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Strong unique continuation for the Lamé system with Lipschitz coefficients*

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Abstract. In this paper we prove the strong unique continuation property for a Lamé system with Lipschitz coefficients in the plane. The proof relies on reducing the Lamé system to a first order elliptic system and suitable Carleman estimates with polynomial weights.

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1. Introduction

In this work we are concerned with the local behavior of weak solutions to a Lamé system in the plane open connected domain $\Omega \subset \mathbb{R}^2$ with Lipschitz coefficients. Let $\lambda(x)$ and $\mu(x)$ be Lamé coefficients in $W^{1,\infty}(\Omega)$ satisfying

$$\mu(x) \geq \delta_0 > 0 \quad \text{and} \quad \lambda(x) + 2\mu(x) \geq \delta_0 > 0 \quad \forall\, x \in \Omega. \tag{1.1}$$

The Lamé system, which represents the displacement equation of equilibrium, is given by

$$\operatorname{div}(\mu(\nabla u + (\nabla u)^t)) + \nabla(\lambda \operatorname{div} u) = 0 \quad \text{in } \Omega, \tag{1.2}$$

where $u = (u_1, u_2)^t$ is the displacement vector and $(\nabla u)_{jk} = \partial_k u_j$ for $j, k = 1, 2$. In this paper, we will prove the strong unique continuation property (SUCP) for solutions of the Lamé system (1.2). More precisely, if $u \in W^{1,2}_{loc}(\Omega)$ satisfying (1.2) vanishes of infinite order at a point $x_0 \in \Omega$, i.e.

$$\int_{|x-x_0|<R} |u|^2 dx = O(R^N) \quad \text{for all } N \text{ as } R \to 0,$$

then $u \equiv 0$ in Ω.

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Results on the weak unique continuation for the Lamé system in $\mathbb{R}^n$, $n \geq 2$, have been proved by Dehman and Robbiano for $\lambda(x), \mu(x) \in C^\infty(\Omega)$ [3], Ang, Ikehata, Trong and Yamamoto for $\lambda \in C^2(\Omega), \mu(x) \in C^3(\Omega)$ [2], and Weck for $\lambda(x), \mu(x) \in C^2(\Omega)$ [15], [16]. For the Lamé system with residual stress, the weak unique continuation were proved by Nakamura and Wang [11] ($n \geq 2$ with twice differentiable coefficients), while the strong unique continuation were established recently by C. Lin [10] ($n = 3$ with twice differentiable coefficients).

Especially, we would like to mention a recent result by Alessandrini and Morassi [1] who proved the (SUCP) for the Lamé system when $n \geq 2$ and $\lambda(x), \mu(x) \in C^{1,1}(\Omega)$. Their proofs are based on ideas developed by Garofalo and Lin [4], [5]. As indicated in the title of the paper, we prove here the (SUCP) for the Lamé system in the plane with $\lambda(x), \mu(x) \in W^{1,\infty}(\Omega)$, which is clearly an improvement on the regularity assumption used in [1] when $n = 2$. Besides, we will approach this problem along a more "classical" line which is based on Carleman's ideas.

One key component in our approach is the reduction of the system (1.2) to a first order elliptic system of two variables. Unlike methods used in [1] (or [2], [16], [11]) where a reduction was performed by introducing an auxiliary function $v = \text{div} u$. In this situation, at least $C^{1,1}$ coefficients are needed in order to guarantee that the lower order terms have essentially bounded coefficients. Our reduction is carried out by using an auxiliary function $\partial_1 u + T\partial_2 u$ with appropriate matrix T. The crucial point is that our new system contains only first derivatives of the Lamé coefficients, which are inherited from writing (1.2) into a non-divergence form. This observation enables us to reduce the smoothness requirement on coefficients. However, it should be pointed out that our new system is *not* uncoupled in the principle part, which creates a new difficulty in proving the (SUCP). Inspired by the idea in [16], our strategy to overcome this difficulty is to use several suitable Carleman estimates.

The way of reducing the Lamé system employed here was motivated by a recent paper by Nakamura and Wang [12] in which they proved the weak unique continuation for the general anisotropic elasticity system in the plane. It turns out the weak unique continuation for the Lamé system in the plane with Lipschitz coefficients is an immediate consequence of Nakamura and Wang's result [12].

Before leaving this section, we would like to mention several key issues in deriving Carleman estimates. Since we will be working with the reduced system (see (2.13)) rather than the original system (1.2). We need to prove that if the solution of (1.2) vanishes of infinite order at x_0, then so is the solution of (2.13). This is clearly true if λ and μ are smooth enough. Since we only consider $\lambda, \mu \in W^{1,\infty}(\Omega)$ here, we want to present some arguments due to Hörmander [7] to justify this matter. Next, we derive two Carleman estimates with polynomial weights $|x|^{-\beta}$, $\beta > 0$. Combining these Carleman estimates with vanishing property, we can show that the solution of (2.13) actually vanishes *exponentially* as $\exp(-B/|x - x_0|)$ at x_0 (see (3.6)). This decay rate allows us to use more singular weights $\exp((\log|x|)^{\beta})$ in deriving another set of Carleman estimates, which play a key role in proving the (SUCP). Similar ideas were also used in [10] and [14].

2. First order elliptic system

This section is devoted to transform the system (1.2) into a first order elliptic system. To begin, let us express (1.2) in the matrix form, namely,

$$\Lambda_{11}\partial_1^2 u + \Lambda_{12}\partial_1\partial_2 u + \Lambda_{22}\partial_2^2 u + Q(u) = 0, \tag{2.1}$$

where

$$\Lambda_{11} = \begin{pmatrix} \lambda + 2\mu & 0 \\ 0 & \mu \end{pmatrix}, \Lambda_{12} = \begin{pmatrix} 0 & \lambda + \mu \\ \lambda + \mu & 0 \end{pmatrix}, \Lambda_{22} = \begin{pmatrix} \mu & 0 \\ 0 & \lambda + 2\mu \end{pmatrix}$$

and

$$Q(u) = \nabla\lambda(\mathrm{div} u) + (\nabla u + (\nabla u)^t)\nabla\mu.$$

From (1.1) it follows that Λ_{11} and Λ_{22} are invertible (in fact, positive definite). We now define $W = (w_1, w_2)^t = (u, \partial_1 u + T\partial_2 u)^t$, where

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus, we can compute

$$\begin{cases} \partial_1 w_1 = \partial_1 u = (\partial_1 u + T\partial_2 u) - T\partial_2 u = -T\partial_2 w_1 + w_2, \\ \partial_1 w_2 = \partial_1^2 u + T\partial_1\partial_2 u. \end{cases} \tag{2.2}$$

Furthermore, it follows from (2.1) that

$$\partial_1^2 u = -\Lambda_{11}^{-1}(\Lambda_{12}\partial_1\partial_2 u + \Lambda_{22}\partial_2^2 u + Q(u)). \tag{2.3}$$

