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Determination of viscosity in the stationary Navier–Stokes equations

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Abstract

In this paper we consider the stationary Navier–Stokes equations in a bounded domain with a variable viscosity. We prove that one can uniquely determine the viscosity function from the knowledge of boundary data.

1. Introduction

In this work we consider the unique determination of the viscosity in an incompressible fluid described by the stationary Navier–Stokes equations. Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with boundary $\partial\Omega \in C^\infty$. Assume that Ω is filled with an incompressible fluid. Let $u = (u^1, u^2, u^3)^T$ be the velocity vector field satisfying the stationary Navier–Stokes system

$$\begin{cases} \operatorname{div} \sigma_\mu(u, p) - (u \cdot \nabla)u = 0 & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where

$$\sigma_\mu(u, p) = 2\mu \operatorname{Sym}(\nabla u) - pI$$

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and $\mathrm{Sym}(A) = (A + A^T)/2$ is the symmetric part of the matrix A. Here $\mu(x) > 0$ is the viscosity function. The exact regularity of μ will be specified as we go along. Precisely, we can write the first equation of (1.1) componentwise, i.e.,

$$\big(\operatorname{div}\sigma_\mu(u,p) - (u\cdot\nabla)u\big)^k = \sum_j \partial_j\big(2\mu\,\mathrm{Sym}(\nabla u)^{jk}\big) - \partial_k p - \sum_j u^j\partial_j u^k.$$

Here and below, all Roman indices are from 1 to 3 unless otherwise indicated. The second equation of (1.1) is the well-known incompressibility condition.

We are interested in the inverse problem in this paper. We first define the meaning of boundary measurements. Mathematically, the boundary measurements are encoded in the Cauchy data of all solutions satisfying (1.1). Precisely, we define

$$\tilde S_\mu := \big\{\big(u|_{\partial\Omega}, \sigma_\mu(u,p)\mathbf{n}|_{\partial\Omega}\big)\big\},$$

where (u, p) solves (1.1) with well-defined boundary traces $u|_{\partial\Omega}$ and $\sigma_\mu(u,p)\mathbf{n}|_{\partial\Omega}$, $\mathbf{n}$ is the unit outer normal of the $\partial\Omega$, and $u|_{\partial\Omega}$ satisfies the compatibility condition

$$\int_{\partial\Omega} u|_{\partial\Omega}\cdot\mathbf{n}\,ds = 0. \tag{1.2}$$

We want to remark that a solution (u, p) satisfies (1.1) with nonhomogeneous Dirichlet condition $u|_{\partial\Omega}$ is not necessarily unique (see [23]). In the physical sense, $\sigma_\mu(u,p)\mathbf{n}|_{\partial\Omega}$ represents the Cauchy force acting on $\partial\Omega$. The inverse problem now is to determine μ from the knowledge of $\tilde S_\mu$.

To study our inverse problem, we will not consider the general Dirichlet data $u|_{\partial\Omega} = \phi$. Instead, we shall take

$$\phi = \varepsilon\psi \tag{1.3}$$

with $|\varepsilon|$ sufficiently small and $\psi \in H^{3/2}(\partial\Omega)$ satisfying the compatibility condition (1.2). For such a choice of Dirichlet data, we can show that there exists a solution (u, p) to (1.1) with $u|_{\partial\Omega} = \varepsilon\psi$ and the boundary trace $\sigma_\mu(u,p)\mathbf{n}|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$. Thus the Cauchy data $\tilde S_\mu$ is meaningful in this case. When $|\varepsilon|$ is sufficiently small, we even know that the solution (u, p) to (1.1) is unique (p is unique up to a constant), but we do not need it. The main result of this paper is the following global uniqueness theorem of the inverse problem.

Theorem 1. *Assume that $\mu_1(x)$ and $\mu_2(x)$ are two viscosity functions satisfying $\mu_1, \mu_2 \in C^{n_0}(\bar\Omega)$ for $n_0 \geqslant 8$ and*

$$\partial^\alpha\mu_1(x) = \partial^\alpha\mu_2(x) \quad \forall x \in \partial\Omega,\ |\alpha| \leqslant 1. \tag{1.4}$$

Let $\tilde S_{\mu_1}$ and $\tilde S_{\mu_2}$ be the Cauchy data associated with μ_1 and μ_2, respectively. If $\tilde S_{\mu_1} = \tilde S_{\mu_2}$ then $\mu_1 = \mu_2$.

When the boundary $\partial\Omega$ is convex and has nonvanishing Gauss curvature, we can remove the assumption (1.4) from Theorem 1.

Theorem 2. *Let $\partial\Omega$ be convex with nonvanishing Gauss curvature. Assume that $\mu_1(x)$ and $\mu_2(x)$ are two viscosity functions satisfying $\mu_1, \mu_2 \in C^{n_0}(\bar{\Omega})$ for $n_0 \geqslant 8$. If $\tilde{S}_{\mu_1} = \tilde{S}_{\mu_2}$ then $\mu_1 = \mu_2$.*

The parameter determination problem by boundary measurements is a rather well-studied field. Since Calderón's pioneer contribution [1], a key method has been the construction of complex geometrical optics solutions with a large parameter which was introduced by Sylvester and Uhlmann [22] and has become a standard method. We also refer the readers to Uhlmann's survey article [24]. The inverse problem for incompressible fluid governed by the Stokes equations was studied by Heck, Li and Wang [7]. They proved a global identifiability of the viscosity parameter by boundary measurements using the method introduced by Eskin and Ralston [3–5] and also related work [16,17]. To study the Navier–Stokes equations we shall apply the linearization method. This method was first introduced by Isakov [9] in a semilinear parabolic inverse problem. This technique allows for the reduction of the semilinear inverse boundary value problem to the corresponding linear one. The linearization strategy has been used by many authors to treat the inverse problem for nonlinear equations, see for example [8,10–15,18–21].

The difficulty in implementing the linearization technique to our problem lies in the existence of particular solutions to (1.1) which possess some controlled asymptotic properties. This is why we introduce the Dirichlet condition with small parameter ε as in (1.3). The key step in the proofs of Theorems 1 and 2 is to show that there exists a solution $(u_\varepsilon, p_\varepsilon)$ to (1.1) with boundary condition (1.3) and $(\varepsilon^{-1}u_\varepsilon, \varepsilon^{-1}p_\varepsilon)$ converges to (v_0, q_0) in suitable Sobolev spaces, where (v_0, q_0) satisfies the Stokes equations. Subsequently, one can determine the Cauchy data associated with the Stokes equations from $\tilde{S}_\mu$. Then the inverse problem for the Navier–Stokes equations (1.1) is reduced to the same problem for the Stokes equations. We would like to mention that Theorems 1 and 2 for Navier–Stokes equations are counterparts of Theorem 1.1 and Corollary 1.4 in [7] for Stokes equations.

