-1}.
\end{aligned}\tag{4.17}
$$

To deal with the other term on the right side of (4.16), we will use the condition (4.9) on D. Let $0<\tilde{\delta}<\min\{\delta_\omega,\varepsilon_\omega\}$ be chosen such that $D_\omega(\tilde{\delta})\subset\{x\colon\ x'=(x\cdot\eta,x\cdot\zeta)\in \mathrm{supp}(\chi_{t_0}),\ x\cdot\omega\geqslant t_0\}$, then using Korn's inequality we can obtain that for $\tau\gg 1$:

$$
\begin{aligned}
&\int\limits_{D_\omega(\delta_\omega)} \bigl|\mathrm{Sym}\,\nabla u_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)\bigr|^2\,\mathrm{d}x\\
&\quad\geqslant \int\limits_{D_\omega(\tilde{\delta})} \bigl|\mathrm{Sym}\,\nabla u_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)\bigr|^2\,\mathrm{d}x\\
&\quad\geqslant c\int\limits_{D_\omega(\tilde{\delta})} \bigl|\nabla u_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)\bigr|^2\,\mathrm{d}x - c'\int\limits_{D_\omega(\tilde{\delta})} \bigl|u_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)\bigr|^2\,\mathrm{d}x\\
&\quad= c\int\limits_{D_\omega(\tilde{\delta})} \bigl|\nabla(\chi_{t_0}Q_{t_0}\mathrm{e}^{\mathrm{i}\tau x\cdot\xi}\mathrm{e}^{-\tau(x\cdot\omega-t_0)A_{t_0}(x')}b+\gamma_{\chi_{t_0},b,t_0,N,\omega}+r_{\chi_{t_0},b,t_0,N,\omega})\bigr|^2\,\mathrm{d}x\\
&\qquad- c'\int\limits_{D_\omega(\tilde{\delta})} \bigl|\chi_{t_0}Q_{t_0}\mathrm{e}^{\mathrm{i}\tau x\cdot\xi}\mathrm{e}^{-\tau(x\cdot\omega-t_0)A_{t_0}(x')}b+\gamma_{\chi_{t_0},b,t_0,N,\omega}+r_{\chi_{t_0},b,t_0,N,\omega}\bigr|^2\,\mathrm{d}x
\end{aligned}
$$

$$\begin{aligned}
&\geqslant c\tau^2 \int\limits_{D_\omega(\tilde\delta)} \left(\left|Q_{t_0}\mathrm{e}^{-\tau(x\cdot\omega-t_0)A_{t_0}}b\right|^2 + \left|Q_{t_0}\mathrm{e}^{-\tau(x\cdot\omega-t_0)A_{t_0}}A_{t_0}b|^2\right)\mathrm{d}x \\
&\quad - c\tau \int\limits_{D_\omega(\tilde\delta)} |\mathrm{e}^{-\tau(x\cdot\omega-t_0)A_{t_0}}|^2\,\mathrm{d}x \\
&\quad - c\tau\left\|\mathrm{e}^{-\tau(x\cdot\omega-t_0)A_{t_0}}\right\|_{L^2(D_\omega(\tilde\delta))}\left\|\nabla\gamma_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)\right\|_{L^2(D_\omega(\tilde\delta))} - c/\tau \\
&\geqslant c\tau^2 \int\limits_{D_\omega(\tilde\delta)} \left|Q_{t_0}\mathrm{e}^{-\tau(x\cdot\omega-t_0)A_{t_0}}b\right|^2\mathrm{d}x - c\tau \int\limits_{D_\omega(\tilde\delta)} \left|\mathrm{e}^{-\tau(x\cdot\omega-t_0)A_{t_0}}\right|^2\mathrm{d}x \\
&\quad - c\tau\left\|\mathrm{e}^{-\tau(x\cdot\omega-t_0)A_{t_0}}\right\|_{L^2(D_\omega(\tilde\delta))}\left\|\nabla\gamma_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)\right\|_{L^2(D_\omega(\tilde\delta))} - c/\tau. \qquad (4.18)
\end{aligned}$$

In deriving (4.18), we have used the fact that $|\xi| = 1$, $|\omega| = 1$ and $\xi \perp \omega$. Now we want to treat the first term on the right side of (4.18). Using the first half of (4.9), we have that