Using the definition of w_2, we immediately see that

$$\partial_2 w_2 = \partial_1\partial_2 u + T\partial_2^2 u,$$

which implies

$$\partial_1\partial_2 u = -T\partial_2^2 u + \partial_2 w_2. \tag{2.4}$$

Plugging (2.4) into (2.3) yields

$$\partial_1^2 u = -\Lambda_{11}^{-1}\Lambda_{12}(-T\partial_2^2 u + \partial_2 w_2) - \Lambda_{11}^{-1}\Lambda_{22}\partial_2^2 u - \Lambda_{11}^{-1}Q(u). \tag{2.5}$$

Now substituting (2.4) and (2.5) into the second equation of (2.2) we have that

$$\begin{aligned}\partial_1 w_2 &= -\Lambda_{11}^{-1}\Lambda_{12}(-T\partial_2^2 u + \partial_2 w_2) - \Lambda_{11}^{-1}\Lambda_{22}\partial_2^2 u - \Lambda_{11}^{-1}Q(u)\\ &\quad -T^2\partial_2^2 u + T\partial_2 w_2\\ &= -(T^2 - \Lambda_{11}^{-1}\Lambda_{12}T + \Lambda_{11}^{-1}\Lambda_{22})\partial_2^2 u + (T - \Lambda_{11}^{-1}\Lambda_{12})\partial_2 w_2 - \Lambda_{11}^{-1}Q(u).\end{aligned} \tag{2.6}$$

By the relations $\partial_1 u = \partial_1 w_1$, $\partial_2 u = -T^{-1}\partial_1 w_1 + T^{-1}w_2$, and denoting

$$\Phi = \begin{pmatrix} 0 & \mu \\ \lambda & 0 \end{pmatrix},$$

we can see that

$$\begin{aligned}Q(u) &= (\partial_1\Lambda_{11} + \partial_2\Phi)\partial_1 u + (\partial_2\Lambda_{22} + \partial_1\Phi^t)\partial_2 u\\ &= (\partial_1\Lambda_{11} + \partial_2\Phi)\partial_1 w_1 + (\partial_2\Lambda_{22} + \partial_1\Phi^t)(-T^{-1}\partial_1 w_1 + T^{-1}w_2)\\ &= (\partial_1\Lambda_{11} + \partial_2\Phi - \partial_2\Lambda_{22}T^{-1} - \partial_1\Phi^t T^{-1})\partial_1 w_1 + (\partial_2\Lambda_{22} + \partial_1\Phi^t)T^{-1}w_2\\ &= A\partial_1 w_1 + Hw_2,\end{aligned} \tag{2.7}$$

where

$$A := \partial_1\Lambda_{11} + \partial_2\Phi - \partial_2\Lambda_{22}T^{-1} - \partial_1\Phi^t T^{-1}$$

and

$$H := (\partial_2\Lambda_{22} + \partial_1\Phi^t)T^{-1}.$$

Note that A, H are in $L^\infty(\Omega)$ and in general not zero.

Now it is readily seen that T satisfies

$$\Lambda_{11}T^2 - \Lambda_{12}T + \Lambda_{22} = 0$$

or equivalently

$$T^2 - \Lambda_{11}^{-1}\Lambda_{12}T + \Lambda_{11}^{-1}\Lambda_{22} = 0. \tag{2.8}$$

Replacing $Q(u)$ in (2.6) by (2.7) and taking (2.8) into account, we have that

$$\partial_1 w_2 = (T - \Lambda_{11}^{-1}\Lambda_{12})\partial_2 w_2 - \Lambda_{11}^{-1}(A\partial_1 w_1 + Hw_2).$$

Consequently, we get from (2.2) that

$$\begin{cases}\partial_1 w_1 + T\partial_2 w_1 - w_2 = 0\\ \Lambda_{11}^{-1}A\partial_1 w_1 + \partial_1 w_2 + (\Lambda_{11}^{-1}\Lambda_{12} - T)\partial_2 w_2 + \Lambda_{11}^{-1}Hw_2 = 0\end{cases}$$

Equivalently, in the matrix form, we have that

$$E\partial_1 W + F\partial_2 W + GW = 0, \tag{2.9}$$

where

$$E = \begin{pmatrix} I & 0 \\ \Lambda_{11}^{-1}A & I \end{pmatrix}, \; F = \begin{pmatrix} T & 0 \\ 0 & \Lambda_{11}^{-1}\Lambda_{12} - T \end{pmatrix}, \text{ and } \quad G = \begin{pmatrix} 0 & -I \\ 0 & \Lambda_{11}^{-1}H \end{pmatrix}.$$

The matrix E is obviously invertible. We immediately deduce the following first elliptic system from (2.9)

$$\partial_1 W + J\partial_2 W + MW = 0 \tag{2.10}$$

with

$$J = E^{-1}F = \begin{pmatrix} T & 0 \\ K & \Lambda_{11}^{-1}\Lambda_{12} - T \end{pmatrix} \text{ and } M = E^{-1}G = \begin{pmatrix} 0 & -I \\ 0 & \Lambda_{11}^{-1}(A+H) \end{pmatrix},$$

where

$$K = -\Lambda_{11}^{-1}AT \in L^\infty(\Omega).$$

We want to further simplify the elliptic system (2.10). To this end, we first observe that

$$\Lambda_{11}^{-1}\Lambda_{12} - T = \begin{pmatrix} 0 & -\frac{\mu}{\lambda+2\mu} \\ \frac{\lambda+2\mu}{\mu} & 0 \end{pmatrix}.$$

It is not hard to check that T and $\Lambda_{11}^{-1}\Lambda_{12} - T$ are diagonalizable. Indeed, direct computations show that

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \tag{2.11}$$

and

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} \frac{-ia(\lambda+2\mu)}{2\mu} & \frac{a}{2} \\ \frac{ia(\lambda+2\mu)}{2\mu} & \frac{a}{2} \end{pmatrix} \begin{pmatrix} 0 & \frac{-\mu}{\lambda+2\mu} \\ \frac{\lambda+2\mu}{\mu} & 0 \end{pmatrix} \begin{pmatrix} \frac{i\mu}{a(\lambda+2\mu)} & \frac{-i\mu}{a(\lambda+2\mu)} \\ \frac{1}{a} & \frac{1}{a} \end{pmatrix}, \tag{2.12}$$

where $a = \sqrt{(\frac{\mu}{\lambda+2\mu})^2 + 1}$. In view of (2.11) and (2.12) , we define an invertible matrix

$$P(x) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{i\mu}{a(\lambda+2\mu)} & \frac{-i\mu}{a(\lambda+2\mu)} \\ 0 & 0 & \frac{1}{a} & \frac{1}{a} \end{bmatrix}$$

and set $W = PV$. Then from (2.10) we obtain that

$$\partial_1 V + \widetilde{J}\partial_2 V + \widetilde{M}V = 0, \tag{2.13}$$

where

$$\widetilde{J} = \begin{bmatrix} \operatorname{diag}(i,-i) & 0 \\ K & \operatorname{diag}(i,-i) \end{bmatrix} \quad \text{and}$$

$$\widetilde{M} = P^{-1}\partial_1 P + P^{-1}J\partial_2 P + P^{-1}MP \in L^\infty(\Omega).$$

It should be noted that the lower left block K can not be eliminated by this diagonalization process since T and $\Lambda_{11}^{-1}\Lambda_{12} - T$ have exactly the same eigenvalues.