This paper is organized as follows. In Section 2, we will prove the existence of the boundary value problem for (1.1). In Section 3, we linearize the Cauchy data $\tilde{S}_\mu$ and prove Theorems 1 and 2. In Appendix A we provide the existence, uniqueness and regularity results of the solution of the Stokes equations which we use in Section 2.

2. Direct problem

In this section we aim to prove the existence of the boundary value problem

$$\begin{cases} \operatorname{div}\big(\sigma_\mu(u, p)\big) - (u \cdot \nabla)u = 0 & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = \phi \in H^{3/2}(\Omega) & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

with the compatibility condition (1.2). When μ is a constant, this problem has been well documented in the literature, see for example [6,23]. Here we study the case where μ is a function and the boundary value contains a small parameter.

As mentioned in the introduction, we choose $\phi = \varepsilon\psi$ with $\psi \in H^{3/2}(\partial\Omega)$ and look for $(u_\varepsilon, p_\varepsilon) = (\varepsilon v_\varepsilon, \varepsilon q_\varepsilon)$ satisfies (2.1). The problem (2.1) is reduced to

$$\begin{cases} \operatorname{div}\big(\sigma_\mu(v_\varepsilon, q_\varepsilon)\big) - \varepsilon(v_\varepsilon \cdot \nabla)v_\varepsilon = 0 & \text{in } \Omega, \\ \operatorname{div} v_\varepsilon = 0 & \text{in } \Omega, \\ v_\varepsilon = \psi & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

We will solve for (2.2) with the form $v_\varepsilon = v_0 + \varepsilon v$ and $q_\varepsilon = q_0 + \varepsilon q$, where (v_0, q_0) satisfies the Stokes equations

$$\begin{cases} \operatorname{div}\big(\sigma_\mu(v_0, q_0)\big) = 0 & \text{in } \Omega, \\ \operatorname{div} v_0 = 0 & \text{in } \Omega, \\ v_0 = \psi & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

and (v, q) satisfies

$$\begin{cases} -\operatorname{div}\big(\sigma_\mu(v, q)\big) + \varepsilon(v_0 \cdot \nabla)v + \varepsilon(v \cdot \nabla)v_0 + \varepsilon^2(v \cdot \nabla)v = f & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

with $f = -(v_0 \cdot \nabla)v_0$.

For (2.3), we know (see Theorem 11 in Appendix A) that for each $\psi \in H^{3/2}(\partial\Omega)$ there exists a unique $(v_0, q_0) \in H^2(\Omega) \times H^1(\Omega)$ (q_0 is unique up to a constant) satisfying (2.3) and the estimate

$$\|v_0\|_{H^2(\Omega)} + \|q_0\|_{H^1(\Omega)/\mathbb{R}} \leqslant C\|\psi\|_{H^{3/2}(\partial\Omega)}, \tag{2.5}$$

where $\|q_0\|_{H^1(\Omega)/\mathbb{R}} := \inf_{c\in\mathbb{R}} \|q_0 + c\|_{H^1(\Omega)}$. In view of the Sobolev imbedding theorem $H^2(\Omega) \hookrightarrow C^0(\bar{\Omega})$, we have that

$$\|f\|_{H^1(\Omega)} = \big\|(v_0 \cdot \nabla)v_0\big\|_{H^1(\Omega)} \leqslant C\|v_0\|^2_{H^2(\Omega)} \leqslant C\|\psi\|^2_{H^{3/2}(\partial\Omega)}. \tag{2.6}$$

Now we need to solve (2.4). We first prove an existence theorem.

Theorem 3. *There exists a positive number ε_0 depending on ψ such that for any $|\varepsilon| \leqslant \varepsilon_0$,* (2.4) *has at least one weak solution $(v, q) \in H_0^1(\Omega) \times L^2(\Omega)$.*

Proof. As outlined in [23] for the standard Navier–Stokes equations, we solve (2.4) by the Galerkin method. Let us denote

$$V = \big\{v \in H_0^1(\Omega)\colon \operatorname{div} v = 0\big\}.$$

Combining Korn's inequality and Poincaré's inequality, $H_0^1(\Omega)$ is a separable Hilbert space with respect to the inner product

$$\langle u, w\rangle = \int_\Omega S(u) \cdot S(\bar{w})\, dx. \tag{2.7}$$

Note that V is a closed subspace of $H_0^1(\Omega)$, which is also separable. Let $w_1, w_2, \ldots$ be elements of $\mathcal{V}$ which form a complete orthonormal system of V, where

$$\mathcal{V} := \big\{w \in C_0^\infty(\Omega)\colon \operatorname{div} w = 0\big\}.$$

Let

$$v_n = \sum_{j=1}^{n} \xi_{j,n} w_j \quad \text{with } \xi_{j,n} \in \mathbb{C} \tag{2.8}$$

satisfy

$$\mu\langle v_n, w_j\rangle + \varepsilon b(v_0, v_n, w_j) + \varepsilon b(v_n, v_0, w_j) + \varepsilon^2 b(v_n, v_n, w_j) = (f, w_j) \tag{2.9}$$

for $j = 1, \dots, n$, where

$$b(u, v, w) = \int_{\Omega} (u \cdot \nabla)v \cdot \bar{w}\, dx$$

and

$$(f, w) = \int_{\Omega} f \cdot \bar{w}\, dx.$$

Arguing as in [23, Lemma 1.3, Chapter II], one can easily prove two properties of $b(u, v, w)$:

$$b(u, v, v) = 0 \quad \text{for all } u \in V,\ v \in H_0^1(\Omega) \tag{2.10}$$

and

$$b(u, v, w) = -b(u, w, v) \quad \text{for all } u \in V,\ v, w \in H_0^1(\Omega). \tag{2.11}$$

Moreover, using the imbedding $H^2(\Omega) \hookrightarrow C^0(\bar{\Omega})$, we can see that

$$\left|b(v, v, v_0)\right| = \left|\int_{\Omega} (v \cdot \nabla)v \cdot v_0\, dx\right| \leqslant C\|v_0\|_{H^2(\Omega)}\|v\|^2_{H^1(\Omega)}. \tag{2.12}$$

Next we recall a technical lemma proved in [23].

