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AN INVERSE PROBLEM FOR A DYNAMICAL LAMÉ SYSTEM WITH RESIDUAL STRESS*

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Abstract. In this paper we prove Hölder and Lipschitz stability estimates for determining all coefficients of a dynamical Lamé system with residual stress, including the density, Lamé parameters, and the residual stress, by three pairs of observations from the whole boundary or from a part of it. These estimates imply first uniqueness results for determination of all parameters in the residual stress systems from few boundary measurements. Our essential assumptions are that the Lamé system possesses a suitable pseudoconvex function, residual stress is small, and three sets of the initial data satisfy some independence condition.

Key words. elasticity system with residual stress, inverse problem, Carleman estimates

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1. Introduction.

We consider an elasticity system with residual stress. This system is anisotropic; i.e., it exhibits elastic properties with different values when measured in various directions. The assumption about isotropy is too restrictive in most important applications, although it allows deeper mathematical analysis of direct, and especially inverse, problems. While the theory of the unique solvability of direct problems in a quite general anisotropic case is relatively well developed [3], almost nothing is known about determination of anisotropic elastic parameters from additional boundary value data (i.e., about inverse problems).

We handle the simplest anisotropy, known as the Lamé system with residual stress, which is a small perturbation of the classical isotropic Lamé system, by a scalar anisotropic second order operator. Smallness of perturbation is motivated by applications to material science [14]. Assuming that speeds of propagation of shear and compression waves in an unperturbed system satisfy some pseudoconvexity-type conditions (which exclude trapped elastic rays) and that three sets of initial conditions are in a certain sense independent, we obtain first uniqueness and stability results about identification of all nine elastic parameters of an isotropic medium with residual stress from lateral boundary observations. When observation time and the observed part of the boundary are arbitrary, we explicitly describe a domain where coefficients are guaranteed to be unique, and we give a Hölder stability estimate. When observation time is sufficiently large and observation is from the whole lateral boundary, we derive Lipschitz stability estimates. These estimates indicate the possibility of a numerical solution of an inverse problem with high resolution and therefore

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a substantial applied potential.

While our assumptions exclude zero initial data (most natural in many applications), recent progress in generating wave fields by interior sources in geophysics, material sciences, and medicine, and also by a substantial amount of historical seismic data from earthquakes (which are interior sources), make our assumptions more realistic.

Let Ω be an open bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega \in C^8$. The residual stress is modeled by a symmetric second-rank tensor $R(x) = (r_{jk}(x))_{j,k=1}^3 \in C^7(\overline{\Omega})$ which is divergence free

$$\mathrm{div} R = 0 \quad \text{in} \quad \Omega \tag{1.1}$$

and satisfies the boundary condition

$$R\nu = 0 \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

where $\mathrm{div}R$ is a vector-valued function with components given by

$$(\mathrm{div}R)_j = \sum_{k=1}^{3} \partial_k r_{jk}, \quad 1 \le j \le 3.$$

In this paper, $x = (x_1, x_2, x_3) \in \mathbf{R}^3$ and $\nu = (\nu_1, \nu_2, \nu_3)^\top$ is the unit outer normal vector to $\partial\Omega$. Here and below, differential operators ∇ and Δ, without subscripts, are with respect to x variables. Let $Q = \Omega \times (-T, T)$ and $\mathbf{u} = (u_1, u_2, u_3)^\top : Q \to \mathbb{R}^3$ be the displacement vector in Q. We note that $\epsilon(\mathbf{u}) = (\nabla\mathbf{u} + \nabla\mathbf{u}^\top)/2$ is the strain tensor. We consider the initial boundary value problem

$$\mathbf{A}_E\mathbf{u} := \rho\partial_t^2\mathbf{u} - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla(\mathrm{div}\,\mathbf{u}) - (\nabla\lambda)\mathrm{div}\,\mathbf{u} - 2\epsilon(\mathbf{u})\nabla\mu - \mathrm{div}((\nabla\mathbf{u})R) = 0 \text{ in } Q, \tag{1.3}$$

$$\mathbf{u} = \mathbf{u}_0, \quad \partial_t\mathbf{u} = \mathbf{u}_1 \quad \text{on} \quad \Omega \times \{0\}, \tag{1.4}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on} \quad \partial\Omega \times (-T, T), \tag{1.5}$$

where ρ is density and λ and μ are Lamé parameters depending only on x and satisfying inequalities

$$\varepsilon_1 < \mu, \quad \varepsilon_1 < \rho, \quad \varepsilon_1 < \lambda + \mu \quad \text{on} \quad \Omega \tag{1.6}$$

for some positive constant ε_1. Hereafter, we use E to represent the set of elastic coefficients in (1.3), i.e., $E = (\rho, \lambda, \mu, R)$. We will assume that $\rho \in C^6(\overline{\Omega})$ and $\lambda, \mu \in C^7(\overline{\Omega})$. The system (1.3) can be written as

$$\rho\,\partial_t^2\mathbf{u} - \mathrm{div}\,\sigma(\mathbf{u}) = 0,$$

where $\sigma(\mathbf{u}) = \lambda(\mathrm{tr}\,\epsilon)I + 2\mu\epsilon + R + (\nabla\mathbf{u})R$ is a stress tensor. Note that the term div R does not appear in (1.3) due to (1.1). Also, due to the same condition, we can see that

$$(\mathrm{div}((\nabla\mathbf{u})R))_i = \sum_{j,k=1}^{3} r_{jk}\partial_j\partial_k u_i, \quad 1 \le i \le 3.$$

To make sure that problem (1.3) with (1.4) and (1.5) is well-posed, it suffices to assume that

$$\|R\|_{C^1(\overline{\Omega})} < \varepsilon_0 \tag{1.7}$$

for some (small) constant $\varepsilon_0 > 0$. Assumption (1.7) is also motivated by material science applications [14]. Indeed, residual stress of interest to engineers is due to past thermal changes in steel production which do not significantly change the elastic properties of steel. It is not hard to see that if ε_0 is sufficiently small, then the boundary value problem (1.3)–(1.5) is hyperbolic, and hence for any initial data $(\mathbf{u}_0, \mathbf{u}_1) \in H^1(\Omega) \times L^2(\Omega)$ and lateral Dirichlet data $\mathbf{g} \in C^1([-T,T]; H^1(\Omega))$, $\mathbf{u}_0 = \mathbf{g}$ on $\partial\Omega \times \{0\}$, there exists a unique solution $\mathbf{u}(\cdot; E; \mathbf{u}_0, \mathbf{u}_1, \mathbf{g}) \in C([-T,T]; H^1(\Omega))$ to (1.3)–(1.5).

In this paper we are interested in the following inverse problem: Let Γ be an open subset of $\partial\Omega$ with $\partial\Gamma \in C^1$. Determine density ρ, Lamé parameters λ, μ, and the residual stress R (a total of nine functions) from Cauchy-type data $(\mathbf{u}, \sigma(\mathbf{u})\nu)$ on $\Gamma \times (-T, T)$, where $\mathbf{u} = \mathbf{u}(\cdot; E; \mathbf{u}_0, \mathbf{u}_1, \mathbf{g})$, given for a finite number of pairs of initial data $(\mathbf{u}_0, \mathbf{u}_1)$.

We will address uniqueness and stability issues. Our main focus is on stability, since stability implies uniqueness. This work is a sequel to our recent paper [11], where we demonstrated uniqueness of only R assuming known constant ρ, λ, μ. Our method is based on Carleman estimates techniques initiated by Bukhgeim and Klibanov [2]. For works on Carleman estimates and related inverse problems for scalar equations, we refer to books [1] and [12] for further details and references. The method of [2] was modified for scalar equations in the paper of Imanuvilov and Yamamoto [6]. It was found by Imanuvilov, Isakov, and Yamamoto [8] that this modification allows one to obtain uniqueness and stability for coefficients of systems of equations; in particular, in [8] there is a first uniqueness result for all three elastic parameters ρ, λ, μ of isotropic elasticity. For further results on identification of the isotropic Lamé system we refer to [7]. For Carleman estimates and uniqueness of the continuation for the residual stress system (1.3) and for identification of the source term and R with given constant ρ, λ, μ, we refer to [10], [11], [13]. In the case of many boundary measurements and zero initial data, there are only partial results on identification of residual stress [5], [15]. In the present work we will show that we can determine all nine parameters in (1.3)–(1.5) by three pairs of Cauchy data. We will derive a Hölder stability estimate in the convex hull of the observation surface Γ and a Lipschitz stability estimate for (ρ, λ, μ, R) in Ω when $\Gamma = \partial\Omega$ and observation time T is large.