Therefore, to prove the (SUCP) for (1.2) via the reduced system (2.13), we have to deal with the coupled principle part, i.e. $K \neq 0$. Also, we want to mention that $\widetilde{J}$ is not normal. For the first order elliptic system similar to (2.13) with a normal matrix $\widetilde{J}$, a strong unique continuation result was proved by Ōkaji [13]. It should be pointed out that the unique continuation property is not always true for general first order elliptic systems in two variables (see [9]).

3. Local behavior of *V*

In this section, we will prove that if a solution u of (1.2) vanishes of infinite order at x_0, then $D^\alpha V$ for $|\alpha| \leq 1$ vanishes exponentially as $\exp(-B/|x - x_0|)$ at x_0. Without loss of generality, from now on we take $x_0 = 0$. To get this exponentially decaying property, we need two Carleman estimates, which will be derived below. But, we first want to show that

Lemma 3.1. *Let* $u \in W^{1,2}_{loc}(\Omega)$ *be a solution of* (1.2) *vanishing of infinite order at* 0*, then so is* $D^\alpha u$ *for* $|\alpha| \leq 2$.

Proof. Here we will follows closely Hörmander's arguments [7, page 6–8]. Although Hörmander's arguments are given for the second order elliptic operator, they can be applied to the Lamé system (1.2) with the strong ellipticity condition (1.1) without essential modifications. First of all, by the regularity theorem with Lipschitz coefficients, we know that $u \in W^{2,2}_{loc}(\Omega)$ [6, Theorem 2.1]. Therefore, following Hörmander [7, Corollary 17.1.4.] we have that for all $|\alpha| \leq 2$

$$\int_{|x|<R} |D^\alpha u|^2 dx = O(R^N) \quad \text{for all } N \text{ as } R \to 0.$$

□

Lemma 3.1 immediately implies that $D^\alpha V$ vanishes of infinite order at 0 for $|\alpha| \leq 1$. That V vanishes of infinite order allows us to use the polynomial weights $|x|^{-\beta}$, $\beta > 0$, in the Carleman estimates. However, this is not strong enough to establish the (SUCP). Thus, we would like to enhance the decaying property of V in order to accommodate more singular weights in Carleman estimates. To this end, we will derive two Carleman estimates with polynomial weights.

Let $L_\pm = \partial_1 \pm i\partial_2$ be the first order scalar elliptic operators. We first recall a Carleman estimate proved by Ōkaji.

Lemma 3.2. [13, Lemma 3.2] *For any* $s \in \mathbb{N} + 1/2$ *and* $v \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ *we have*

$$\int |x|^{-2s-2}|v|^2 dx \leq 4 \int |x|^{-2s}|L_\pm v|^2 dx. \tag{3.1}$$

Since the system (2.13) is coupled, we need another Carleman estimate to handle this situation.

Lemma 3.3. *For any* $s > 0$ *and* $v \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ *we have*

$$\int |x|^{-2s}|\nabla v|^2 dx \le 2\int |x|^{-2s}|L_\pm v|^2 dx + 16s^2 \int |x|^{-2s-2}|v|^2 dx. \tag{3.2}$$

Proof. Let $v = v_R + iv_I$ then

$$\begin{aligned} |L_\pm v|^2 &= |\partial_1 v \pm i\partial_2 v|^2 = |\partial_1 v|^2 + |\partial_2 v|^2 \mp 2\mathrm{Re}(i\partial_1 v \partial_2 \bar{v}) \\ &= |\partial_1 v|^2 \mp 2(\partial_1 v_R \partial_2 v_I - \partial_1 v_I \partial_2 v_R) + |\partial_2 v|^2. \end{aligned} \tag{3.3}$$

Using the integration by parts and the inequality $2|a||b| \le \varepsilon^{-1}|a|^2 + \varepsilon|b|^2$ with $0 < \varepsilon < 1$, we can estimate

$$\begin{aligned} &|\int |x|^{-2s}(\partial_1 v_R \partial_2 v_I - \partial_1 v_I \partial_2 v_R)dx| \\ &= |-\int \partial_1(|x|^{-2s})v_R \partial_2 v_I dx + \int \partial_2(|x|^{-2s})v_R \partial_1 v_I dx| \\ &\le \int |2s|x|^{-2s-2}x_1 v_R \partial_2 v_I|dx + \int |2s|x|^{-2s-2}x_2 v_R \partial_1 v_I|dx \\ &\le 2\int (s|x|^{-s-2}|x_1 v_R|)(|x|^{-s}\partial_2 v_I|)dx \\ &\quad +2\int (s|x|^{-s-2}|x_2 v_R|)(|x|^{-s}|\partial_1 v_I|)dx. \end{aligned} \tag{3.4}$$

Applying the inequality $2|a||b| \le \varepsilon^{-1}|a|^2 + \varepsilon|b|^2$ with $0 < \varepsilon < 1$ twice on the right side of (3.4) gives

$$\begin{aligned} &2|\int |x|^{-2s}(\partial_1 v_R \partial_2 v_I - \partial_1 v_I \partial_2 v_R)dx| \\ &\le 2\varepsilon^{-1}\int (s|x|^{-s-2}|x_1 v_R|)^2 dx + 2\varepsilon \int (|x|^{-s}\partial_2 v_I|)^2 dx \\ &\quad +2\varepsilon^{-1}\int (s|x|^{-s-2}|x_2 v_R|)^2 dx + 2\varepsilon \int (|x|^{-s}|\partial_1 v_I|)^2 dx \\ &\le 2\varepsilon^{-1}\int s^2|x|^{-2s-4}|x_1|^2|v|^2 dx + 2\varepsilon\int |x|^{-2s}|\partial_2 v|^2 dx \\ &\quad +2\varepsilon^{-1}\int s^2|x|^{-2s-4}|x_2|^2|v|^2 dx + 2\varepsilon\int |x|^{-2s}|\partial_1 v|^2 dx \\ &\le 2\varepsilon^{-1}\int s^2|x|^{-2s-2}|v|^2 dx + 2\varepsilon\int |x|^{-2s}(|\partial_1 v|^2 + |\partial_2 v|^2)dx. \end{aligned} \tag{3.5}$$

Now choosing $\varepsilon = \frac{1}{4}$ and combining (3.3), (3.4) and (3.5), we get that

$$2\int |x|^{-2s}|L_\pm v|^2 dx \ge \int |x|^{-2s}(|\partial_1 v|^2 + |\partial_2 v|)^2 dx - 16s^2\int |x|^{-2s-2}|v|^2 dx,$$

from which (3.2) follows. □

Remark 3.1. The estimates (3.1) and (3.2) remain valid if $v \in W^{1,2}_{loc}(\mathbb{R}^2)$ is compactly supported and v and ∇v vanish of infinite order at 0. This can be easily seen by cutting u off for small $|x|$ and regularizing.