Lemma 4. *(See [23, Lemma 1.4, Chapter II].) Let X be a finite-dimensional Hilbert space with inner product $[\cdot,\cdot]$ and norm $\|\cdot\|$ and let P be a continuous map from X to itself such that*

$$\big[P(\zeta), \zeta\big] > 0 \quad \text{for } \|\zeta\| = k > 0. \tag{2.13}$$

Then there exists $\zeta \in X$ with $\|\zeta\| \leqslant k$ so that $P(\zeta) = 0$.

In applying Lemma 4, we take $X =$ the space spanned by $w_1, \dots, w_n$ and the inner product $[\cdot,\cdot]$ is induced by that of V, namely, $\langle\cdot,\cdot\rangle$ given in (2.7). Here the norm $\|\cdot\| = \|\cdot\|_{H^1(\Omega)}$. We now define $P = P_n$ by

$$\begin{aligned}\big[P_n(v), w\big] &= \big\langle P_n(v), w\big\rangle \\ &= \mu\langle v, w\rangle + \varepsilon b(v_0, v, w) + \varepsilon b(v, v_0, w) + \varepsilon^2 b(v, v, w) - (f, w)\end{aligned}$$

for $v, w \in X$. The continuity of P_n is obvious. To verify (2.13), we can see with the help of (2.10), (2.11), and (2.12) that

$$\begin{aligned}\big[P_n(v), v\big] &= \mu\langle v, v\rangle + \varepsilon b(v_0, v, v) + \varepsilon b(v, v_0, v) + \varepsilon^2 b(v, v, v) - (f, v) \\ &\geqslant C\|v\|^2_{H^1(\Omega)} - \big|\varepsilon b(v, v, v_0)\big| - \big|(f, v)\big| \\ &\geqslant C\|v\|^2_{H^1(\Omega)} - |\varepsilon|C'\|v_0\|_{H^2(\Omega)}\|v\|^2_{H^1(\Omega)} - \|f\|_{H^{-1}(\Omega)}\|v\|_{H^1(\Omega)} \\ &\geqslant \|v\|_{H^1(\Omega)}\big\{\big(C - |\varepsilon|C'\|v_0\|_{H^2(\Omega)}\big)\|v\|_{H^1(\Omega)} - \|f\|_{H^{-1}(\Omega)}\big\},\end{aligned}$$

where C and C' are positive numbers. Therefore, if we choose a small ε_0, depending on ψ, such that

$$C - |\varepsilon|C'\|v_0\|_{H^2(\Omega)} > 0 \quad \forall|\varepsilon| \leqslant \varepsilon_0,$$

then $[P_n(v), v] > 0$ for $\|v\|_{H^1(\Omega)} = k$ with

$$k > \frac{\|f\|_{H^{-1}(\Omega)}}{C - |\varepsilon|C'\|v_0\|_{H^2(\Omega)}} \quad \text{for } |\varepsilon| \leqslant \varepsilon_0.$$

Hence, Lemma 4 guarantees the existence of v_n satisfying (2.8) and (2.9).

Now we want to pass the limit of v_n. Multiplying (2.9) by $\bar{\xi}_{j,n}$ and summing the corresponding equalities from 1 to n gives

$$\mu\langle v_n, v_n\rangle + \varepsilon b(v_0, v_n, v_n) + \varepsilon b(v_n, v_0, v_n) + \varepsilon^2 b(v_n, v_n, v_n) = (f, v_n).$$

Using (2.10), (2.11), and (2.12) again, we obtain that

$$\|v_n\|_{H^1(\Omega)} \leqslant C_0\|f\|_{H^{-1}(\Omega)},$$

where $C_0 > 0$ is uniformly in ε provided $|\varepsilon| \leqslant \varepsilon_0$. Therefore, there exist v in V and a subsequence $\{v_{n'}\}$ such that

$$v_{n'} \to v \quad \text{weakly in } V. \tag{2.14}$$

By the Rellich theorem, we have that

$$v_{n'} \to v \quad \text{strongly in } L^2(\Omega). \tag{2.15}$$

With the help of (2.14) and (2.15), we can derive that

$$b(v_0, v_{n'}, w) \to b(v_0, v, w), \tag{2.16}$$

$$b(v_{n'}, v_0, w) \to b(v, v_0, w), \tag{2.17}$$

and

$$b(v_{n'}, v_{n'}, w) \to b(v, v, w) \tag{2.18}$$

for all $w \in \mathcal{V}$. The proof of (2.18) was already given in [23, Lemma 1.5, Chapter II]. As for (2.16) and (2.17), they are easy consequences of (2.15). Therefore, passing the limit in $v_{n'}$, we obtain that

$$\mu\langle v, w_j\rangle + \varepsilon b(v_0, v, w_j) + \varepsilon b(v, v_0, w_j) + \varepsilon^2 b(v, v, w_j) = (f, w_j)$$

for $j = 1, 2, \ldots$. Since $w_1, w_2, \ldots$ is a complete orthonormal system of V, we conclude that

$$\mu\langle v, w\rangle + \varepsilon b(v_0, v, w) + \varepsilon b(v, v_0, w) + \varepsilon^2 b(v, v, w) = (f, w)$$

for all $w \in V$. Thus, there exists $q \in L^2(\Omega)$ (and $v \in V$) such that

$$-\operatorname{div}\bigl(\sigma_\mu(v, q)\bigr) + \varepsilon(v_0 \cdot \nabla)v + \varepsilon(v \cdot \nabla)v_0 + \varepsilon^2(v \cdot \nabla)v = f$$

in the weak sense. □

Now we have the existence of weak solution (v, q) to (2.4). To indicate the dependence of (v, q) on ε, we denote $v = \tilde{v}_\varepsilon$ and $q = \tilde{q}_\varepsilon$. Our next task is to derive the regularity of $(\tilde{v}_\varepsilon, \tilde{q}_\varepsilon)$. The aim is to show that $(\tilde{v}_\varepsilon, \tilde{q}_\varepsilon)$ are uniformly bounded in ε with respect to some Sobolev norms. This enables us to consider the limiting behavior of $(\varepsilon^{-1} v_\varepsilon, \varepsilon^{-1} p_\varepsilon)$. The proof of regularity for $(\tilde{v}_\varepsilon, \tilde{q}_\varepsilon)$ relies on the regularity result for the Stokes equations and the "bootstrapping" technique. Some arguments used here are inspired by [23].