$$\begin{aligned}
\int\limits_{D_\omega(\tilde\delta)} \left|Q_{t_0}\mathrm{e}^{-\tau(x\cdot\omega-t_0)A_{t_0}}b\right|^2\mathrm{d}x &= \int_0^{\tilde\delta} \int\limits_{\{x\in D:\ x\cdot\omega=t_0+s\}} \left|Q_{t_0}\mathrm{e}^{-\tau sA_{t_0}}b\right|^2\mathrm{d}s\,\mathrm{d}\mu \\
&\geqslant \int_0^{\tilde\delta} \int\limits_{\{x\in D:\ x\cdot\omega=t_0+s\}} \left|Q_{t_0}^{-1}\right|^{-2}\left|\mathrm{e}^{\tau sA_{t_0}}\right|^{-2}|b|^2\,\mathrm{d}s\,\mathrm{d}\mu \\
&\geqslant c\int_0^{\tilde\delta} \int\limits_{\{x\in D:\ x\cdot\omega=t_0+s\}} \mathrm{e}^{-2\tau s\lambda}\,\mathrm{d}s\,\mathrm{d}\mu \qquad (\text{for some } \lambda > 0) \\
&= c\int_0^{\tilde\delta} \mu\left(\left\{x\in D:\ x\cdot\omega = t_0+s\right\}\right)\mathrm{e}^{-2\tau s\lambda}\,\mathrm{d}s \\
&\geqslant (c/c_\omega)\int_0^{\tilde\delta} s^{p_\omega}\mathrm{e}^{-2\tau s\lambda}\,\mathrm{d}s = (c/c_\omega)\tau^{-1-p_\omega}\int_0^{\tau\tilde\delta} \tilde s^{p_\omega}\mathrm{e}^{-2\tilde s\lambda}\,\mathrm{d}\tilde s \\
&\geqslant \tilde c\tau^{-1-p_\omega}. \qquad (4.19)
\end{aligned}$$

To treat the second term on the right side of (4.18), we use the second half of (4.9) and the fact that $\mathrm{spec}(A_{t_0}) \subset \mathbb{C}_r$ to obtain:

$$\int_{D_\omega(\tilde{\delta})} \left|\mathrm{e}^{-\tau(x\cdot\omega-t_0)A_{t_0}}\right|^2 \mathrm{d}x \leqslant c\int_0^{\tilde{\delta}} \mathrm{e}^{-\tau s\lambda}\mu\big(\{x\in D\colon\ x\cdot\omega=t_0+s\}\big)\,\mathrm{d}s \leqslant c\int_0^{\tilde{\delta}} s^{p_\omega}\mathrm{e}^{-\tau s\lambda}$$
$$\leqslant c\tau^{-1-p_\omega}. \tag{4.20}$$

Consequently, combining (4.20) and (2.5) gives:

$$\left\|\mathrm{e}^{-\tau(x\cdot\omega-t_0)A_{t_0}}\right\|_{L^2(D_\omega(\tilde{\delta}))}\left\|\nabla\gamma_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)\right\|_{L^2(D_\omega(\tilde{\delta}))} \leqslant c\tau^{-1-p_\omega/2}. \tag{4.21}$$

Now substituting estimates (4.19), (4.20) and (4.21) into (4.18), we immediately get that

$$\int_{D_\omega(\delta_\omega)} \left|\nabla u_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)\right|^2 \mathrm{d}x \geqslant c\tau^{1-p_\omega} \tag{4.22}$$

for $\tau \gg 1$. Finally, combining (4.16), (4.17) and (4.22) yields:

$$I(\tau,\chi_{t_0},b,t_0,\omega) \geqslant c\tau^{1-p_\omega} \quad \text{as } \tau\to\infty.$$

Next, for the case of jump condition (4.8), we use the second half of (4.11), i.e.,

$$I(\tau,\chi_{t_0},b,t_0,\omega) \leqslant \int_D H\nabla u_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)\cdot\overline{\nabla u_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)}\,\mathrm{d}x$$
$$= \int_D H\,\mathrm{Sym}\,\nabla u_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)\cdot\overline{\mathrm{Sym}\,\nabla u_{\chi_{t_0},b,t_0,N,\omega}(x,\tau)}\,\mathrm{d}x,$$

and proceed in the same way. The proof of (ii) is now complete. □

In view of Theorem 4.1, we can give an algorithm for reconstructing the convex hull of an inclusion D by the Dirichlet-to-Neumann map Λ_D as long as H and D satisfy the described conditions.

Reconstruction algorithm.

(i) Give $\omega\in\mathbb{S}^2$ and choose $\eta,\zeta,\xi\in\mathbb{S}^2$ so that $\{\eta,\zeta,\omega\}$ form a basis of $\mathbb{R}^3$ and ξ lies in the span of η and ζ;
(ii) Choose a starting t such that $\Omega\subset\{x\cdot\omega\geqslant t\}$;
(iii) Choose a ball B such that the center of B lies on $\{x\cdot\omega=s\}$ for some $s<t$ and $\Omega\subset\overline{B}_t(\omega)$ (note that in this case $B\cap\{x\cdot\omega=t\}=:\Sigma_t(\omega)\subset\Pi_t\Sigma_{t-\varepsilon}(\omega)$ for all sufficiently small ε); take $0\neq b\in\mathbb{C}^3$;
(iv) Choose $\chi_t\in C_0^\infty(\mathbb{R}^2)$ such that $\chi_t>0$ in $\Sigma_t(\omega)$ and $\chi_t=0$ on $\partial\Sigma_t(\omega)$;
(v) Construct the oscillating–decaying solution $u_{\chi_{t-\varepsilon},b,t-\varepsilon,N,\omega}$ in $B_{t-\varepsilon}(\omega)$ with $\chi_{t-\varepsilon}=\chi_t$ and the approximation sequence $\tilde{u}_{\varepsilon,j}$ in $\widetilde{\Omega}$;