With the help of Lemma 3.1 and 3.2, we are ready to show that $D^\alpha V$ vanishes exponentially at 0 for $|\alpha| \le 1$.

Proposition 3.1. *If $V \in W^{1,2}_{loc}(\Omega)$ satisfies* (2.13) *and $D^\alpha V$ vanishes of infinite order at* 0 *for $|\alpha| \le 1$, then there exist positive constants B and C such that*

$$\int_{|x|<R} (|V|^2 + |\nabla V|^2)dx \le C\exp(-BR^{-1}), \tag{3.6}$$

for all sufficiently small $R > 0$. Here the constant C depends on V but is independent of R provided R is small, while the constant B is independent of u and R.

Proof. To begin, we write $V = (v_1, v_2, v_3, v_4)^t$. In view of Remark 3.1, we can apply (3.1) and (3.2) to the function ξv_j for $j = 1, \cdots, 4$, where $\xi(x) \in C_0^\infty(\mathbb{R}^2)$ such that $\xi(x) = 1$ for $|x| \le R$ and $\xi(x) = 0$ for $|x| \ge 2R$ ($R > 0$ sufficiently small). Here the number R is not yet fixed and is given by $R = \gamma^{-2}s^{-1}$, where $\gamma > 0$ is a large constant which will be chosen later. Using the estimate (3.1) and the first two equations of (2.13), we can derive that

$$\begin{aligned}
&\gamma^4 s^2 \int_{|x|<R} |x|^{-2s-2}(|v_1|^2 + |v_2|^2)dx \\
&= R^{-2}\int_{|x|<R} |x|^{-2s-2}(|v_1|^2 + |v_2|^2)dx \\
&\le \int_{|x|<R} |x|^{-2s-4}(|v_1|^2 + |v_2|^2)dx \\
&\le \int |x|^{-2s-4}(|\xi v_1|^2 + |\xi v_2|^2)dx \\
&\le 4\int |x|^{-2s-2}(|L_+(\xi v_1)|^2 + |L_-(\xi v_2)|^2)dx \\
&\le 4\int_{|x|<R} |x|^{-2s-2}(|L_+ v_1|^2 + |L_- v_2|^2)dx + 4\int_{|x|>R} |x|^{-2s-2}|G_1|^2 dx \\
&\le 8\|\widetilde{M}\|_\infty^2 \int_{|x|<R} |x|^{-2s-2}|V|^2 dx + 4\int_{|x|>R} |x|^{-2s-2}|G_1|^2 dx,
\end{aligned} \tag{3.7}$$

where $G_1 \in L^2(\mathbb{R}^2)$ is supported in $B_{2R}\backslash B_R$. Hereafter, B_r denotes the disc centered at 0 with radius $r > 0$. Note that in applying (3.1) in (3.7) we take $s + 1 \in \mathbb{N} + \frac{1}{2}$. On the other hand, using (3.2) we have that

$$
\begin{aligned}
&\int_{|x|<R} |x|^{-2s}(|\nabla v_1|^2+|\nabla v_2|^2)dx\\
&\quad\le \int |x|^{-2s}(|\nabla(\xi v_1)|^2+|\nabla(\xi v_2)|^2)dx\\
&\quad\le 2\int |x|^{-2s}(|L_+(\xi v_1)|^2+|L_-(\xi v_2)|^2)dx\\
&\qquad+16s^2\int |x|^{-2s-2}(|\xi v_1|^2+|\xi v_2|^2)dx\\
&\quad\le 4\|\widetilde{M}\|_\infty^2\int_{|x|<R}|x|^{-2s}|V|^2dx+16s^2\int_{|x|<R}|x|^{-2s-2}(|v_1|^2+|v_2|^2)dx\\
&\qquad+4\int_{|x|>R}|x|^{-2s}|G_2|^2dx+16s^2\int_{|x|>R}|x|^{-2s-2}|G_3|^2dx, \qquad (3.8)
\end{aligned}
$$

where $G_2, G_3 \in L^2(\mathbb{R}^2)$ are supported in $B_{2R}\backslash B_R$.

Next, we will carry out above arguments for v_1 and v_2 to v_3 and v_4. But here we need to take into account of $K \neq 0$ in (2.13). We first estimate

$$
\begin{aligned}
&\int_{|x|<R} |x|^{-2s-2}(|v_3|^2+|v_4|^2)dx\\
&\quad\le \int |x|^{-2s-2}(|\xi v_3|^2+|\xi v_4|^2)dx\\
&\quad\le 4\int |x|^{-2s}(|L_+(\xi v_3)|^2+|L_-(\xi v_4)|^2)dx\\
&\quad\le 4\int_{|x|<R}|x|^{-2s}(|L_+v_3|^2+|L_-v_4|^2)dx+4\int_{|x|>R}|x|^{-2s}|G_4|^2dx\\
&\quad\le 8\|K\|_\infty^2\int_{|x|<R}|x|^{-2s}(|\nabla v_1|^2+|\nabla v_2|^2)dx+8\|\widetilde{M}\|_\infty^2\int_{|x|<R}|x|^{-2s}|V|^2dx\\
&\qquad+4\int_{|x|>R}|x|^{-2s}|G_4|^2dx, \qquad (3.9)
\end{aligned}
$$

where $G_4 \in L^2(\mathbb{R}^2)$ is supported in $B_{2R}\backslash B_R$. Note that here we take $s \in \mathbb{N}+\frac{1}{2}$. Similarly, we can compute that

$$
\begin{aligned}
&\int_{|x|<R} |x|^{-2s+2}(|\nabla v_3|^2+|\nabla v_4|^2)dx\\
&\quad\le \int |x|^{-2s+2}(|\nabla(\xi v_3)|^2+|\nabla(\xi v_4)|^2)dx\\
&\quad\le 2\int |x|^{-2s+2}(|L_+(\xi v_3)|^2+|L_-(\xi v_4)|^2)dx\\
&\qquad+16s^2\int |x|^{-2s}(|\xi v_3|^2+|\xi v_4|^2)dx
\end{aligned}
$$

$$
\begin{aligned}
&\le 4\|K\|_\infty^2\int_{|x|<R}|x|^{-2s+2}(|\nabla v_1|^2+|\nabla v_2|^2)dx\\
&\quad +4\|\widetilde{M}\|_\infty^2\int_{|x|<R}|x|^{-2s+2}|V|^2dx+16s^2\int_{|x|<R}|x|^{-2s}(|v_3|^2+|v_4|^2)dx\\
&\quad +4\int_{|x|>R}|x|^{-2s+2}|G_5|^2dx+16s^2\int_{|x|>R}|x|^{-2s}|G_6|^2dx\\
&\le 4\|K\|_\infty^2\int_{|x|<R}|x|^{-2s+2}(|\nabla v_1|^2+|\nabla v_2|^2)dx\\
&\quad +4\|\widetilde{M}\|_\infty^2\int_{|x|<R}|x|^{-2s+2}|V|^2dx+16\gamma^{-4}\int_{|x|<R}|x|^{-2s-2}(|v_3|^2+|v_4|^2)dx\\
&\quad +4\int_{|x|>R}|x|^{-2s+2}|G_5|^2dx+16s^2\int_{|x|>R}|x|^{-2s}|G_6|^2dx, \qquad (3.10)
\end{aligned}
$$

where $G_5, G_6 \in L^2(\mathbb{R}^2)$ are supported in $B_{2R}\backslash B_R$. Here in using (3.2), we have replaced the parameter s by $s-1$ provided $s>1$.