We are now ready to state our main results. Let $d = \inf |x|$ and $D = \sup |x|$ over $x \in \Omega$. We will assume that

$$0 < d. \tag{1.8}$$

Let θ be a positive number. For a function $c \in C^1(\overline{\Omega})$ we introduce the condition

$$\theta^2 < c \quad \text{and} \quad \theta^2 c + D\theta\sqrt{c}|\nabla c| + \frac{1}{2} c\, x \cdot \nabla c < c^2 \quad \text{on} \quad \overline{\Omega}. \tag{1.9}$$

Let $\varepsilon_0 > 0$ be given as in (1.7), $M > 0$ be arbitrarily fixed, and $\mathcal{E}_{\varepsilon_0,M}$ be the class of functions (elastic parameters) defined by

$$\begin{aligned} \mathcal{E}_{\varepsilon_0,M} = \quad & \{(\rho, \lambda, \mu, R) : \ \|\rho\|_{C^6(\overline{\Omega})} + \|\lambda\|_{C^7(\overline{\Omega})} + \|\mu\|_{C^7(\overline{\Omega})} + \|R\|_{C^6(\overline{\Omega})} < M : \\ & \rho, \lambda, \mu \text{ satisfy (1.6) and } c = \frac{\mu}{\rho},\ c = \frac{\lambda + 2\mu}{\rho} \text{ satisfy (1.9)}, \\ & R \text{ is symmetric and satisfies (1.1), (1.2), and (1.7)}\}. \end{aligned}$$

To study the inverse problem, we need not only the well-posedness of (1.3)–(1.5) but also some extra regularity of the solution $\mathbf{u}$. To achieve the latter property, the initial and Dirichlet data $(\mathbf{u}_0, \mathbf{u}_1, \mathbf{g})$ are required to satisfy some smoothness and compatibility conditions. More precisely, we will assume that $\mathbf{u}_0 \in H^9(\Omega), \mathbf{u}_1 \in H^8(\Omega)$, and $\mathbf{g} \in C^8([-T,T]; H^1(\partial\Omega)) \cap C^5([-T,T]; H^4(\partial\Omega))$ and that they satisfy standard compatibility conditions of order 8 at $\partial\Omega \times \{0\}$. By using energy estimates [3] and Sobolev embedding theorems as in [8], one can show that

$$\|\partial_x^\alpha \partial_t^\beta \mathbf{u}\|_{C^0(\overline{Q})} \leq C \tag{1.10}$$

for $|\alpha| \leq 2$ and $0 \leq \beta \leq 5$. We will use three sets of initial data $(\mathbf{u}_0(\cdot;j), \mathbf{u}_1(\cdot;j))$, $j = 1,2,3$. To guarantee uniqueness in the inverse problem, we impose some nondegeneracy condition on the initial data. Namely, let $\mathbf{M}$ denote the 18×13 matrix

$$\begin{pmatrix}
\mu\Delta\mathbf{u}_0(\cdot;1) + (\lambda+\mu)\nabla(\operatorname{div}\mathbf{u}_0(\cdot;1)) & \operatorname{div}\mathbf{u}_0(\cdot;1)I_3 & 2\epsilon(\mathbf{u}_0(\cdot;1)) & \mathbf{R}(\mathbf{u}_0(\cdot;1)) \\
\mu\Delta\mathbf{u}_1(\cdot;1) + (\lambda+\mu)\nabla(\operatorname{div}\mathbf{u}_1(\cdot;1)) & \operatorname{div}\mathbf{u}_1(\cdot;1)I_3 & 2\epsilon(\mathbf{u}_1(\cdot;1)) & \mathbf{R}(\mathbf{u}_1(\cdot;1)) \\
\mu\Delta\mathbf{u}_0(\cdot;2) + (\lambda+\mu)\nabla(\operatorname{div}\mathbf{u}_0(\cdot;2)) & \operatorname{div}\mathbf{u}_0(\cdot;2)I_3 & 2\epsilon(\mathbf{u}_0(\cdot;2)) & \mathbf{R}(\mathbf{u}_0(\cdot;2)) \\
\mu\Delta\mathbf{u}_1(\cdot;2) + (\lambda+\mu)\nabla(\operatorname{div}\mathbf{u}_1(\cdot;2)) & \operatorname{div}\mathbf{u}_1(\cdot;2)I_3 & 2\epsilon(\mathbf{u}_1(\cdot;2)) & \mathbf{R}(\mathbf{u}_1(\cdot;2)) \\
\mu\Delta\mathbf{u}_0(\cdot;3) + (\lambda+\mu)\nabla(\operatorname{div}\mathbf{u}_0(\cdot;3)) & \operatorname{div}\mathbf{u}_0(\cdot;3)I_3 & 2\epsilon(\mathbf{u}_0(\cdot;3)) & \mathbf{R}(\mathbf{u}_0(\cdot;3)) \\
\mu\Delta\mathbf{u}_1(\cdot;3) + (\lambda+\mu)\nabla(\operatorname{div}\mathbf{u}_1(\cdot;3)) & \operatorname{div}\mathbf{u}_1(\cdot;3)I_3 & 2\epsilon(\mathbf{u}_1(\cdot;3)) & \mathbf{R}(\mathbf{u}_1(\cdot;3))
\end{pmatrix},$$

where I_3 is the 3×3 identity matrix, and where $\mathbf{R}(\mathbf{v})$ is a 3×6 matrix defined by

$$\mathbf{R}(\mathbf{v}) = \begin{pmatrix} \partial_1^2\mathbf{v} & 2\partial_1\partial_2\mathbf{v} & 2\partial_1\partial_3\mathbf{v} & \partial_2^2\mathbf{v} & 2\partial_2\partial_3\mathbf{v} & \partial_3^2\mathbf{v} \end{pmatrix}. \tag{1.11}$$

We will assume that

(1.12) there exists a 13×13 minor of $\mathbf{M}$ such that the absolute value of its determinant is greater than some constant $\varepsilon_2 > 0$ on $\overline{\Omega}$.

One can check that $\mathbf{u}_0(\cdot;1) = (x_1x_2,0,0)^\top$, $\mathbf{u}_1(\cdot;1) = (0,0,0)^\top$, $\mathbf{u}_0(\cdot;2) = (x_1,x_2,x_3)^\top$, $\mathbf{u}_1(\cdot;2) = (0,x_2,x_3)$, $\mathbf{u}_0(\cdot;3) = (x_1^2,x_2^2,x_3^2)^\top$, and $\mathbf{u}_1(\cdot;3) = (x_2x_3,x_1x_3,x_1x_2)^\top$ satisfy (1.12). Here, 13 row vectors from rows 2 and 7–18 are linearly independent on $\overline{\Omega}$. In fact, the direct calculations yield that the absolute value of the determinant of 13×13 minor is $2^{10}(\lambda(x) + \mu(x))$, and we can choose $\varepsilon_2 = 2^{10}\varepsilon_1$ in (1.12), where $\varepsilon_1 > 0$ is given in (1.6).

Condition (1.12) does not hold physically, but for the identification of the residual stress, the density, and the Lamé coefficients we have to set up the system by choosing initial values artificially, e.g., in a laboratory. The above example of such initial values suggests that there may be many choices for it.

We will use the following notation.