Theorem 5. *Let* $(\tilde{v}_\varepsilon, \tilde{q}_\varepsilon)$ *be a weak solution of* (2.4) *for* $|\varepsilon| \leqslant \varepsilon_0$. *We may choose* $\varepsilon_0 < 1$. *Then* $(\tilde{v}_\varepsilon, \tilde{q}_\varepsilon) \in H^2(\Omega) \times H^1(\Omega)$ *and satisfies*

$$\|\tilde{v}_\varepsilon\|_{H^2(\Omega)} + \|\tilde{q}_\varepsilon\|_{H^1(\Omega)/\mathbb{R}} \leqslant C \sum_{j=2}^{16} \|\psi\|^j_{H^{3/2}(\partial\Omega)} \tag{2.19}$$

where C *is independent of* ε.

Proof. To simply the notations in the proof, we reuse $v = \tilde{v}_\varepsilon$ and $q = \tilde{q}_\varepsilon$. We now write the first equation of (2.4) in the form of Stokes equations

$$\operatorname{div}\bigl(\sigma_\mu(v, q)\bigr) = g, \quad \operatorname{div} v = 0$$

with

$$g = \varepsilon(v_0 \cdot \nabla)v + \varepsilon(v \cdot \nabla)v_0 + \varepsilon^2(v \cdot \nabla)v - f.$$

From the proof of the existence, we see that $v \in H_0^1(\Omega)$ and

$$\|v\|_{H^1(\Omega)} \leqslant C\|f\|_{H^{-1}(\Omega)} \leqslant C\|\psi\|^2_{H^{3/2}(\partial\Omega)}.$$

Hereafter, C, C', C'', and $\tilde{C}$ represent constants independent of ε. Their exact values are not important. In view of the Sobolev imbedding, we have $v \in L^6(\Omega)$. Subsequently, we get that $(v \cdot \nabla)v \in L^{3/2}(\Omega)$ and

$$\big\|(v \cdot \nabla)v\big\|_{L^{3/2}(\Omega)} \leqslant C\|v\|^2_{H^1(\Omega)} \leqslant C\|\psi\|^4_{H^{3/2}(\partial\Omega)}.$$

Similarly, we have that

$$\big\|(v \cdot \nabla)v_0\big\|_{L^{3/2}(\Omega)} \leqslant C\|v\|_{L^6(\Omega)}\|v_0\|_{H^1(\Omega)} \leqslant C\|\psi\|^4_{H^{3/2}(\partial\Omega)}$$

and

$$\big\|(v_0 \cdot \nabla)v\big\|_{L^{3/2}(\Omega)} \leqslant C\big\|(v_0 \cdot \nabla)v\big\|_{L^2(\Omega)} \leqslant C\|\psi\|^4_{H^{3/2}(\partial\Omega)}.$$

Therefore, from (2.6) we have $g \in L^{3/2}(\Omega)$ and

$$\|g\|_{L^{3/2}(\Omega)} \leqslant C\big(\varepsilon_0\|\psi\|^4_{H^{3/2}(\partial\Omega)} + \|\psi\|^2_{H^{3/2}(\partial\Omega)}\big).$$

From now on, we choose $\varepsilon_0 < 1$ and hence

$$\|g\|_{L^{3/2}(\Omega)} \leqslant C\big(\|\psi\|^4_{H^{3/2}(\partial\Omega)} + \|\psi\|^2_{H^{3/2}(\partial\Omega)}\big).$$

The regularity theorem for the Stokes equations (Theorem 11 in Appendix A) implies

$$\|v\|_{W^{2,3/2}(\Omega)} + \|q\|_{W^{1,3/2}(\Omega)/\mathbb{R}} \leqslant C\|g\|_{L^{3/2}(\Omega)} \leqslant C\big(\|\psi\|^4_{H^{3/2}(\partial\Omega)} + \|\psi\|^2_{H^{3/2}(\partial\Omega)}\big). \tag{2.20}$$

The estimate (2.20) is not exactly what we want. We need to improve $L^{3/2}$-base Sobolev norms to L^2-base ones on the left-hand side of (2.20). This can be achieved by the "bootstrapping" argument. In view of Sobolev imbedding, $W^{2,3/2}(\Omega) \hookrightarrow L^r(\Omega)$ for any $1 < r < \infty$, we thus obtain that $v \otimes v \in L^s$ for any $s \in (1, \infty)$ and

$$\begin{aligned}
\big\|(v \cdot \nabla)v\big\|_{W^{-1,s}(\Omega)} &= \big\|\nabla(v \otimes v)\big\|_{W^{-1,s}(\Omega)} \\
&\leqslant C\|v\|_{L^{r_1}(\Omega)}\|v\|_{L^{r_2}(\Omega)} \\
&\leqslant C'\|v\|^2_{W^{2,3/2}(\Omega)} \\
&\leqslant C''\big(\|\psi\|^8_{H^{3/2}(\partial\Omega)} + \|\psi\|^4_{H^{3/2}(\partial\Omega)}\big),
\end{aligned} \tag{2.21}$$

where $s^{-1} = r_1^{-1} + r_2^{-1}$ and $C'' = C''(\Omega, r_1, r_2)$. In the first equality of (2.21), we have used $\operatorname{div} v = 0$. On the other hand, $v_0 \in H^2(\Omega) = W^{2,2}(\Omega) \hookrightarrow W^{2,3/2}(\Omega)$. Likewise, we have that

$$\begin{cases}
\big\|(v \cdot \nabla)v_0\big\|_{W^{-1,s}(\Omega)} \leqslant \tilde{C}\big(\|\psi\|^5_{H^{3/2}(\partial\Omega)} + \|\psi\|^3_{H^{3/2}(\partial\Omega)}\big), \\
\big\|(v_0 \cdot \nabla)v\big\|_{W^{-1,s}(\Omega)} \leqslant \tilde{C}\big(\|\psi\|^5_{H^{3/2}(\partial\Omega)} + \|\psi\|^3_{H^{3/2}(\partial\Omega)}\big), \\
\|f\|_{W^{-1,s}(\Omega)} = \big\|(v_0 \cdot \nabla)v_0\big\|_{W^{-1,s}(\Omega)} \leqslant \tilde{C}\|\psi\|^2_{H^{3/2}(\partial\Omega)},
\end{cases} \tag{2.22}$$

where $s^{-1} = r_1^{-1} + r_2^{-1}$ and $\tilde{C} = \tilde{C}(\Omega, r_1, r_2)$. Combining (2.21) and (2.22) leads to

$$\|g\|_{W^{-1,s}(\Omega)} \leqslant C \sum_{j=2}^{8} \|\psi\|^j_{H^{3/2}(\partial\Omega)}.$$

Thus from Theorem 10 in Appendix A we have

$$\|v\|_{W^{1,s}(\Omega)} + \|q\|_{L^s(\Omega)/\mathbb{R}} \leqslant C \sum_{j=2}^{8} \|\psi\|^j_{H^{3/2}(\partial\Omega)}$$

for any $s \in (1, \infty)$.