Now combining (3.7), γ^2(3.8), γ(3.9), and (3.10) yields

$$
\begin{aligned}
&\gamma^4s^2\int_{|x|<R}|x|^{-2s-2}(|v_1|^2+|v_2|^2)dx+\gamma^2\int_{|x|<R}|x|^{-2s}(|\nabla v_1|^2+|\nabla v_2|^2)dx\\
&\quad +\gamma\int_{|x|<R}|x|^{-2s-2}(|v_3|^2+|v_4|^2)dx+\int_{|x|<R}|x|^{-2s+2}(|\nabla v_3|^2+|\nabla v_4|^2)dx\\
&\le 8\|\widetilde{M}\|_\infty^2\int_{|x|<R}|x|^{-2s-2}|V|^2dx+4\gamma^2\|\widetilde{M}\|_\infty^2\int_{|x|<R}|x|^{-2s}|V|^2dx\\
&\quad +16\gamma^2s^2\int_{|x|<R}|x|^{-2s-2}(|v_1|^2+|v_2|^2)dx\\
&\quad +8\gamma\|K\|_\infty^2\int_{|x|<R}|x|^{-2s}(|\nabla v_1|^2+|\nabla v_2|^2)dx\\
&\quad +8\gamma\|\widetilde{M}\|_\infty^2\int_{|x|<R}|x|^{-2s}|V|^2dx\\
&\quad +4\|K\|_\infty^2\int_{|x|<R}|x|^{-2s+2}(|\nabla v_1|^2+|\nabla v_2|^2)dx\\
&\quad +4\|\widetilde{M}\|_\infty^2\int_{|x|<R}|x|^{-2s+2}|V|^2dx+16\gamma^{-4}\int_{|x|<R}|x|^{-2s-2}(|v_3|^2+|v_4|^2)dx\\
&\quad +4\int_{|x|>R}|x|^{-2s-2}|G_1|^2dx+4\gamma^2\int_{|x|>R}|x|^{-2s}|G_2|^2dx\\
&\quad +16\gamma^2s^2\int_{|x|>R}|x|^{-2s-2}|G_3|^2dx+4\gamma\int_{|x|>R}|x|^{-2s}|G_4|^2dx\\
&\quad +4\int_{|x|>R}|x|^{-2s+2}|G_5|^2dx+16s^2\int_{|x|>R}|x|^{-2s}|G_6|^2dx. \qquad (3.11)
\end{aligned}
$$

In order to compare terms on both sides of (3.11), we estimate the second term on the right side of (3.11) and obtain

$$4\gamma^2\|\widetilde{M}\|_\infty^2\int_{|x|<R}|x|^{-2s}|V|^2dx\le 4\gamma^{-2}s^{-2}\|\widetilde{M}\|_\infty^2\int_{|x|<R}|x|^{-2s-2}|V|^2dx.$$

Note that $|x|^{-s_1}<|x|^{-s_2}$ for $0<s_1<s_2$ provided that $|x|<R<1$. Therefore, carefully checking terms on both sides of (3.11), we can choose $\gamma>0$ large enough such that all terms with v_j and ∇v_j, $j=1,\cdots,4$, on the right side of (3.11) are absorbed by the left hand side. We now fix such γ. Consequently, we get that for $s>0$

$$\begin{aligned}
&\left(\frac{R}{2}\right)^{-2s+2}\int_{|x|<R/2}(|V|^2+|\nabla V|^2)dx\\
&\quad\le\int_{|x|<R/2}|x|^{-2s+2}(|V|^2+|\nabla V|^2)dx\\
&\quad\le\int_{|x|<R}|x|^{-2s+2}(|V|^2+|\nabla V|^2)dx\\
&\quad\le C_1s^2\int_{R<|x|<2R}\sum_{k=1}^6|x|^{-2s-2}|G_k|^2dx\\
&\quad\le C_1R^{-2s-2}s^2\int_{R<|x|<2R}\sum_{k=1}^6|G_k|^2dx,
\end{aligned}\tag{3.12}$$

where C_1 is a positive constant. Recall that $R=\gamma^{-2}s^{-1}$. We therefore deduce from (3.12) that

$$\begin{aligned}
&\int_{|x|<R/2}(|V|^2+|\nabla V|^2)dx\\
&\quad\le 4C_1\gamma^{-4}R^{-6}(2^{-2\gamma^{-2}R^{-1}})\int_{|x|>R}\sum_{k=1}^6|G_i|^2dx\\
&\quad\le CR^{-6}(2^{-2\gamma^{-2}R^{-1}}),
\end{aligned}$$

where $C\le 24C_1\gamma^{-4}\|V\|_{L^2(B_1)}^2$. In other words, we have

$$\int_{|x|<R/2}(|V|^2+|\nabla V|^2)dx\le C\exp(-\tilde{B}R^{-1})\tag{3.13}$$

for some constant $\tilde{B}>0$.

It should be noted that (3.13) is valid for $s\in\mathbb{N}+\frac12$ and $R=\gamma^{-2}s^{-1}$. Therefore, if we choose $s\in\{j+\frac12:j\in\mathbb{N}\}$, then (3.13) only holds for $R_j=\gamma^{-2}(j+\frac12)^{-1}$. Nevertheless, we can see that

$$R_{j+1}<R_j<2R_{j+1}\quad\text{and}\quad R_j\to 0\text{ as }j\to\infty.$$

Thus, we can conclude that

$$\int_{|x|<R} (|V|^2 + |\nabla V|^2) dx \le C \exp(-BR^{-1}),$$

for all sufficiently small $R > 0$ with $B = \tilde{B}/2$. □

4. Carleman estimates with more singular weights

In view of Proposition 3.1, we derive two Carleman estimates with weights $\phi_s = \exp((\log|x|)^{2s})$, which will be used to complete the proof of (SUCP).