C, γ are generic constants depending only on $\Omega, T, \delta, \varepsilon_0, \varepsilon_1, \varepsilon_2, M, \mathbf{u}_0(\cdot;j), \mathbf{u}_1(\cdot;j)$, $j = 1,2,3$ (any other dependence will be indicated). $\|\cdot\|_{(k)}(Q)$ is the norm in the Sobolev space $H^k(Q)$. $Q(\varepsilon) = Q \cap \{\varepsilon < |x|^2 - \theta^2t^2 - d_1^2\}$ and $\Omega(\varepsilon) = \Omega \cap \{\varepsilon < |x|^2 - d_1^2\}$, where $d_1 \geq d$. Let $\mathbf{u}(;1;j)$ and $\mathbf{u}(;2;j)$ be solutions of (1.3), (1.4) with initial data $(\mathbf{u}_0(;j), \mathbf{u}_1(;j))$, for $j = 1,2,3$, corresponding to sets of coefficients $E_1 = (\rho_1, \lambda_1, \mu_1, R_1)$ and $E_2 = (\rho_2, \lambda_2, \mu_2, R_2)$, respectively. We will consider the Dirichlet data (displacements) as measurements (observations). We introduce the norm of the differences of the data as

$$F = \sum_{j=1}^{3}\sum_{\beta=2}^{4}(\|\partial_t^\beta(\mathbf{u}(;2;j) - \mathbf{u}(;1;j))\|_{(\frac{5}{2})}(\Gamma \times (-T,T))$$
$$+\|\partial_t^\beta\sigma(\mathbf{u}(;2;j) - \mathbf{u}(;1;j))\nu\|_{(\frac{3}{2})}(\Gamma \times (-T,T))).$$

This data norm includes the fourth order time derivatives and is technically necessary for our proof of the Lipschitz stability in the inverse problem. Because we have to obtain a suitable number of equations in 13 unknown functions ρ, λ, μ, r_{jk} $(1 \le j \le k \le 3)$ in terms of data, we will use also $\partial_t^3 \mathbf{u}(\ ;k;j)$, $j = 1,2,3$ at $t = 0$ (see (3.9)). For that, we need L^2-estimates of $\partial_t^4 \mathbf{u}$ in (x,t) in the Carleman estimate, which yield estimates of $\partial_t^3 \mathbf{u}(\ ;k;j)$ at $t = 0$ (see (3.12)).

We first state the Hölder-type estimate for determining coefficients in $\Omega(\varepsilon)$.

THEOREM 1.1. *Assume that the domain Ω satisfies* (1.8) *and for some d_1 $(\ge d)$,*

$$|x|^2 - d_1^2 < 0 \text{ when } x \in (\partial\Omega \setminus \Gamma) \text{ and } D^2 - \theta^2 T^2 - d_1^2 < 0. \tag{1.13}$$

Let the initial data $(\mathbf{u}_0(;j), \mathbf{u}_1(;j))$, $j = 1,2,3$, satisfy (1.12) *with $\lambda = \lambda_1, \mu = \mu_1$.*

Then there exist ε_0 and constants $C, \gamma \in (0,1)$ such that for $E_1, E_2 \in \mathcal{E}_{\varepsilon_0,M}$ with

$$\lambda_1 = \lambda_2 \quad \text{and} \quad \mu_1 = \mu_2 \quad \text{on} \quad \Gamma, \tag{1.14}$$

one has

$$\|\rho_1-\rho_2\|_{(0)}(\Omega(\varepsilon))+\|\lambda_1-\lambda_2\|_{(0)}(\Omega(\varepsilon))+\|\mu_1-\mu_2\|_{(0)}(\Omega(\varepsilon))+\|R_1-R_2\|_{(0)}(\Omega(\varepsilon)) \le CF^\gamma. \tag{1.15}$$

Remark 1.2. If $d_1 < D$, then the second condition in (1.13) implies that

$$\frac{D^2 - d_1^2}{\theta^2} < T^2.$$

In other words, the observation time T needs to be sufficiently large. In this case, we can determine elastic parameters in the domain $\Omega(\varepsilon)$. The domain $\Omega(\varepsilon)$ is discussed in [9, section 3.4].

If Γ is the whole lateral boundary and T is sufficiently large, then a much stronger (and in a certain sense the best possible) Lipschitz stability estimate holds.

THEOREM 1.3. *Let $d_1 = d$. Assume that*

$$\frac{D^2 - d^2}{\theta^2} < T^2 < \frac{d^2}{\theta^2}. \tag{1.16}$$

Let the initial data $(\mathbf{u}_0(;j), \mathbf{u}_1(;j))$, $j = 1,2,3$, satisfy (1.12) *with $\lambda = \lambda_1, \mu = \mu_1$, and $\Gamma = \partial\Omega$.*

Then there exists ε_0 in (1.7) *and C such that for $E_1, E_2 \in \mathcal{E}_{\varepsilon_0,M}$ satisfying the conditions*

$$\rho_1 = \rho_2,\ R_1 = R_2,\ \partial^\alpha \lambda_1 = \partial^\alpha \lambda_2, \text{ and } \partial^\alpha \mu_1 = \partial^\alpha \mu_2 \text{ on } \Gamma \text{ when } |\alpha| \le 1, \tag{1.17}$$

one has

$$\|\rho_1 - \rho_2\|_{(0)}(\Omega) + \|\lambda_1 - \lambda_2\|_{(0)}(\Omega) + \|\mu_1 - \mu_2\|_{(0)}(\Omega) + \|R_1 - R_2\|_{(0)}(\Omega) \le CF. \tag{1.18}$$

Remark 1.4. Condition (1.16) is needed for pseudoconvexity of weight function φ in Carleman estimates in the next sections and generally cannot be removed. Existence of T is guaranteed by the condition $D^2 < 2d^2$. Under the additional assumption $e \cdot \nabla c(x) \le 0, x \in \Omega$ for some direction e, the condition $D^2 < 2d^2$ can be achieved by using translation $x = y + Le$ with large L.

As mentioned previously, the proofs of these theorems rely on Carleman estimates. We briefly describe the needed Carleman estimates in section 2. Using this estimate we will prove in section 3 the Hölder stability estimate (1.15). In section 4, we derive the Lipschitz stability estimate for our inverse problem.

2. Carleman estimate. In this section we will collect Carleman estimates needed to solve our inverse problem. Their proofs can be found in [10] and [11]. Let $\psi(x,t) = |x|^2 - \theta^2 t^2 - d_1^2$ and $\varphi(x,t) = \exp(\frac{\eta}{2}\psi(x,t))$. Due to conditions (1.9) and (1.13) and known sufficient conditions of pseudoconvexity [9, Theorem 3.4.1], we can fix (large) $\eta > 0$ so that the phase function φ is strongly pseudoconvex on $\overline{Q(0)}$ with respect to

$$\frac{\rho}{\mu}\partial_t^2 - \Delta, \qquad \frac{\rho}{\lambda + 2\mu}\partial_t^2 - \Delta.$$

Similarly, (1.9) and the second inequality in (1.16) guarantee strong pseudoconvexity of φ on $\bar{Q}$.

THEOREM 2.1. *There are constants ε_0 and C such that under the conditions of Theorem 1.3 for $E \in \mathcal{E}_{\varepsilon_0,M}$,*

(2.1)
$$\int_Q (\tau|\nabla_{x,t}\mathbf{u}|^2 + \tau|\nabla_{x,t}\mathrm{div}\,\mathbf{u}|^2 + \tau|\nabla_{x,t}\mathrm{curl}\,\mathbf{u}|^2 + \tau^3|\mathbf{u}|^2 + \tau^3|\mathrm{div}\,\mathbf{u}|^2 + \tau^3|\mathrm{curl}\,\mathbf{u}|^2)e^{2\tau\varphi}$$
$$\leq C \int_Q (|\mathbf{A}_E\mathbf{u}|^2 + |\nabla(\mathbf{A}_E\mathbf{u})|^2)e^{2\tau\varphi}$$

for all $\mathbf{u} \in H_0^3(Q)$, and under the conditions of Theorem 1.1,

$$\int_{Q(0)} (\tau^2|\mathbf{u}|^2 + |\mathrm{div}\,\mathbf{u}|^2 + |\mathrm{curl}\,\mathbf{u}|^2 + \tau^{-1}|\nabla\mathbf{u}|^2)e^{2\tau\varphi} \leq C\int_{Q(0)} |\mathbf{A}_E\mathbf{u}|^2 e^{2\tau\varphi} \tag{2.2}$$

for all $\mathbf{u} \in H_0^2(Q(0))$.

The Carleman estimates of Theorem 2.1 form our basic tool for treating the inverse problem. The basic idea in proving Theorem 2.1 is to reduce (1.3) to an extended system of dimension 7 for $(\mathbf{u}, \mathrm{div}\,\mathbf{u}, \mathrm{curl}\,\mathbf{u})$. The resulting new system is not principally diagonalized. However, when the residual stress R is small, the second derivatives of $\mathbf{u}$ can be bounded by first derivatives of $\mathrm{div}\,\mathbf{u}$ and $\mathrm{curl}\,\mathbf{u}$. We refer to [10] and [11] for detailed computations. For the case considered here, we need only verify the strong pseudoconvexity of φ on $\overline{Q}$ or on $\bar{Q}$. Under conditions (1.9), (1.13) or (1.16) one can check that φ satisfies the required property when ε_0 is small and η is large (see [9] or [10]). An estimate similar to (2.2) was also derived in [8].