We need one more iteration. Due to the imbedding theorem, i.e., $W^{1,s}(\Omega) \hookrightarrow C^0(\bar{\Omega})$ for $s > 3$, we can see that $(v \cdot \nabla)v \in L^s(\Omega)$ for any $s \in (1, \infty)$ since Ω is bounded. In particular, we have $(v \cdot \nabla)v \in L^2(\Omega)$ and

$$\big\|(v \cdot \nabla)v\big\|_{L^2(\Omega)} \leqslant C \sum_{j=4}^{16} \|\psi\|^j_{H^{3/2}(\partial\Omega)}.$$

We already knew that $v_0 \in W^{2,2}(\Omega) \hookrightarrow C^0(\bar{\Omega})$. So we immediately have

$$\begin{cases} \big\|(v \cdot \nabla)v_0\big\|_{L^2(\Omega)} \leqslant C \sum_{j=3}^{9} \|\psi\|^j_{H^{3/2}(\partial\Omega)}, \\ \big\|(v_0 \cdot \nabla)v\big\|_{L^2(\Omega)} \leqslant C \sum_{j=3}^{9} \|\psi\|^j_{H^{3/2}(\partial\Omega)}, \\ \|f\|_{L^2(\Omega)} \leqslant C\|\psi\|^2_{H^{3/2}(\partial\Omega)}. \end{cases}$$

In other words, we get

$$\|g\|_{L^2(\Omega)} \leqslant C \sum_{j=2}^{16} \|\psi\|^j_{H^{3/2}(\partial\Omega)}$$

and therefore Theorem 11 in Appendix A implies

$$\|v\|_{H^2(\Omega)} + \|q\|_{H^1(\Omega)/\mathbb{R}} \leqslant C \sum_{j=2}^{16} \|\psi\|^j_{H^{3/2}(\partial\Omega)}. \qquad \square$$

Remark 6. Even though we will not need the uniqueness result of direct problem (2.1) in our inverse problem, the solution $(u_\varepsilon, p_\varepsilon)$ of (2.1) with boundary condition $\phi = \varepsilon\psi$ is indeed unique (p_ε is unique up to constants) when $|\varepsilon|$ is sufficiently small. This fact can be proved by modifying arguments in [23, Theorem 1.6, Chapter II].

3. Inverse problem

We already proved the existence of the solution to the system (1.1) with the boundary data $\varepsilon\psi$ in Section 2. More importantly, we derived the asymptotic behavior of solutions as $\varepsilon \to 0$. In this section we linearize the Cauchy data $\tilde{S}_\mu$ and prove Theorems 1 and 2.

Given any $\psi \in H^{3/2}(\partial\Omega)$, let $(v_0, q_0) \in H^2(\Omega) \times H^1(\Omega)$ be the unique solution (q_0 is unique up to a constant) of the Stokes equations

$$\begin{cases} \operatorname{div}\big(\sigma_\mu(v_0, q_0)\big) = 0 & \text{in } \Omega, \\ \operatorname{div} v_0 = 0 & \text{in } \Omega \end{cases} \tag{3.1}$$

with boundary data $v_0|_{\partial\Omega} = \psi$ (see Theorem 11 in Appendix A). As proved in the previous section, there exists $(u_\varepsilon, p_\varepsilon)$ with the form

$$u_\varepsilon = \varepsilon v_0 + \varepsilon^2 \tilde{v}_\varepsilon, \qquad p_\varepsilon = \varepsilon q_0 + \varepsilon^2 \tilde{q}_\varepsilon$$

satisfying (1.1) with boundary data $u_\varepsilon|_{\partial\Omega} = \varepsilon\psi$ for all $|\varepsilon| < \varepsilon_0$, where ε_0 depends on $\|\psi\|_{H^{3/2}(\partial\Omega)}$. Moreover, $\tilde{v}_\varepsilon$ and $\tilde{q}_\varepsilon$ satisfy (2.19). We immediately see that

$$\begin{aligned} \big\|\varepsilon^{-1} u_\varepsilon - v_0\big\|_{H^2(\Omega)} &= \|\varepsilon \tilde{v}_\varepsilon\|_{H^2(\Omega)} \to 0, \\ \big\|\varepsilon^{-1} p_\varepsilon - q_0\big\|_{H^1(\Omega)/\mathbb{R}} &= \|\varepsilon \tilde{q}_\varepsilon\|_{H^1(\Omega)/\mathbb{R}} \to 0, \end{aligned}$$

and therefore

$$\big\|\varepsilon^{-1} u_\varepsilon|_{\partial\Omega} - v_0|_{\partial\Omega}\big\|_{H^{3/2}(\partial\Omega)} \to 0, \tag{3.2}$$

$$\big\|\varepsilon^{-1} \sigma_\mu(u_\varepsilon, p_\varepsilon)\mathbf{n}|_{\partial\Omega} - \sigma_\mu(v_0, q_0)\mathbf{n}|_{\partial\Omega}\big\|_{H^{1/2}(\partial\Omega)} \to 0 \tag{3.3}$$

provided

$$\int_\Omega p_\varepsilon \, dx = \int_\Omega q_0 \, dx = 0.$$

As in [7], we define the Cauchy data associated to (3.1)

$$S_\mu = \big\{\big(v_0|_{\partial\Omega}, \sigma_\mu(v_0, q_0)\mathbf{n}|_{\partial\Omega}\big)\big\} \subset H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega), \tag{3.4}$$

where (v_0, q_0) satisfies (3.1). Now combining (3.2) and (3.3) leads to the following result:

Theorem 7. *The Cauchy data S_μ of the Stokes equations can be uniquely determined from the Cauchy data $\tilde{S}_\mu$ of the Navier–Stokes equations.*

In other words, let μ_1 and μ_2 be two viscosities, then $\tilde{S}_{\mu_1} = \tilde{S}_{\mu_2}$ implies $S_{\mu_1} = S_{\mu_2}$. So we reduce the uniqueness question of the inverse problem for the Navier–Stokes equations to that for the Stokes equations. Therefore, Theorems 1 and 2 follow from Theorem 7 and the unique determination of viscosity for the Stokes equations proved in [7].