Lemma 4.1. *There exist a sufficiently large number $s_0 > 0$ such that for all $v \in U_{r_0}$ with $0 < r_0 < e^{-1}$, $s \ge s_0$, and $\sigma \in \mathbb{R}$ satisfying $\sigma = O(|s|)$, we have that*

$$s^2 \int \phi_s^2 (\log|x|)^{2\sigma+2s-2} |x|^{-2} |v|^2 dx \le \int (\log|x|)^{2\sigma} \phi_s^2 |L_\pm v|^2 dx, \tag{4.1}$$

where $U_{r_0} = \{v \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}) : \mathrm{supp}(v) \subset B_{r_0}\}$.

Proof. We will prove (4.1) only for L_+. The proof of (4.1) for L_- is similar. Denote $\psi(|x|) = (\log|x|)^{2s} + \log[(-\log|x|)^\sigma]$ for $|x| < 1$ and set $v = e^{-\psi} w$. Then we can find that

$$\begin{aligned}
\int e^{2\psi} |L_+ v|^2 dx &= \int |\partial_1 w - i\partial_2\psi w + i\partial_2 w - \partial_1\psi w|^2 dx \\
&= \int |\partial_1 w - i\partial_2\psi w|^2 + |i\partial_2 w - \partial_1\psi w|^2 \\
&\quad + 2\mathrm{Re}\big((\partial_1 w - i\partial_2\psi w) \cdot \overline{(i\partial_2 w - \partial_1\psi w)}\big) dx \\
&\ge 2\mathrm{Re} \int (\partial_1 w - i\partial_2\psi w) \cdot \overline{(i\partial_2 w - \partial_1\psi w)} dx \\
&= 2\mathrm{Re} \int (i\partial_2\psi w - \partial_1 w) \cdot (i\partial_2\bar{w} + \partial_1\psi\bar{w}) dx \\
&= 2\mathrm{Re} \int i\partial_2\psi w \partial_1\psi\bar{w} dx - 2\mathrm{Re} \int i\partial_1 w \partial_2\bar{w} dx \\
&\quad - 2\mathrm{Re} \int \partial_1 w \partial_1\psi\bar{w} dx - 2\mathrm{Re} \int \partial_2\psi w \partial_2\bar{w} dx.
\end{aligned} \tag{4.2}$$

It is readily seen that

$$2\mathrm{Re} \int i\partial_2\psi w \partial_1\psi\bar{w} dx = 2\mathrm{Re} \int i\partial_2\psi\partial_1\psi |w|^2 dx = 0. \tag{4.3}$$

Performing the integration by parts yields

$$\int \partial_1 w \partial_2 \bar{w} dx = \int \overline{\partial_1 w \partial_2 \bar{w}} dx$$

and thus

$$2\mathrm{Re} \int i \partial_1 w \partial_2 \bar{w} dx = 0. \tag{4.4}$$

We now deal with the last two terms in (4.2). Using the integration by parts, we get that

$$-2\mathrm{Re} \int \partial_1 w \partial_1 \psi \bar{w} dx = -\int \partial_1 \psi \partial_1 |w|^2 dx = \int \partial_1^2 \psi |w|^2 dx. \tag{4.5}$$

Similarly, we can find that

$$-2\mathrm{Re} \int \partial_2 \psi w \partial_2 \bar{w} dx = -\int \partial_2 \psi \partial_2 |w|^2 dx = \int \partial_2^2 \psi |w|^2 dx. \tag{4.6}$$

Combining (4.2), (4.3), (4.4), (4.5) and (4.6) gives

$$\int (\partial_1^2 \psi + \partial_2^2 \psi) |w|^2 dx \le \int e^{2\psi} |L_+ v|^2 dx. \tag{4.7}$$

Through direct computations, we obtain that for $j = 1, 2$

$$\begin{aligned}
\partial_j \psi &= 2s(\log|x|)^{2s-1}|x|^{-2} x_j + \sigma(\log|x|)^{-1}|x|^{-2} x_j, \\
\partial_j^2 \psi &= 2s(2s-1)(\log|x|)^{2s-2}|x|^{-4} x_j^2 - 4s(\log|x|)^{2s-1}|x|^{-4} x_j^2 \\
&\quad + 2s(\log|x|)^{2s-1}|x|^{-2} - \sigma(\log|x|)^{-2}|x|^{-4} x_j^2 \\
&\quad - 2\sigma(\log|x|)^{-1}|x|^{-4} x_j^2 + \sigma(\log|x|)^{-1}|x|^{-2}
\end{aligned}$$

and therefore

$$\begin{aligned}
\partial_1^2 \psi + \partial_2^2 \psi &= 2s(2s-1)(\log|x|)^{2s-2}|x|^{-2} - \sigma(\log|x|)^{-2}|x|^{-2} \\
&\quad - 2\sigma(\log|x|)^{-1}|x|^{-2} + 2\sigma(\log|x|)^{-1}|x|^{-2}.
\end{aligned}$$

In other words, $\partial_1^2 \psi + \partial_2^2 \psi$ dominates $s^2(\log|x|)^{2s-2}|x|^{-2}$ in $|x| < e^{-1}$ when s is sufficiently large. Consequently, by taking s large enough, we can get from (4.7) that

$$\int s^2 (\log|x|)^{2s-2}|x|^{-2}|w|^2 dx \le \int (\partial_1^2 \psi + \partial_2^2 \psi)|w|^2 dx \le \int e^{2\psi}|L_+ v|^2 dx \tag{4.8}$$

for $v, w \in U_{r_o}$ with $r_0 < e^{-1}$. Substituting $w = e^{\psi} v$ and $\psi = (\log|x|)^{2s} + \log[(-\log|x|)^{\sigma}]$ into (4.8), we immediately obtain that

$$
\begin{aligned}
s^2 & \int (\log|x|)^{2\sigma+2s-2}|x|^{-2}\phi_s^2|v|^2dx \\
&= s^2 \int (\log|x|)^{2s-2}|x|^{-2}e^{2\psi}|v|^2dx \\
&= s^2 \int (\log|x|)^{2s-2}|x|^{-2}|w|^2dx \\
&\le \int e^{2\psi}|L_+v|^2dx \\
&= \int (\log|x|)^{2\sigma}\phi_s^2|L_+v|^2dx
\end{aligned}
$$

for $v \in U_{r_0}$. The proof is complete. □

As before, to handle the possible non-zero off-diagonal block K in (2.13), we need another Carleman estimate.