In order to use (2.1), it is required that the Cauchy data of the solution and the source term vanish on the lateral boundary. To handle nonvanishing Cauchy data, the following lemma is useful.

LEMMA 2.2. *For any pair of $(\mathbf{g}_0, \mathbf{g}_1) \in H^{\frac{5}{2}}(\partial\Omega \times (-T,T)) \times H^{\frac{3}{2}}(\partial\Omega \times (-T,T))$, we can find a vector-valued function $\mathbf{u}^* \in H^3(Q)$ such that*

$$\mathbf{u}^* = \mathbf{g}_0, \quad \sigma(\mathbf{u}^*)\nu = \mathbf{g}_1, \quad \mathbf{A}_E\mathbf{u}^* = 0 \quad \text{on} \quad \partial\Omega \times (-T,T),$$

and

$$\|\mathbf{u}^*\|_{(3)}(Q) \leq C(\|\mathbf{g}_0\|_{(\frac{5}{2})}(\partial\Omega \times (-T,T)) + \|\mathbf{g}_1\|_{(\frac{3}{2})}(\partial\Omega \times (-T,T))) \tag{2.3}$$

for some $C > 0$, provided ε_0 in (1.7) is sufficiently small.

Proof. By standard extension theorems for any $\mathbf{g}_2 \in H^{\frac{1}{2}}(\partial\Omega \times (-T,T))$ we can find $\mathbf{u}^{**} \in H^3(Q)$ so that

$$\mathbf{u}^{**} = \mathbf{g}_0, \quad \sigma(\mathbf{u}^{**})\nu = \mathbf{g}_1, \quad \partial_\nu^2\mathbf{u}^{**} = \mathbf{g}_2 \quad \text{on} \quad \partial\Omega \times (-T,T)$$

and

$$\|\mathbf{u}^{**}\|_{(3)}(Q) \le C(\|\mathbf{g}_2\|_{(\frac{1}{2})}(\partial\Omega \times (-T,T)) + \|\mathbf{g}_1\|_{(\frac{3}{2})}(\partial\Omega \times (-T,T)) + \|\mathbf{g}_0\|_{(\frac{5}{2})}(\partial\Omega \times (-T,T))).$$

Since $\partial\Omega \times (-T,T)$ is noncharacteristic with respect to $\mathbf{A}_E$ provided (1.6) holds and ε_0 is small, the condition $\mathbf{A}_E\mathbf{u}^{**} = 0$ on $\partial\Omega \times (-T,T)$ is equivalent to the fact that $\mathbf{g}_2$ can be written as a linear combination (with C^1 coefficients) of $\partial_t^2\mathbf{g}_0$ and tangential derivatives of $\mathbf{g}_0$ (up to second order) and of $\mathbf{g}_1$ (up to first order) along $\partial\Omega$. In particular,

$$\|\mathbf{g}_2\|_{(\frac{1}{2})}(\partial\Omega \times (-T,T)) \le C(\|\mathbf{g}_1\|_{(\frac{3}{2})}(\partial\Omega \times (-T,T)) + \|\mathbf{g}_0\|_{(\frac{5}{2})}(\partial\Omega \times (-T,T))).$$

Choosing $\mathbf{g}_2$ as this linear combination, we obtain (2.3). ☐

To handle $\nabla\lambda$ and $\nabla\mu$ in (1.3), we need other Carleman estimates. We first derive the estimate needed in the proof of Theorem 1.1. Let d_1 be given as in Theorem 1.1. Then we can see that $\partial\Omega(\varepsilon) = (\Gamma \cup \{|x|^2 = d_1^2 + \varepsilon\}) \cap \bar{\Omega}$.

LEMMA 2.3. *Let $f \in C^1(\overline{\Omega})$ satisfy $f|_\Gamma = 0$. Then*

$$\tau \int_{\Omega(\varepsilon)} |f(x)|^2 e^{2\tau\varphi(x,0)} dx \le C \int_{\Omega(\varepsilon)} |\nabla f(x)|^2 e^{2\tau\varphi(x,0)} dx. \tag{2.4}$$

Proof. Denote $\varphi_0(x) = \varphi(x,0)$. Let $g = e^{\tau\varphi_0} f$; then $e^{\tau\varphi_0}\nabla f = \nabla g - \tau\nabla\varphi_0 g$. Note that $g|_\Gamma = 0$. We observe that $\nabla\varphi_0(x) = \eta x\varphi_0(x)$, and thus on $\partial\Omega(\varepsilon) \setminus \Gamma$ with the unit outer normal $\nu(= -x/|x|)$,

$$\partial_\nu\varphi_0(x) = \nabla\varphi_0 \cdot \nu = -\eta|x|\varphi_0(x). \tag{2.5}$$

Using integration by parts and (2.5), we have that

$$\begin{aligned}
&\int_{\Omega(\varepsilon)} |\nabla g - \tau\nabla\varphi_0 g|^2 \\
&\quad= \int_{\Omega(\varepsilon)} |\nabla g|^2 + \tau^2 \int_{\Omega(\varepsilon)} |\nabla\varphi_0 g|^2 - 2\tau \int_{\Omega(\varepsilon)} \nabla g \cdot \nabla\varphi_0 g \\
&\quad\ge -\tau \int_{\Omega(\varepsilon)} \nabla\varphi_0 \cdot \nabla(g^2) \\
&\quad= -\tau \int_{\partial\Omega(\varepsilon)\setminus\Gamma} \partial_\nu\varphi_0 g^2 + \tau \int_{\Omega(\varepsilon)} \Delta\varphi_0 g^2 \\
&\quad= \tau \int_{\partial\Omega(\varepsilon)\setminus\Gamma} \eta|x|\varphi_0(x) g^2(x) d\Gamma(x) + \tau \int_{\Omega(\varepsilon)} (3\eta + \eta^2|x|^2)\varphi_0 g^2(x) dx \\
&\quad\ge C \int_{\Omega(\varepsilon)} g^2,
\end{aligned}$$

which implies (2.4). ☐

The following estimate is useful in proving Theorem 1.3 (see also [8, Lemma 3.6]).

COROLLARY 2.4. *Let $f \in C^1(\overline{\Omega})$ and $f = 0$ on $\partial\Omega$. Then we have*

$$\tau \int_\Omega |f(x)|^2 e^{2\tau\varphi(x,0)} dx \le C \int_\Omega |\nabla f(x)|^2 e^{2\tau\varphi(x,0)} dx.$$

3. Hölder stability for the determination of coefficients. In this section we prove the first main result of the paper, Theorem 1.1. Let us denote $\mathbf{u}(;j) = \mathbf{u}(;2;j) - \mathbf{u}(;1;j)$ for $j = 1,2,3$, and $\mathbf{F} = (f_1, f_2, \dots, f_9, R)^\top$, where $f_1 = \rho_1 - \rho_2$, $f_2 = \lambda_1 - \lambda_2$, $f_3 = \mu_1 - \mu_2$, $(f_4, f_5, f_6)^\top = \nabla f_2$, $(f_7, f_8, f_9)^\top = \nabla f_3$, and

$$R^\top = \begin{pmatrix} r_{11} \\ r_{12} \\ r_{13} \\ r_{22} \\ r_{23} \\ r_{33} \end{pmatrix} = \begin{pmatrix} r_{1,11} - r_{2,11} \\ r_{1,12} - r_{2,12} \\ r_{1,13} - r_{2,13} \\ r_{1,22} - r_{2,22} \\ r_{1,23} - r_{2,23} \\ r_{1,33} - r_{2,33} \end{pmatrix}.$$

Subtracting equations (1.3) for $\mathbf{u}(;1;j)$ from the equations for $\mathbf{u}(;2;j)$ yields

$$\mathbf{A}_{E_2}\mathbf{u}(;j) = \mathcal{A}(\mathbf{u}(;1;j))\mathbf{F} \quad \text{on} \quad Q, \tag{3.1}$$

where

$$\begin{aligned} \mathcal{A}(\mathbf{v})\mathbf{F} = \quad & -f_1\partial_t^2\mathbf{v} + (f_2 + f_3)\nabla(\operatorname{div}\mathbf{v}) + f_3\Delta\mathbf{v} + \operatorname{div}\mathbf{v}(f_4, f_5, f_6)^\top \\ & + 2\epsilon(\mathbf{v})(f_7, f_8, f_9)^\top + \sum_{j,k=1}^{3} r_{jk}\partial_j\partial_k\mathbf{v} \end{aligned}$$

and

$$\mathbf{u}(;j) = \partial_t\mathbf{u}(;j) = 0 \quad \text{on} \quad \Omega \times \{0\}. \tag{3.2}$$