Theorem 8. *(See [7, Theorem 1.1].) Assume that $\mu_1(x)$ and $\mu_2(x)$ are two viscosity functions satisfying $\mu_1, \mu_2 \in C^{n_0}(\bar{\Omega})$ for $n_0 \geqslant 8$ and*

$$\partial^\alpha \mu_1(x) = \partial^\alpha \mu_2(x) \quad \forall x \in \partial\Omega,\ |\alpha| \leqslant 1.$$

Let S_{μ_1} and S_{μ_2} be the Cauchy data associated with μ_1 and μ_2, respectively. If $S_{\mu_1} = S_{\mu_2}$ then $\mu_1 = \mu_2$.

Theorem 9. *(See [7, Corollary 1.4].) Let $\partial\Omega$ be convex with nonvanishing Gauss curvature. Assume that $\mu_1(x)$ and $\mu_2(x)$ are two viscosity functions satisfying $\mu_1, \mu_2 \in C^{n_0}(\bar{\Omega})$ for $n_0 \geqslant 8$. If $S_{\mu_1} = S_{\mu_2}$ then $\mu_1 = \mu_2$.*

The regularity requirement in Theorem 8 is to make sure that Eskin's method of [3] works in our case. We do not know whether it is optimal. Also, we want to remark that in [7] we define the Cauchy data of the Stokes equations as a subset of $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$, i.e. $S_\mu \subset H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$. Nevertheless, the same proof in [7] still holds true when we consider S_μ as given in (3.4).

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Appendix A. L^s $(s > 1)$ theory for the Stokes equations

In this appendix we will prove the existence, uniqueness, and regularity of the solution to the Stokes equations in the category of L^s when the viscosity is a function. When μ is a constant, this problem has been well documented in the literature, see for example [6,23]. However, we were not able to find any reference for the case where μ is a variable function. For the sake of completeness, we provide a proof here. As before, let Ω be an open bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Consider the Stokes equations

$$\begin{cases} \operatorname{div} \sigma_\mu(v, q) = f & \text{in } \Omega, \\ \operatorname{div} v = g & \text{in } \Omega, \\ v = \psi & \text{on } \partial\Omega, \end{cases} \tag{A.1}$$

where the viscosity $\mu(x) > 0$ and the following compatibility condition holds:

$$\int_\Omega g\,dx = \int_{\partial\Omega} \psi \cdot \mathbf{n}\,ds.$$

The system (A.1) will not be affected if we add a constant to q. We define the following norm

$$\|q\|_{W^{k,s}(\Omega)/\mathbb{R}} := \inf_{c \in \mathbb{R}} \|q + c\|_{W^{k,s}(\Omega)}, \quad k \geqslant -1.$$

For a suitable $\mu(x)$, one can prove that

Theorem 10. *Suppose $\mu \in C^{1,1}(\bar{\Omega})$. For any*

$$f \in W^{-1,s}(\Omega), \quad g \in L^s(\Omega), \quad \psi \in W^{1-1/s,s}(\partial\Omega), \quad 1 < s < \infty,$$

there exists a unique solution $(v, q) \in W^{1,s}(\Omega) \times L^s(\Omega)$ (q is unique up to a constant) satisfying (A.1). *Moreover this solution obeys the estimate*

$$\begin{aligned} &\|v\|_{W^{1,s}(\Omega)} + \|q\|_{L^s(\Omega)/\mathbb{R}} \\ &\quad \leqslant C\big(\|f\|_{W^{-1,s}(\Omega)} + \|g\|_{L^s(\Omega)} + \|\psi\|_{W^{1-1/s,s}(\partial\Omega)}\big) \end{aligned} \tag{A.2}$$

where C depends on s, Ω, $\min_{x\in\bar{\Omega}} \mu$ and $\|\mu\|_{C^{1,1}(\bar{\Omega})}$.

Theorem 11. *Suppose $\mu \in C^{r,1}(\bar{\Omega})$ with the integer $r = \max\{m, 1\}$, $m \geqslant 0$. For any*

$$f \in W^{m,s}(\Omega), \quad g \in W^{m+1,s}(\Omega), \quad \psi \in W^{m+2-1/s,s}(\partial\Omega), \quad 1 < s < \infty,$$

there exists a unique solution $(v, q) \in W^{m+2,s}(\Omega) \times W^{m+1,s}(\Omega)$ (q is unique up to a constant) satisfying (A.1). *Moreover this solution obeys the estimate*

$$\begin{aligned} &\|v\|_{W^{m+2,s}(\Omega)} + \|q\|_{W^{m+1,s}(\Omega)/\mathbb{R}} \\ &\quad \leqslant C\big(\|f\|_{W^{m,s}(\Omega)} + \|g\|_{W^{m+1,s}(\Omega)} + \|\psi\|_{W^{m+2-1/s,s}(\partial\Omega)}\big) \end{aligned} \tag{A.3}$$

where C depends on m, s, Ω, $\min_{x\in\bar{\Omega}} \mu$ and $\|\mu\|_{C^{r,1}(\bar{\Omega})}$.

Remark 12. In Theorems 10 and 11, the regularity assumptions on μ are not necessarily optimal. We impose the least smoothness $\mu \in C^{1,1}(\bar{\Omega})$ due to the consideration of uniqueness in Lemma 14.

When μ is a constant, these results were originally proved by Cattabriga [2], and a nice presentation could be found for example in [6, Theorem IV.6.1 and Ex. IV.6.2]. Here we shall provide a proof for the variable viscosity following general procedures used in [2,6]. From [6, Theorem III.3.2], we know there exists at least one vector field $w \in W^{m+2,s}(\Omega)$ ($m \geqslant -1$) such that

$$\begin{cases} \operatorname{div} w = g & \text{in } \Omega, \\ w = \psi & \text{on } \partial\Omega. \end{cases}$$

Moreover, this solution satisfies the estimate

$$\|w\|_{W^{m+2,s}(\Omega)} \leqslant C(m, s, \Omega)\big(\|g\|_{W^{m+1,s}(\Omega)} + \|\psi\|_{W^{m+2-1/s,s}(\partial\Omega)}\big).$$

So we can always assume that g and ψ are zeros in system (A.1). Theorems 10 and 11 will be proved together. We first show a priori estimates.

Lemma 13. *Let* $(v,q)\in W^{1,s}(\Omega)\times L^s(\Omega)$ *be a weak solution of* (A.1).