Lemma 4.2. *For all $s > 0$, $\sigma \in \mathbb{R}$, and all $v \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ we have that*

$$
\begin{aligned}
& \int \phi_s^2(\log|x|)^{2\sigma}|\nabla v|^2dx \\
& \quad \le 2\int \phi_s^2(\log|x|)^{2\sigma}|L_\pm v|^2dx \\
& \qquad +32(\sigma^2+4s^2)\int \phi_s^2(\log|x|)^{2\sigma+4s-2}|x|^{-2}|v|^2dx.
\end{aligned} \tag{4.9}
$$

Proof. This lemma will be proved along the lines of the proof for Lemma 3.3. As in Lemma 3.3, let $v = v_R + iv_I$ then

$$
|L_\pm v|^2 = |\partial_1 v \pm i\partial_2 v|^2 = |\partial_1 v|^2 \mp 2(\partial_1 v_R\partial_2 v_I - \partial_1 v_I\partial_2 v_R) + |\partial_2 v|^2. \tag{4.10}
$$

It suffices to estimate the second term on the very right side of (4.10). Simple integration by parts implies

$$
\begin{aligned}
& |\int (\log|x|)^{2\sigma}\phi_s^2(\partial_1 v_R\partial_2 v_I - \partial_1 v_I\partial_2 v_R)dx| \\
& \quad = |-\int \partial_1((\log|x|)^{2\sigma}\phi_s^2)v_R\partial_2 v_I dx + \int \partial_2((\log|x|)^{2\sigma}\phi_s^2)v_R\partial_1 v_I dx| \\
& \quad \le \int |2\sigma(\log|x|)^{2\sigma-1}\phi_s^2|x|^{-2}x_1 v_R\partial_2 v_I|dx \\
& \qquad + \int |2\sigma(\log|x|)^{2\sigma-1}\phi_s^2|x|^{-2}x_2 v_R\partial_1 v_I|dx
\end{aligned}
$$

$$+\int |4s(\log|x|)^{2\sigma+2s-1}\phi_s^2|x|^{-2}x_1v_R\partial_2v_I|dx$$
$$+\int |4s(\log|x|)^{2\sigma+2s-1}\phi_s^2|x|^{-2}x_2v_R\partial_1v_I|dx. \tag{4.11}$$

We will estimate four terms on the right side of (4.11) one by one. The main tool is the inequality $2|a||b| \le \varepsilon|a|^2 + \varepsilon^{-1}|b|^2$ with $0 < \varepsilon < 1$. We begin with the first term

$$\begin{aligned}&\int |2\sigma(\log|x|)^{2\sigma-1}\phi_s^2|x|^{-2}x_1v_R\partial_2v_I|dx\\ &= \int |\,[(\log|x|)^{\sigma}\phi_s\partial_2v_I][2\sigma(\log|x|)^{\sigma-1}\phi_s|x|^{-2}x_1v_R]|dx\\ &\le \frac{\varepsilon}{2}\int(\log|x|)^{2\sigma}\phi_s^2(\partial_2v_I)^2dx + \frac{2\sigma^2}{\varepsilon}\int(\log|x|)^{2\sigma-2}\phi_s^2|x|^{-4}x_1^2v_R^2dx.\end{aligned} \tag{4.12}$$

For other three terms, we have that

$$\begin{aligned}&\int |2\sigma(\log|x|)^{2\sigma-1}\phi_s^2|x|^{-2}x_2v_R\partial_1v_I|dx\\ &= \int |\,[(\log|x|)^{\sigma}\phi_s\partial_1v_I][2\sigma(\log|x|)^{\sigma-1}\phi_s|x|^{-2}x_2v_R]|dx\\ &\le \frac{\varepsilon}{2}\int(\log|x|)^{2\sigma}\phi_s^2(\partial_1v_I)^2dx + \frac{2\sigma^2}{\varepsilon}\int(\log|x|)^{2\sigma-2}\phi_s^2|x|^{-4}x_2^2v_R^2dx,\end{aligned} \tag{4.13}$$

$$\begin{aligned}&\int |4s(\log|x|)^{2\sigma+2s-1}\phi_s^2|x|^{-2}x_1v_R\partial_2v_I|dx\\ &= \int |\,[(\log|x|)^{\sigma}\phi_s\partial_2v_I][4s(\log|x|)^{\sigma+2s-1}\phi_s|x|^{-2}x_1v_R]|dx\\ &\le \frac{\varepsilon}{2}\int(\log|x|)^{2\sigma}\phi_s^2(\partial_2v_I)^2dx + \frac{8s^2}{\varepsilon}\int(\log|x|)^{2\sigma+4s-2}\phi_s^2|x|^{-4}x_1^2v_R^2dx,\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}&\int |4s(\log|x|)^{2\sigma+2s-1}\phi_s^2|x|^{-2}x_2v_R\partial_1v_I|dx\\ &= \int |\,[(\log|x|)^{\sigma}\phi_s\partial_1v_I][4s(\log|x|)^{\sigma+2s-1}\phi_s|x|^{-2}x_2v_R]|dx\\ &\le \frac{\varepsilon}{2}\int(\log|x|)^{2\sigma}\phi_s^2(\partial_1v_I)^2dx + \frac{8s^2}{\varepsilon}\int(\log|x|)^{2\sigma+4s-2}\phi_s^2|x|^{-4}x_2^2v_R^2dx.\end{aligned} \tag{4.15}$$

Using (4.11) to (4.15) we see that

$$
\begin{aligned}
&|\int (\log|x|)^{2\sigma}\phi_s^2(\partial_1 v_R\partial_2 v_I-\partial_1 v_I\partial_2 v_R)dx| \\
&\quad \le \varepsilon\int(\log|x|)^{2\sigma}\phi_s^2|\nabla v|^2dx+\frac{(2\sigma^2+8s^2)}{\varepsilon}\int(\log|x|)^{2\sigma+4s-2}\phi_s^2|x|^{-2}v^2dx.
\end{aligned}
\tag{4.16}
$$

Now the estimate (4.9) follows from (4.10) and (4.16) with $\varepsilon=\frac{1}{4}$. □

Remark 4.1. Similar to Remark 3.1, Carleman estimates (4.1) and (4.9) remain valid if we assume $v\in W^{1,2}_{loc}(\mathbb{R}^2)$ with $\operatorname{supp}(v)\subset B_{r_0}$, $r_0<e^{-1}$, and satisfies that $D^\alpha v$ for all $|\alpha|\le 1$ vanish of exponential order at 0, i.e., (3.6).

5. Proof of (SUCP)

We will prove the (SUCP) for any solution V of (2.13) arising from $V=P^{-1}(u,\partial_1 u+T\partial_2 u)^t$. This result immediately implies the (SUCP) for (1.2). Let V be a solution of (2.13) . We have shown that $V\in W^{1,2}_{loc}(\Omega)$ and satisfies (3.6). In view of Remark 4.1, we can use the Carleman estimates (4.1) and (4.9) provided that we cut-off V. So let $\xi(x)\in C_0^\infty(\mathbb{R}^2)$ be a cut-off function satisfying $\xi(x)=1$ for $|x|\le R$ and $\xi(x)=0$ for $|x|\ge 2R$, $R>0$ is sufficiently small and $2R<e^{-1}$. As before, we set $V=(v_1,v_2,v_3,v_4)^t$.