(vi) Compute the indicator function $I(\tau, \chi_t, b, t, \omega)$ which is determined by boundary measurements;
(vii) If $I(\tau, \chi_t, b, t, \omega) \to 0$ as $\tau \to \infty$, then choose $t' > t$ and repeat (iv), (v), (vi);
(viii) If $I(\tau, \chi_{t'}, b, t', \omega) \nrightarrow 0$ for some $\chi_{t'}$, then $t' = t_0 = h_D(\omega)$;
(ix) Varying $\omega \in \mathbb{S}^2$ and repeat (i)–(viii), we can determine the convex hull of D.

4.2. Reconstruction of cavities

Let domains Ω, D and the elasticity tensor C_0 be described as in Section 4.1. However, here we assume $\overline{D} \subset \Omega$. For any $f \in H^{1/2}(\partial\Omega)$, there exists a unique solution v to:

$$\begin{cases} \mathcal{L}_{C_0} v = 0 \quad \text{in } \Omega \setminus \overline{D}, \\ v|_{\partial\Omega} = f, \ (C_0 \nabla v)\nu|_{\partial D} = 0; \end{cases} \tag{4.23}$$

also, there exists a unique solution u solving:

$$\begin{cases} \mathcal{L}_{C_0} u = 0 \quad \text{in } \Omega \setminus \overline{D}, \\ u|_{\partial\Omega} = f, \end{cases} \tag{4.24}$$

where ν is the unit outer normal to ∂D. Associated with (4.23) and (4.24), we can define the Dirichlet-to-Neumann map Λ_D and Λ_0, respectively, by:

$$\langle \Lambda_D(f), g \rangle = \int_{\Omega \setminus \overline{D}} C_0 \nabla v \cdot \nabla \tilde{w} \, \mathrm{d}x$$

and

$$\langle \Lambda_0(f), g \rangle = \int_{\Omega} C_0 \nabla u \cdot \nabla w \, \mathrm{d}x,$$

where $\tilde{w} \in H^1(\Omega \setminus \overline{D})$ with $\tilde{w}|_{\partial\Omega} = g$ and $w \in H^1(\Omega)$ with $w|_{\partial\Omega} = g$. The inverse problem considered here is to determine D from $\Lambda_D(f)$ for infinitely many $f \in H^{1/2}(\partial\Omega)$.

As in the inclusion case, we define the indicator function:

$$I(\tau, \chi_t, b, t, \omega) = \lim_{\varepsilon \to 0} \lim_{j \to \infty} \langle (\Lambda_0 - \Lambda_D) \tilde{u}_{\varepsilon,j}|_{\partial\Omega}, \overline{\tilde{u}_{\varepsilon,j}|_{\partial\Omega}} \rangle.$$

Here we also assume that D satisfies the estimate (4.9). Then we can show

Theorem 4.2. (i) *If $t < t_0$, then for any $\chi_t \in C_0^\infty(\mathbb{R}^2)$ and $b \in \mathbb{C}^3$ we have that $I(\tau, \chi_t, b, t, \omega) \to 0$ as $\tau \to \infty$;*

(ii) *If $t = t_0$, then for any $\chi_{t_0} \in C_0^\infty(\mathbb{R}^2)$ with $x_0' = (x_0 \cdot \eta, x_0 \cdot \zeta)$ being an interior point of* $\operatorname{supp}(\chi_{t_0})$ *and $0 \neq b \in \mathbb{C}^3$, we get that $|I(\tau, \chi_{t_0}, b, t_0, \omega)| \geqslant c\tau^{1-p_\omega}$ (as $\tau \to \infty$) for some positive constant c.*

Proof. We recall that

$$\langle(\Lambda_0-\Lambda_D)f,\bar{f}\rangle=\int\limits_{\Omega\setminus\overline{D}} C_0\nabla(v-u)\cdot\overline{\nabla(v-u)}\,\mathrm{d}x+\int\limits_D C_0\nabla u\cdot\overline{\nabla u}\,\mathrm{d}x, \tag{4.25}$$

where v and u are solutions to (4.23) and (4.24), respectively. The formula (4.25) leads to

$$\frac{1}{\delta'}\int\limits_D |\operatorname{Sym}\nabla u_{\chi_t,b,t,N,\omega}|^2\,\mathrm{d}x\leqslant I(\tau,\chi_t,b,t,\omega)\leqslant\delta'\int\limits_D |\operatorname{Sym}\nabla u_{\chi_t,b,t,N,\omega}|^2\,\mathrm{d}x \tag{4.26}$$

for some positive constant δ' (see [8]). Now the rest of the proof is carried out as in that of Theorem 4.1. □

Using Theorem 4.2, it is clear that the algorithm described above can be used to reconstruct the convex hull of a cavity.

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