(i) *Suppose* $\mu\in C^0(\bar{\Omega})$, f, g, *and* ψ *are as in Theorem* 10. *We have*

$$\|v\|_{W^{1,s}(\Omega)}+\|q\|_{L^s(\Omega)/\mathbb{R}} \leqslant C\big(\|f\|_{W^{-1,s}(\Omega)}+\|g\|_{L^s(\Omega)}+\|\psi\|_{W^{1-1/s,s}(\partial\Omega)}+\|v\|_{L^s(\Omega)}+\|q\|_{W^{-1,s}(\Omega)}\big). \tag{A.4}$$

(ii) *Suppose* $\mu\in C^{m,1}(\bar{\Omega})$ $(m\geqslant 0)$, f, g, *and* ψ *are as in Theorem* 11. *We have*

$$\|v\|_{W^{m+2,s}(\Omega)}+\|q\|_{W^{m+1,s}(\Omega)/\mathbb{R}} \leqslant C\big(\|f\|_{W^{m,s}(\Omega)}+\|g\|_{W^{m+1,s}(\Omega)}+\|\psi\|_{W^{m+2-1/s,s}(\partial\Omega)}+\|v\|_{L^s(\Omega)}+\|q\|_{W^{-1,s}(\Omega)}\big). \tag{A.5}$$

Proof. We only need to consider the case when g and ψ are zeros. The estimate (A.4) can be proved by freezing the coefficient and the result for the Stokes equations with constant viscosity. One only needs to show (A.4) holds in a ball $B_R(x^{(0)})$ and a half ball $B_R^+(x^{(1)}):=B_R(x^{(1)})\cap\{x_3>0\}$ which correspond to the interior estimate and the estimate near the boundary, respectively. Then a finite covering of $\bar{\Omega}$ implies that (A.4) holds. Consider a ball $B_{2R}(x^{(0)})\subset\Omega$ and take a cutoff function $\eta\in C_0^\infty(B_{2R}(x^{(0)}))$ with $\eta=1$ in $B_R(x^{(0)})$. Direct computation shows that ηv and ηq satisfy the following constant coefficient Stokes equations

$$\begin{cases} \mu\big(x^{(0)}\big)\Delta(\eta v)-\nabla(\eta q)=\tilde{f} & \text{in } B_{2R}\big(x^{(0)}\big),\\ \operatorname{div}(\eta v)=\tilde{g} & \text{in } B_{2R}\big(x^{(0)}\big),\\ \eta v=0 & \text{on } \partial B_{2R}\big(x^{(0)}\big), \end{cases} \tag{A.6}$$

where

$$\begin{aligned}\tilde{f}=&-\operatorname{div}\big[2\big(\mu(x)-\mu\big(x^{(0)}\big)\big)\operatorname{Sym}\big(\nabla(\eta v)\big)\big]+\eta f\\ &+\operatorname{div}\big[2\big(\mu(x)-\mu\big(x^{(0)}\big)\big)\operatorname{Sym}(\nabla\eta\otimes v)\big]+2\big(\mu(x)-\mu\big(x^{(0)}\big)\big)\nabla\eta\cdot\operatorname{Sym}(\nabla v)\\ &+2\mu\big(x^{(0)}\big)(\nabla\eta\cdot\nabla)\cdot v+\mu\big(x^{(0)}\big)(\Delta\eta)v+q\nabla\eta\end{aligned}$$

and $\tilde{g}=\nabla\eta\cdot v$.

From [6, Theorem IV.6.1 and Ex. IV.6.2], we have

$$\begin{aligned}&\|\eta v\|_{W^{1,s}(B_{2R}(x^{(0)}))}+\|\eta q\|_{L^s(B_{2R}(x^{(0)}))/\mathbb{R}}\\ &\quad\leqslant C_1\big(\|\tilde{f}\|_{W^{-1,s}(B_{2R}(x^{(0)}))}+\|\tilde{g}\|_{L^s(B_{2R}(x^{(0)}))}\big)\\ &\quad\leqslant C_2\sup_{x\in B_{2R}(x^{(0)})}\big|\mu(x)-\mu\big(x^{(0)}\big)\big|\|\eta v\|_{W^{1,s}(B_{2Rl}(x^{(0)}))}\\ &\qquad+C_3\big(\|f\|_{W^{-1,s}(B_{2R}(x^{(0)}))}+\|v\|_{L^s(B_{2R}(x^{(0)}))}+\|q\|_{W^{-1,s}(B_{2R}(x^{(0)}))}\big).\end{aligned}$$

If the radius $2R$ of the ball is small, then $\sup_{x\in B_{2R}(x^{(0)})}|\mu(x)-\mu(x^{(0)})|$ will be small too because of the continuity of μ. So the term $\|\eta v\|_{W^{1,s}(B_{2R}(x^{(0)}))}$ can be absorbed to the left-hand side. Furthermore, from $\eta=1$ in $B_R(x^{(0)})$ we conclude

$$\begin{aligned}&\|v\|_{W^{1,s}(B_R(x^{(0)}))} + \|q\|_{L^s(B_R(x^{(0)}))/\mathbb{R}} \\ &\quad \leqslant C\big(\|f\|_{W^{-1,s}(B_{2R}(x^{(0)}))} + \|v\|_{L^s(B_{2R}(x^{(0)}))} + \|q\|_{W^{-1,s}(B_{2R}(x^{(0)}))}\big).\end{aligned}$$

A similar argument works for the half ball. Then (A.4) holds by a finite covering of $\bar{\Omega}$.

The estimate (A.5) can be proved by the iteration for $m > 0$ provided it is true for $m = 0$. If $(v, q) \in W^{2,s}(\Omega) \times W^{1,s}(\Omega)$, then (A.5) holds for $m = 0$ by a similar proof as for (A.4). We use the difference quotient method to show $(v, q) \in W^{2,s}(\Omega) \times W^{1,s}(\Omega)$. The definition of the difference quotient of a function $w \in W^{k,s}(\Omega)$ $(k \geqslant 0)$ in the direction e_j is given by

$$w^h(x) = \frac{w(x + he_j) - w(x)}{h}, \quad h \neq 0.$$

A useful estimate is

$$\|w^h\|_{W^{k-1,s}(\Omega')} \leqslant C\|w\|_{W^{k,s}(\Omega)} \tag{A.7}$$

for any Ω' satisfying $\bar{\Omega}' \subset \Omega$ and $|h| < \operatorname{dist}(\Omega', \partial\Omega)$.