Differentiating (3.1) in t and using the time-independence of the coefficients of the system, we get

$$\mathbf{A}_{E_2}\mathbf{U}(;j) = \mathcal{A}(\mathbf{U}(;1;j))\mathbf{F} \quad \text{on} \quad Q, \tag{3.3}$$

where

$$\mathbf{U}(;j) = \begin{pmatrix} \partial_t^2\mathbf{u}(;j) \\ \partial_t^3\mathbf{u}(;j) \\ \partial_t^4\mathbf{u}(;j) \end{pmatrix}, \quad \mathbf{U}(;1;j) = \begin{pmatrix} \partial_t^2\mathbf{u}(;1;j) \\ \partial_t^3\mathbf{u}(;1;j) \\ \partial_t^4\mathbf{u}(;1;j) \end{pmatrix},$$

and

$$\mathcal{A}(\mathbf{U}(;1;j)) = \begin{pmatrix} \mathcal{A}(\partial_t^2\mathbf{u}(;1;j)) \\ \mathcal{A}(\partial_t^3\mathbf{u}(;1;j)) \\ \mathcal{A}(\partial_t^4\mathbf{u}(;1;j)) \end{pmatrix}.$$

By extension theorems for Sobolev spaces, there exists $\mathbf{U}^*(;j) \in H^2(Q)$ such that

$$\mathbf{U}^*(;j) = \mathbf{U}(;j),\ \sigma(\mathbf{U}^*(;j))\nu = \sigma(\mathbf{U}(;j))\nu \quad \text{on} \quad \Gamma \times (-T,T) \tag{3.4}$$

and

$$\|\mathbf{U}^*(;j)\|_{(2)}(Q) \le C(\|\mathbf{U}(;j)\|_{(\frac{3}{2})}(\Gamma \times (-T,T)) + \|\sigma(\mathbf{U})(;j)\nu\|_{(\frac{1}{2})}(\Gamma \times (-T,T))) \le CF \tag{3.5}$$

for all $j = 1,2,3$ due to the definitions of $\mathbf{u}(;j), \mathbf{U}(;j)$, and F. We now introduce $\mathbf{V}(;j) = \mathbf{U}(;j) - \mathbf{U}^*(;j)$. Then

$$\mathbf{A}_{E_2}\mathbf{V}(;j) = \mathcal{A}(\mathbf{U}(;1;j))\mathbf{F} - \mathbf{A}_{E_2}\mathbf{U}^*(;j) \quad \text{on} \quad Q \tag{3.6}$$

and

$$\mathbf{V}(;j) = \sigma(\mathbf{V})(;j)\nu = 0 \quad \text{on} \quad \Gamma \times (-T, T). \tag{3.7}$$

To use the Carleman estimate (2.2), we introduce a cut-off function $\chi \in C^2(\mathbb{R}^4)$ such that $0 \le \chi \le 1$, $\chi = 1$ on $Q(\frac{\varepsilon}{2})$, and $\chi = 0$ on $Q \setminus Q(0)$. By the Leibniz formula,

$$\mathbf{A}_{E_2}(\chi \mathbf{V}(;j)) = \chi \mathbf{A}_{E_2}(\mathbf{V}(;j)) + \mathbf{A}_1 \mathbf{V}(;j) = \chi \mathcal{A}\mathbf{F} - \chi \mathbf{A}_{E_2} \mathbf{U}^*(;j) + \mathbf{A}_1 \mathbf{V}(;j)$$

due to (3.6). Here (and below) $\mathbf{A}_1$ denotes a first order matrix differential operator with coefficients uniformly bounded by $C(\varepsilon)$. By the choice of χ, $\mathbf{A}_1\mathbf{V}(;j) = 0$ on $Q(\frac{\varepsilon}{2})$. It is not hard to see that (3.7) implies that $\mathbf{V}(;j) = \partial_\nu \mathbf{V}(;j) = 0$ on $\Gamma \times (-T, T)$. Hence due to the first condition of (1.13), the function $\chi\mathbf{V}(;j) \in H_0^2(Q(0))$ (see, for example [4, Corollary 1.5.1.6, p. 39]). Observe, that $\chi = 0$ is zero near the non-C^8-smooth part of $\partial Q(0)$, and therefore we can use results for C^8-smooth domains by slightly extending $Q(0)$. So we can apply to $\chi\mathbf{V}(;j)$ the Carleman estimate (2.2) to get

$$\int_Q \tau |\chi \mathbf{V}(;j)|^2 e^{2\tau\varphi}$$

$$\le C \int_Q (|\mathbf{F}|^2 + |\mathbf{A}_{E_2}(\mathbf{U}^*(;j))|^2) e^{2\tau\varphi} + C \int_{Q \setminus Q(\frac{\varepsilon}{2})} |\mathbf{A}_1 \mathbf{V}(;j)|^2 e^{2\tau\varphi}$$

$$\le C \left(\int_Q |\mathbf{F}|^2 e^{2\tau\varphi} + F^2 e^{2\tau\Phi} + C(\varepsilon) e^{2\tau\varepsilon_1} \right), \tag{3.8}$$

where $\Phi = \sup \varphi$ over Q and $\varepsilon_1 = e^{\frac{\eta\varepsilon}{4}}$. To get the last inequality, we used the bounds (3.5) and (1.10).

On the other hand, from (1.3), (3.1), (3.2) we have

$$\rho_2 \partial_t^2 \mathbf{u}(;j) = \mathcal{A}(\mathbf{u}(;1;j))\mathbf{F},$$

$$\rho_2 \partial_t^3 \mathbf{u}(;j) = \mathcal{A}(\partial_t \mathbf{u}(;1;j))\mathbf{F}$$

on $\Omega \times \{0\}$. We now want to rearrange the formulas above. Let $\mathbf{a}_{kj} = -\partial_t^{2+k}\mathbf{u}(0;1;j)$, $\mathbf{b}_{kj} = \nabla(\operatorname{div} \mathbf{u}_k(;j))$, $\mathbf{c}_{kj} = \Delta \mathbf{u}_k(;j) + \nabla \operatorname{div} \mathbf{u}_k(;j)$, $\mathbf{B}_{kj} = \operatorname{div} \mathbf{u}_k(;j)$, $\mathbf{C}_{kj} = 2\epsilon(\mathbf{u}_k(;j))$, and $\mathbf{R}_{kj} = \mathbf{R}(\mathbf{u}_k(;j))$ (see the definition of $\mathbf{R}$ in (1.11)), where $k = 0, 1$ and $j = 1, 2, 3$. Using that $\mathbf{u}(;1;j) = \mathbf{u}_0(;j), \partial_t \mathbf{u}(;1;j) = \mathbf{u}_1(;j)$ on $\Omega \times \{0\}$, we have

$$\begin{pmatrix} \mathbf{a}_{01} & \mathbf{B}_{01}I_3 & \mathbf{C}_{01} & \mathbf{R}_{01} \\ \mathbf{a}_{11} & \mathbf{B}_{11}I_3 & \mathbf{C}_{11} & \mathbf{R}_{11} \\ \mathbf{a}_{02} & \mathbf{B}_{02}I_3 & \mathbf{C}_{02} & \mathbf{R}_{02} \\ \mathbf{a}_{12} & \mathbf{B}_{12}I_3 & \mathbf{C}_{12} & \mathbf{R}_{12} \\ \mathbf{a}_{03} & \mathbf{B}_{03}I_3 & \mathbf{C}_{03} & \mathbf{R}_{03} \\ \mathbf{a}_{13} & \mathbf{B}_{13}I_3 & \mathbf{C}_{13} & \mathbf{R}_{13} \end{pmatrix} \begin{pmatrix} f_1 \\ f_4 \\ \vdots \\ f_9 \\ r_{11} \\ \vdots \\ r_{33} \end{pmatrix} = \rho_2 \begin{pmatrix} \partial_t^2 \mathbf{u}(,0;1) \\ \partial_t^3 \mathbf{u}(,0;1) \\ \partial_t^2 \mathbf{u}(,0;2) \\ \partial_t^3 \mathbf{u}(,0;2) \\ \partial_t^2 \mathbf{u}(,0;3) \\ \partial_t^3 \mathbf{u}(,0;3) \end{pmatrix} - \begin{pmatrix} \mathbf{b}_{01} & \mathbf{c}_{01} \\ \mathbf{b}_{11} & \mathbf{c}_{11} \\ \mathbf{b}_{02} & \mathbf{c}_{02} \\ \mathbf{b}_{12} & \mathbf{c}_{12} \\ \mathbf{b}_{03} & \mathbf{c}_{03} \\ \mathbf{b}_{13} & \mathbf{c}_{13} \end{pmatrix} \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \tag{3.9}$$