Now let $\sigma=0$ in (4.1) for the operators L_+ and L_- acting on ξv_1 and ξv_2, respectively, we obtain from (2.13) that

$$
\begin{aligned}
&s^2\int(\log|x|)^{2s-2}\phi_s^2|x|^{-2}(|\xi v_1|^2+|\xi v_2|^2)dx \\
&\quad\le\int\phi_s^2(|L_+(\xi v_1)|^2+|L_-(\xi v_2)|^2)dx \\
&\quad\le\int_{|x|<R}\phi_s^2(|L_+v_1|^2+|L_-v_2|^2)dx+\int_{R<|x|<2R}\phi_s^2|F_1|^2dx \\
&\quad\le 2\|\widetilde{M}\|_\infty^2\int_{|x|<R}\phi_s^2|V|^2dx+\int_{R<|x|<2R}\phi_s^2|F_1|^2dx,
\end{aligned}
\tag{5.1}
$$

where $F_1\in L^2(B_{2R}\setminus B_R)$. Letting $\sigma=-s$ in (4.1) for the operators L_+ and L_- acting on ξv_3 and ξv_4, respectively, we have from (2.13) that

$$
\begin{aligned}
&s^2\int(\log|x|)^{-2}\phi_s^2|x|^{-2}(|\xi v_3|^2+|\xi v_4|^2)dx \\
&\quad\le\int(\log|x|)^{-2s}\phi_s^2(|L_+(\xi v_3)|^2+|L_-(\xi v_4)|^2)dx
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_{|x|<R} (\log|x|)^{-2s}\phi_s^2(|L_+ v_3|^2 + |L_- v_4|^2)dx \\
&\quad + \int_{R<|x|<2R} (\log|x|)^{-2s}\phi_s^2|F_2|^2 dx \\
&\le 2\|K\|_\infty^2 \int_{|x|<R} (\log|x|)^{-2s}\phi_s^2(|\nabla v_1|^2 + |\nabla v_2|^2)dx \\
&\quad +2\|\widetilde{M}\|_\infty^2 \int_{|x|<R} (\log|x|)^{-2s}\phi_s^2|V|^2 dx + \int_{R<|x|<2R} (\log|x|)^{-2s}\phi_s^2|F_2|^2 dx,
\end{aligned}
\tag{5.2}
$$

where $F_3 \in L^2(B_{2R} \setminus B_R)$.

Next choosing $\sigma = -s$ in (4.9) for the operators L_+, L_- and using (2.13), we get that

$$
\begin{aligned}
&\int_{|x|<R} (\log|x|)^{-2s}\phi_s^2(|\nabla v_1|^2 + |\nabla v_2|^2)dx \\
&\quad \le \int (\log|x|)^{-2s}\phi_s^2(|\nabla(\xi v_1)|^2 + |\nabla(\xi v_2)|^2)dx \\
&\quad \le 2\int (\log|x|)^{-2s}\phi_s^2(|L_+(\xi v_1)|^2 + |L_-(\xi v_2)|^2)dx \\
&\qquad +160s^2 \int (\log|x|)^{2s-2}\phi_s^2|x|^{-2}(|\xi v_1|^2 + |\xi v_2|^2)dx \\
&\quad \le 4\|\widetilde{M}\|_\infty^2 \int_{|x|<R} (\log|x|)^{-2s}\phi_s^2|V|^2 dx \\
&\qquad +160s^2 \int (\log|x|)^{2s-2}\phi_s^2|x|^{-2}(|\xi v_1|^2 + |\xi v_2|^2)dx \\
&\qquad + \int_{R<|x|<2R} (\log|x|)^{-2s}\phi_s^2|F_3|^2 dx,
\end{aligned}
\tag{5.3}
$$

where $F_2 \in L^2(B_{2R} \setminus B_R)$.

Let $\zeta > 0$ be a large number which will be determined later. Combining $\zeta\times$(5.1) and (5.2) leads to

$$
\begin{aligned}
&\zeta s^2 \int (\log|x|)^{2s-2}\phi_s^2|x|^{-2}(|\xi v_1|^2 + |\xi v_2|^2)dx \\
&\quad +s^2 \int (\log|x|)^{-2}\phi_s^2|x|^{-2}(|\xi v_3|^2 + |\xi v_4|^2)dx \\
&\le 2\zeta\|\widetilde{M}\|_\infty^2 \int_{|x|<R} \phi_s^2|V|^2 dx + \zeta\int_{R<|x|<2R} \phi_s^2|F_1|^2 dx
\end{aligned}
$$

$$+2\|\widetilde{M}\|_\infty^2\int_{|x|<R}(\log|x|)^{-2s}\phi_s^2|V|^2dx+\int_{R<|x|<2R}(\log|x|)^{-2s}\phi_s^2|F_3|^2dx$$
$$+2\|K\|_\infty^2\int_{|x|<R}(\log|x|)^{-2s}\phi_s^2(|\nabla v_1|^2+|\nabla v_2|^2)dx. \tag{5.4}$$

Replacing the very last term of (5.4) by (5.3) implies

$$\zeta s^2\int(\log|x|)^{2s-2}\phi_s^2|x|^{-2}(|\xi v_1|^2+|\xi v_2|^2)dx$$
$$+s^2\int(\log|x|)^{-2}\phi_s^2|x|^{-2}(|\xi v_3|^2+|\xi v_4|^2)dx$$
$$\le 2\zeta\|\widetilde{M}\|_\infty^2\int_{|x|<R}\phi_s^2|V|^2dx+\zeta\int_{R<|x|<2R}\phi_s^2|F_1|^2dx$$
$$+2\|\widetilde{M}\|_\infty^2\int_{|x|<R}(\log|x|)^{-2s}\phi_s^2|V|^2dx+\int_{R<|x|<2R}(\log|x|)^{-2s}\phi_s^2|F_3|^2dx$$
$$+8\|\widetilde{M}\|_\infty^2\|K\|_\infty^2\int_{|x|<R}(\log|x|)^{-2s}\phi_s^2|V|^2dx$$
$$+320\|K\|_\infty^2 s^2\int(\log|x|)^{2s-2}\phi_s^2|x|^{-2}(|\xi v_1|^2+|\xi v_2|^2)dx$$
$$+2\|K\|_\infty^2\int_{R<|x|<2R}(\log|x|)^{-2s}\phi_s^2|F_2|^2dx. \tag{5.5}$$

We now choose $\zeta>320\|K\|_\infty^2$ and fix it. Observe that if $s>0$ then

$$(\log|x|)^{-2s}<1<|x|^{-2}(\log|x|)^{-2} \tag{5.6}$$

for all $|x|<R_0$ with sufficiently small $R_0>0$. So from now on we fix $R=R_0/2$. Taking ζ as indicated and using (5.6), we get from (5.5) that for s large

$$s^2\int_{|x|<R}\phi_s^2|V|^2dx\le C\int_{R<|x|<2R}\phi_s^2\sum_{k=1}^3|F_k|^2dx. \tag{5.7}$$

Note that ϕ_s^2 is a decreasing function of $|x|$ in $|x|<1$. Therefore, it follows from (5.7) that

$$s^2\int_{|x|<R}|V|^2dx\le C\int_{R<|x|<2R}\sum_{k=1}^3|F_k|^2dx$$

and thus $V=0$ in $|x|<R$, which implies $V\equiv 0$ in Ω by standard arguments. This ends the proof.

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