As before, we consider the case for a ball and a half ball only. Direct computation shows that $(\eta v)^h$ and $(\eta q)^h$ satisfy the Stokes equations (A.6) with the data $\tilde{f}^h$, $\tilde{g}^h$. The first term in $\tilde{f}^h$ will bother us a little bit. We compute

$$\begin{aligned}&\big\{\operatorname{div}\big[2\big(\mu(x) - \mu\big(x^{(0)}\big)\big)\operatorname{Sym}\big(\nabla(\eta v)\big)\big]\big\}^h \\ &\quad = \operatorname{div}\big\{\big[2\big(\mu(x) - \mu\big(x^{(0)}\big)\big)\operatorname{Sym}\big(\nabla(\eta v)\big)\big]^h\big\} \\ &\quad = \operatorname{div}\big[2(\mu^h(x)\operatorname{Sym}\big(\nabla(\eta v)(x + he_j)\big)\big] + \operatorname{div}\big[2\big(\mu(x) - \mu\big(x^{(0)}\big)\big)\operatorname{Sym}\big(\nabla(\eta v)^h\big)\big].\end{aligned}$$

From the estimate (A.4), (A.7), letting R be small as before, we know

$$\|(\eta v)^h\|_{W^{1,s}(B_{2R}(x^{(0)}))} + \|(\eta q)^h\|_{L^s(B_{2R}(x^{(0)}))/\mathbb{R}} \leqslant C.$$

So $(v, q) \in W^{2,s}(B_R(x^{(0)})) \times W^{1,s}(B_R(x^{(0)}))$.

For a half ball $B_R^+(x^{(1)}) := B_R(x^{(1)}) \cap \{x_3 > 0\}$, using the same argument as for a ball, we can conclude that the derivatives $D_j D_k v \in L^2(B_R^+(x^{(1)}))$ for all pairs $(j, k) \neq (3, 3)$, and $D_j \in L^2(B_R^+(x^{(1)}))$ for $j \neq 3$. Then using this obtained result and the system (A.1) satisfied by (v, q), we can get $D_3 D_3 v \in L^2(B_R^+(x^{(1)}))$ and $D_3 \in L^2(B_R^+(x^{(1)}))$. The proof is complete. □

We are going to drop the last two terms in (A.4) and in (A.5). We need the following uniqueness result.

Lemma 14. *Suppose that $\mu \in C^{1,1}(\bar{\Omega})$. If $(v, q) \in W^{1,s}(\Omega) \times L^s(\Omega)$ is a weak solution of* (A.1) *with $f = g = 0$ in Ω and $\psi = 0$ on $\partial\Omega$, then $v = 0$, $q =$* const *a.e. in Ω.*

Proof. If $s = 2$, the uniqueness of the solution is proved in Section 2 of [7]. If $s > 2$, it is also true due to $W^{k,s}(\Omega) \subset W^{k,2}(\Omega)$, $k = 0, 1$. If $1 < s < 2$, from the second part of Lemma 13 for

$m=1$, we know $(v,q)\in W^{3,s}(\Omega)\times W^{2,s}(\Omega)$. Then the uniqueness result for $1<s<2$ follows from the Sobolev imbedding

$$W^{3,s}(\Omega)\times W^{2,s}(\Omega)\subset W^{1,2}(\Omega)\times L^2(\Omega),\quad \text{for any } 1<s<2,$$

and the uniqueness result for $s=2$. □

Arguing as in [6, Lemma IV.6.1], one can drop the last two terms in (A.4) and in (A.5) by Lemma 14 and compactness properties of Sobolev spaces. We obtain

Lemma 15. *Let* $(v,q)\in W^{1,s}(\Omega)\times L^s(\Omega)$ *be a weak solution of* (A.1).

(i) *Suppose that* μ, f, g *and* ψ *are as in Theorem* 10. *We have*

$$\begin{aligned}&\|v\|_{W^{1,s}(\Omega)}+\|q\|_{L^s(\Omega)/\mathbb{R}}\\&\quad\leqslant C\big(\|f\|_{W^{-1,s}(\Omega)}+\|g\|_{L^s(\Omega)}+\|\psi\|_{W^{1-1/s,s}(\partial\Omega)}\big).\end{aligned}\tag{A.8}$$

(ii) *Suppose that* μ, f, g *and* ψ *are as in Theorem* 11. *We have*

$$\begin{aligned}&\|v\|_{W^{m+2,s}(\Omega)}+\|q\|_{W^{m+1,s}(\Omega)/\mathbb{R}}\\&\quad\leqslant C\big(\|f\|_{W^{m,s}(\Omega)}+\|g\|_{W^{m+1,s}(\Omega)}+\|\psi\|_{W^{m+2-1/s,s}(\partial\Omega)}\big).\end{aligned}\tag{A.9}$$

To finish the proofs for Theorems 10 and 11, one only needs to show the existence of weak solution. In view of (A.9), we only need to prove Theorem 10.

Lemma 16. *Suppose that* μ, f, g *and* ψ *are as in Theorem* 10. *Then there exists a weak solution* $(v,q)\in W^{1,s}(\Omega)\times L^s(\Omega)$ *of* (A.1).

Proof. Once again, we only consider the case while g and ψ are zeros. When $s=2$, the existence of weak solution is proved in Section 2 of [7]. For a general $s\in(1,\infty)$, we argue as follows. We first prove that the weak solution $(v,q)\in W^{1,s}(\Omega)\times L^s(\Omega)$ exists when $f\in W^{1,2}(\Omega)$. Since $f\in W^{1,2}(\Omega)\subset W^{-1,2}(\Omega)$, we already showed the existence of weak solution $(v,q)\in W^{1,2}(\Omega)\times L^2(\Omega)$. Moreover, from (A.9) of Lemma 15 with $m=1$, $s=2$ and an imbedding theorem, we obtain

$$(v,q)\in W^{3,2}(\Omega)\times W^{2,2}(\Omega)\subset W^{1,s}(\Omega)\times L^s(\Omega).$$

Now we consider the case when $f\in W^{-1,s}(\Omega)$. Let us choose a sequence $\{f_j\}\subset C_0^\infty\subset W^{1,2}(\Omega)$ such that $\|f_j-f\|_{W^{-1,s}(\Omega)}\to 0$ as $j\to\infty$. Then there exist weak solutions $(v_j,q_j)\in W^{1,s}(\Omega)\times L^s(\Omega)$ corresponding to the data f_j. Using (A.8) we get that there exists $(v,q)\in W^{1,s}(\Omega)\times L^s(\Omega)$ such that

$$v_j\to v\quad\text{strongly in } W^{1,s}(\Omega),\qquad q_j\to q\quad\text{strongly in } L^s(\Omega).$$

Obviously, (v,q) is a weak solution of (A.1). □

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