on Ω. From the system (1.3) at $t = 0$ and from this system differentiated in t and taken at $t = 0$, we obtain

$$
\begin{aligned}
\mathbf{a}_{kj} &= -\frac{\mu_1}{\rho_1}\Delta \mathbf{u}_k(;j) - \frac{\lambda_1+\mu_1}{\rho_1}\nabla(\operatorname{div}\mathbf{u}_k(;j)) - \operatorname{div}\mathbf{u}_k(;j)\frac{\nabla\lambda_1}{\rho_1} \\
&\quad -2\epsilon(\mathbf{u}_k(;j))\frac{\nabla\mu_1}{\rho_1} - \sum_{\ell,m=1}^{3} r_{1,\ell m}\partial_\ell\partial_m \mathbf{u}_k(;j) \\
&= -\frac{\mu_1}{\rho_1}\Delta \mathbf{u}_k(;j) - \frac{\lambda_1+\mu_1}{\rho_1}\nabla(\operatorname{div}\mathbf{u}_k(;j)) - \mathbf{B}_{kj}\frac{\nabla\lambda_1}{\rho_1} \\
&\quad -\mathbf{C}_{kj}\frac{\nabla\mu_1}{\rho_1} - \sum_{\ell,m=1}^{3} r_{1,\ell m}\partial_\ell\partial_m \mathbf{u}_k(;j)
\end{aligned}
\tag{3.10}
$$

when $k = 0, 1$ and $j = 1, 2, 3$.

We now consider the matrix on the left-hand side of (3.9). Using (3.10), one can add to the first column the remaining columns multiplied by suitable factors such that $-\operatorname{div}\mathbf{u}_k(;j)\frac{\nabla\lambda_1}{\rho_1}$, $-2\epsilon(\mathbf{u}_k(;j))\frac{\nabla\mu_1}{\rho_1}$, and $-\sum_{\ell,m=1}^{3} r_{1,\ell m}\partial_\ell\partial_m \mathbf{u}_k(;j)$ are eliminated from the first column of this matrix. Then we multiply the first column of the new matrix by $-\rho_1$. We end up with the matrix $\mathbf{M}$ defined in section 1. Obviously, determinants of corresponding minors of the matrix on the left side of (3.9) and of the matrix $\mathbf{M}$ are the same. It follows from condition (1.12) and bounds (1.10) that

$$
|\mathbf{F}|^2 \le C\left(\sum_{j=1}^{3}\sum_{\beta=2}^{3}|\partial_t^\beta \mathbf{u}(0;j)|^2 + |f_2|^2 + |f_3|^2\right) \quad \text{on} \quad \overline{\Omega}. \tag{3.11}
$$

Since $\chi(\cdot, T) = 0$,

$$
\int_\Omega |\chi\partial_t^\beta \mathbf{u}(;j)|^2(x,0)e^{2\tau\varphi(x,0)}dx = -\int_0^T \partial_t\left(\int_\Omega |\chi\partial_t^\beta \mathbf{u}(;j)|^2(x,t)e^{2\tau\varphi(x,t)}dx\right)dt
$$

$$
\le \int_Q 2\chi^2(|\partial_t^{\beta+1}\mathbf{u}(;j)||\partial_t^\beta \mathbf{u}(;j)| + \tau|\partial_t\varphi||\partial_t^\beta \mathbf{u}(;j)|^2)e^{2\tau\varphi} + 2\int_{Q\setminus Q(\frac{\varepsilon}{2})}|\partial_t^\beta \mathbf{u}(;j)|^2\chi|\partial_t\chi|e^{2\tau\varphi},
$$

where $\beta = 2, 3$. The right side does not exceed

$$
C\left(\int_Q \tau|\chi\mathbf{U}(;j)|^2e^{2\tau\varphi} + C(\varepsilon)\int_{Q\setminus Q(\frac{\varepsilon}{2})}|\mathbf{U}(;j)|^2e^{2\tau\varphi}\right)
$$

$$
\le C\left(\int_Q \tau|\chi\mathbf{V}(;j)|^2e^{2\tau\varphi} + C(\varepsilon)\int_{Q\setminus Q(\frac{\varepsilon}{2})}|\mathbf{U}(;j)|^2e^{2\tau\varphi} + \tau\int_Q |\mathbf{U}^*(;j)|^2e^{2\tau\varphi}\right)
$$

because $\mathbf{U}(;j) = \mathbf{V}(;j) + \mathbf{U}^*(;j)$. Using that $\chi = 1$ on $\Omega(\frac{\varepsilon}{2})$, $\varphi < \varepsilon_1$ on $Q \setminus Q(\frac{\varepsilon}{2})$, and $\varphi < \Phi$ on Q from these inequalities, from (3.8), (3.5), and (1.10) we set

$$
\int_{\Omega(\frac{\varepsilon}{2})}|\partial_t^\beta \mathbf{u}(0;j)|^2e^{2\tau\varphi(,0)} \le C\left(\int_Q |\mathbf{F}|^2e^{2\tau\varphi} + C(\varepsilon)e^{2\tau\varepsilon_1} + \tau e^{2\tau\Phi}F^2\right) \tag{3.12}
$$

for $\beta = 2, 3$ and $j = 1, 2, 3$. Using that $\chi = 1$ on $\Omega(\frac{\varepsilon}{2})$, from (3.11) and (3.12) we obtain

$$\int_{\Omega(\frac{\varepsilon}{2})} |\mathbf{F}|^2 e^{2\tau\varphi(,0)} \leq C\bigg(\int_{Q(\frac{\varepsilon}{2})} |\mathbf{F}|^2 e^{2\tau\varphi} + \tau e^{2\tau\Phi} F^2 + C(\varepsilon)e^{2\tau\varepsilon_1} + \int_{\Omega(\frac{\varepsilon}{2})} (|f_2|^2 + |f_3|^2) e^{2\tau\varphi(,0)}\bigg), \tag{3.13}$$

where we also split Q in the right side of (3.12) into $Q(\frac{\varepsilon}{2})$ and its complement and used that $|\mathbf{F}| \leq C$ and $\varphi < \varepsilon_1$ on the complement.

To eliminate the first integral in the right side of (3.13), we observe that

$$\int_{Q(\frac{\varepsilon}{2})} |\mathbf{F}|^2(x) e^{2\tau\varphi(x,t)} dxdt \leq \int_{\Omega(\frac{\varepsilon}{2})} |\mathbf{F}|^2(x) e^{2\tau\varphi(x,0)} \left(\int_{-T}^{T} e^{2\tau(\varphi(x,t)-\varphi(x,0))} dt\right) dx.$$

Due to our choice of function φ, we have $\varphi(x,t) - \varphi(x,0) < 0$ when $t \neq 0$. Hence by the Lebesgue theorem the inner integral (with respect to t) converges to 0 as τ goes to infinity. By reasons of continuity of φ, this convergence is uniform with respect to $x \in \Omega$. Choosing $\tau > C$ we therefore can absorb the integral over $Q(\frac{\varepsilon}{2})$ in the right side of (3.13) by the left side and arrive at the inequality

$$\int_{\Omega(\frac{\varepsilon}{2})} |\mathbf{F}|^2 e^{2\tau\varphi(,0)} \leq C\left(\tau e^{2\tau\Phi} F^2 + C(\varepsilon) e^{2\tau\varepsilon_1} + \int_{\Omega(\frac{\varepsilon}{2})} (|f_2|^2 + |f_3|^2) e^{2\tau\varphi(,0)}\right). \tag{3.14}$$

On the other hand, to eliminate the last integral on the right side of (3.14), we use Lemma 2.3 with condition (1.14) to get

$$\int_{\Omega(\frac{\varepsilon}{2})} (|f_2|^2 + |f_3|^2) e^{2\tau\varphi(,0)} \leq \frac{C}{\tau} \int_{\Omega(\frac{\varepsilon}{2})} (|\nabla f_2|^2 + |\nabla f_3|^2) e^{2\tau\varphi(,0)}. \tag{3.15}$$

Using (3.15) with large τ and the inequality $\tau \leq e^{\tau}$, we absorb the last integral in the right side of (3.14) into the left side and obtain

$$\int_{\Omega(\frac{\varepsilon}{2})} |\mathbf{F}|^2 e^{2\tau\varphi(,0)} \leq C(e^{2\tau(\Phi_1+1)} F^2 + C(\varepsilon) e^{2\tau\varepsilon_1}).$$

Letting $\varepsilon_2 = e^{\frac{\eta\varepsilon}{2}} \leq \varphi$ on $\Omega(\varepsilon)$ and dividing both parts by $e^{2\tau\varepsilon_2}$ yields

$$\int_{\Omega(\varepsilon)} |\mathbf{F}|^2 \leq C(\tau e^{2\tau(\Phi+1-\varepsilon_2)} F^2 + e^{-2\tau(\varepsilon_2-\varepsilon_1)}) \leq C(\varepsilon)(e^{2\tau(\Phi+1)} F^2 + e^{-2\tau(\varepsilon_2-\varepsilon_1)}) \tag{3.16}$$

since $\tau e^{-2\tau\varepsilon_2} < C(\varepsilon)$. If $\frac{1}{C} \leq F$, then bound (1.15) is obvious because the left side in (1.15) is less than C. So to prove (1.15) it suffices to assume that $F < \frac{1}{C}$. Then $\tau = \frac{-logF}{\Phi+1+\varepsilon_2-\varepsilon_1} > C$, and we can use this τ in (3.16). Due to the choice of τ,

$$e^{-2\tau(\varepsilon_2-\varepsilon_1)} = e^{2\tau(\Phi+1)} F^2 = F^{2\frac{\varepsilon_2-\varepsilon_1}{\Phi+1+\varepsilon_2-\varepsilon_1}},$$

and from (3.16) we obtain (1.15) with $\gamma = \frac{\varepsilon_2-\varepsilon_1}{\Phi+1+\varepsilon_2-\varepsilon_1}$. The proof of Theorem 1.1 is now complete.

4. Lipschitz stability for the determination of coefficients. In this section we will prove Theorem 1.3. The key ingredient is the following Lipschitz stability estimate for the Cauchy problem for the system $\mathbf{A}_E\mathbf{u} = \mathbf{f}$.

THEOREM 4.1. *Suppose that Ω and T satisfy the assumptions of Theorem* 1.3. *Let $\mathbf{u} \in (H^3(Q))^3$ solve the Cauchy problem*

$$\begin{cases} \mathbf{A}_E\mathbf{u} = \mathbf{f} \quad in \quad Q, \\ \mathbf{u} = \sigma_\nu(\mathbf{u}) = 0 \quad on \quad \partial\Omega \times (-T,T) \end{cases} \tag{4.1}$$

with $\mathbf{f} \in L^2((-T,T); H^1(\Omega))$ and $\mathbf{f} = 0$ on $\partial\Omega \times (-T,T)$. Furthermore, assume that (1.7) *holds for sufficiently small ε_0.*

Then there exists a constant $C > 0$ such that

$$\|\mathbf{u}\|^2_{H^1(Q)} + \|\operatorname{div}\mathbf{u}\|^2_{H^1(Q)} + \|\operatorname{curl}\mathbf{u}\|^2_{H^1(Q)} \le C\|\mathbf{f}\|^2_{L^2((-T,T);H^1(\Omega))}. \tag{4.2}$$

This estimate was proved in [11].

By virtue of (4.2) and an equivalence of the norms $\|\mathbf{u}\|_{(1)}(\Omega)$ and of

$$\|\operatorname{div}\mathbf{u}\|_{(0)}(\Omega) + \|\operatorname{curl}\mathbf{u}\|_{(0)}(\Omega) + \|\mathbf{u}\|_{(0)}(\Omega)$$

in $H^1_0(\Omega)$ (e.g., [3, pp. 358–359]), it is not hard to derive the following.

COROLLARY 4.2. *Under the conditions of Theorem* 4.1,

$$\|\mathbf{u}\|_{(0)}(Q) + \|\nabla_{x,t}\mathbf{u}\|_{(0)}(Q) + \|\partial_t\nabla\mathbf{u}\|_{(0)}(Q) \le C\|\mathbf{f}\|_{L^2((-T,T);H^1(\Omega))}. \tag{4.3}$$

Now we are ready to prove Theorem 1.3. We will use the notations in section 3. Recall that

$$\mathbf{A}_{E_2}\mathbf{U}(;1;j) = \mathcal{A}(\mathbf{U}(;1;j))\mathbf{F},$$

where

$$\begin{aligned} \mathcal{A}(\mathbf{U}(;1;j))\mathbf{F} \quad = \quad & -f_1\partial_t^2\mathbf{U}(;1;j) + (f_2+f_3)\nabla(\operatorname{div}\mathbf{U}(;1;j)) + f_3\Delta\mathbf{U}(;1;j) \\ & + \operatorname{div}\mathbf{U}(;1;j)(f_4,f_5,f_6)^\top + 2\epsilon(\mathbf{U}(;1;j))(f_7,f_8,f_9)^\top \\ & + \sum_{j,k=1}^{3} r_{jk}\partial_j\partial_k\mathbf{U}(;1;j). \end{aligned}$$

So, from (1.17) we have

$$\mathbf{A}_{E_2}\mathbf{U}(;j) = 0 \quad \text{on} \quad \partial\Omega \times (-T,T). \tag{4.4}$$

Furthermore, in view of Lemma 2.2, there exists $\mathbf{U}^*(;j) \in H^3(Q)$ such that

(4.5)
$$\mathbf{U}^*(;j) = \mathbf{U}(;j), \quad \sigma(\mathbf{U}^*(;j))\nu = \sigma(\mathbf{U}(;j))\nu, \quad \mathbf{A}_{E_2}\mathbf{U}^*(;j) = 0 \quad \text{on} \quad \partial\Omega \times (-T,T)$$

and

(4.6)
$$\|\mathbf{U}^*(;j)\|_{(3)}(Q) \le C(\|\mathbf{U}(;j)\|_{(\frac{5}{2})}(\partial\Omega\times(-T,T)) + \|\sigma(\mathbf{U})(;j)\nu\|_{(\frac{3}{2})}(\partial\Omega\times(-T,T))) \le CF$$

due to the definition of F. As before, we set $\mathbf{V}(;j) = \mathbf{U}(;j) - \mathbf{U}^*(;j)$. Due to (4.4) and (4.5), we get

$$\mathbf{V}(;j) = \sigma(\mathbf{V})(;j)\nu = 0, \quad \mathbf{A}_{E_2}\mathbf{V}(;j) = 0 \quad \text{on} \quad \partial\Omega \times (-T,T). \tag{4.7}$$

With (4.7), applying Corollary 4.2 to (3.6), (3.7) and using (4.6) gives

$$\|\mathbf{V}(;j)\|^2_{(0)}(Q)+\|\nabla_{x,t}\mathbf{V}(;j)\|^2_{(0)}(Q)+\|\partial_t\nabla\mathbf{V}(;j)\|^2_{(0)}(Q)\le C(\|\mathbf{F}\|_{(1)}(\Omega)^2+F^2) \tag{4.8}$$

for $j=1,2,3$.

On the other hand, as in the proof of Theorem 1.1 we will bound the right side of (4.8) by $\mathbf{V}$. To use the Carleman estimate (2.1) we need to cut off $\mathbf{V}(;j)$ near $t=T$ and $t=-T$. We first observe that from the definition,

$$1\le\varphi(x,0),\qquad x\in\Omega,$$

and from condition (1.16),

$$\varphi(x,T)=\varphi(x,-T)<1\quad\text{when}\quad x\in\overline{\Omega}.$$

So there exists a $\delta>\frac{1}{C}$ such that

$$1-\delta<\varphi\quad\text{on}\quad\Omega\times(0,\delta),\qquad\varphi<1-2\delta\quad\text{on}\quad\Omega\times(T-2\delta,T). \tag{4.9}$$

We now choose a smooth cut-off function $0\le\chi_0(t)\le 1$ such that $\chi_0(t)=1$ for $-T+2\delta<t<T-2\delta$ and $\chi(t)=0$ for $|t|>T-\delta$. As in the argument before (3.8) (see also Lemma A.1 in [13]), from (4.7) we derive that $\mathbf{V}(;j)=\partial_\nu\mathbf{V}(;j)=0$ on $\partial\Omega\times(-T,T)$. Then since $\partial\Omega\times(-T,T)$ is not characteristic with respect to $\mathbf{A}_E$ the third equation in (4.7) implies that $\partial^2_\nu\mathbf{V}(;j)=0$ on $\partial\Omega\times(-T,T)$. Summing up, $\mathbf{V}(;j)=\partial_\nu\mathbf{V}(;j)=\partial^2_\nu\mathbf{V}(;j)=0$ on $\partial\Omega\times(-T,T)$. Now from known results about traces in Sobolev spaces [4], as above we conclude that $\chi_0\mathbf{V}(;j)\in H^3_0(Q)$. Using the Leibniz formula

$$\mathbf{A}_{E_2}(\chi_0\mathbf{V}(;j))=\chi_0\mathcal{A}(\mathbf{U}(;1;j))\mathbf{F}-\chi_0\mathbf{A}_{E_2}\mathbf{U}^*(;j)+2\rho_2(\partial_t\chi_0)\partial_t\mathbf{V}(;j)+\rho_2(\partial_t^2\chi_0)\mathbf{V}(;j)$$

and Carleman estimate (2.1) yields

$$\int_Q\chi_0^2(\tau^3|\mathbf{V}(;j)|^2+\tau|\nabla\mathbf{V}(;j)|^2)e^{2\tau\varphi}$$

$$\le C\left(\int_Q(|\mathbf{F}|^2+|\nabla\mathbf{F}|^2+|\mathbf{A}_{E_2}\mathbf{U}^*(;j)|^2+|\nabla(\mathbf{A}_{E_2}\mathbf{U}^*)(;j)|^2)e^{2\tau\varphi}\right.$$

$$\left.+\int_{\Omega\times\{T-2\delta<|t|<T\}}(|\mathbf{V}(;j)|^2+|\nabla_{x,t}\mathbf{V}(;j)|^2+|\partial_t\nabla\mathbf{V}(;j)|^2\right)e^{2\tau\varphi})$$

$$\le C\left(\int_Q(|\mathbf{F}|^2+|\nabla\mathbf{F}|^2)e^{2\tau\varphi}+e^{2\tau\Phi}F^2+e^{2\tau(1-2\delta)}\int_\Omega(|\mathbf{F}|^2+|\nabla\mathbf{F}|^2)\right),$$

where we let $\Phi = \sup_Q \varphi$ and used (4.6), (4.8), (4.9). Since $\mathbf{U}(;j) = \mathbf{V}(;j) + \mathbf{U}^*(;j)$, from (4.6) we obtain

$$\int_Q \chi_0^2(\tau^3|\mathbf{U}(;j)|^2 + \tau|\nabla\mathbf{U}(;j)|^2)e^{2\tau\varphi}$$

$$\leq C\left(\tau^3 e^{2\tau\Phi}F^2 + \int_\Omega\left(\int_{-T}^T e^{2\tau\varphi(x,t)}dt + e^{2\tau(1-2\delta)}\right)(|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)(x)\right)dx. \tag{4.10}$$

Utilizing (3.2) and (1.12), similarly to deriving (3.11), we get from (3.9) that

$$|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2 \leq C\left(\sum_{j=1}^3 \sum_{\substack{\beta=2,3;\\k=0,1}} |\partial_t^\beta\nabla^k\mathbf{u}(0;j)|^2 + \sum_{k=0,1}(|\nabla^k f_2|^2 + |\nabla^k f_3|^2)\right). \tag{4.11}$$

Therefore, by (4.11) and Corollary 2.4 (with conditions (1.17) for Lamé coefficients), we have

$$\int_\Omega (|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)e^{2\tau\varphi(,0)}$$

$$\leq C\left(\int_\Omega \sum_{j=1}^3 \sum_{\substack{\beta=2,3;\\k=0,1}} |\partial_t^\beta\nabla^k\mathbf{u}(0;j)|^2 e^{2\tau\varphi(,0)} + \int_\Omega \sum_{k=0,1}(|\nabla^k f_2|^2 + |\nabla^k f_3|^2)e^{2\tau\varphi(,0)}\right)$$

$$\leq -C\int_0^T \partial_t\left(\int_\Omega \sum_{j=1}^3 \sum_{\substack{\beta=2,3;\\k=0,1}} \chi_0^2|\partial_t^\beta\nabla^k\mathbf{u}(;j)|^2(x,t)e^{2\tau\varphi(x,t)}dx\right)dt$$
$$+\frac{C}{\tau}\int_\Omega(|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)e^{2\tau\varphi(;0)}.$$

Choosing τ large, we eliminate the last term and obtain

$$\int_\Omega (|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)e^{2\tau\varphi(,0)}$$

$$\leq C\int_Q \chi_0^2 \sum_{j=1}^3 \sum_{\substack{\beta=2,3;\\k=0,1}} (|\partial_t^\beta\nabla^k\mathbf{u}(;j)||\partial_t^{\beta+1}\nabla^k\mathbf{u}(;j)| + \tau|\partial_t\varphi||\partial_t^\beta\nabla^k\mathbf{u}(;j)|^2)e^{2\tau\varphi}$$

$$+C\int_{\Omega\times(T-2\delta,T)} \chi_0|\partial_t\chi_0| \sum_{j=1}^3 \sum_{\beta=2,3;k=0,1} |\partial_t^\beta\nabla^k\mathbf{u}(;j)|^2 e^{2\tau\varphi}.$$

Now as in the proofs of section 3, the right side is less than

$$C\left(\int_Q \tau\chi_0^2(|\mathbf{U}(;j)|^2+|\nabla\mathbf{U}(;j)|^2)e^{2\tau\varphi}+\int_{\Omega\times(T-2\delta,T)}(|\mathbf{U}(;j)|^2+|\nabla\mathbf{U}(;j)|^2)e^{2\tau\varphi}\right)$$

$$\leq C\left(\int_Q \tau\chi_0^2(|\mathbf{U}(;j)|^2+|\nabla\mathbf{U}(;j)|^2)e^{2\tau\varphi}+e^{2\tau(1-2\delta)}(\|\mathbf{F}\|^2_{(1)}(\Omega)+F^2)\right),$$

where we used equality $\mathbf{U}(;j)=\mathbf{U}^*(;j)+\mathbf{V}(;j)$ and (4.6), (4.8). From the two previous bounds and (4.10) we conclude that

$$\int_\Omega(|\mathbf{F}|^2+|\nabla\mathbf{F}|^2)e^{2\tau\varphi(,0)} \leq C\left(\tau^3e^{2\tau\Phi}F^2+\int_\Omega\left(\int_{-T}^{T}e^{2\tau\varphi(,t)}dt+e^{2\tau(1-2\delta)}\right)(|\mathbf{F}|^2+|\nabla\mathbf{F}|^2)\right). \tag{4.12}$$

Due to our choice of φ, $1\leq\varphi(,0), \varphi(,t)-\varphi(,0)<0$ when $t\neq 0$. Thus by the Lebesgue theorem as in the proofs of section 3, we have

$$2C\left(\int_{-T}^{T}e^{2\tau\varphi(,t)}dt+e^{2\tau(1-\delta)}\right)\leq e^{2\tau\varphi(,0)}$$

uniformly on Ω when $\tau>C$. Hence choosing and fixing such large τ, we eliminate the second term on the right side of (4.12). The proof of Theorem 1.3 is now complete.

5. Conclusion. While natural in some applications, the assumption about the smallness of residual stress is restrictive. In our opinion it can be relaxed by using the methods of papers [8], [11], and this paper. More restrictive and much more difficult to remove is the condition that the initial data are not zero. At present, even for scalar isotropic hyperbolic equations, global uniqueness of the speed of propagation or of the potential from few lateral boundary measurements is an open and outstanding research problem (see, for example, [9]). Moreover, in the case of zero initial data, general anisotropic hyperbolic operators (and hence systems) cannot be uniquely determined by all lateral boundary measurements (Dirichlet-to-Neumann map). In fact, a large gauge transformation group changes equations inside Ω without affecting the lateral boundary data. Hence (special) nonzero initial data are necessary for the complete identification of such equations and systems.

Of substantial interest is uniqueness in inverse problems for more general anisotropic systems, for example, for dynamical elasticity systems with transversal isotropy. For such systems there are no Carleman estimates or uniqueness of the continuation results. On the other hand, such systems are quite important for applications to geophysics, material science, and medicine, and they are notorious mathematical challenges.

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