# CONSTRUCTION OF OSCILLATING-DECAYING SOLUTIONS FOR ANISOTROPIC ELASTICITY SYSTEMS

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**ABSTRACT**

In this paper, we present a framework of constructing oscillating-decaying solutions for the general inhomogeneous anisotropic elasticity system. These oscillating-decaying solutions can be used in solving inverse problems concerning the identification of cavities or inclusions embedded in an elastic body.

**Keywords :** Oscillating-decaying solutions, Anisotropic elasticity system.

## 1. INTRODUCTION

Special type solutions for the elliptic equation or system has played an important role in solving related inverse problems. In 1987, Sylvester and Uhlmann [12] introduced exponentially growing solutions (they called complex geometrical optics solutions) to solve the inverse boundary value problem for the conductivity equation. Recently, Ikehata used exponentially growing solutions for $\Delta$ having the form $u = \exp\{\tau(x\cdot\omega - t + ix\cdot\omega^{\perp})\}$, where $\omega$ and $\omega^{\perp}$ are unit vectors orthogonal to each other and $\tau >> 1$, to solve several inverse problems concerning the reconstruction of the convex hull of a polygonal or polyhedral source domain, a polygonal inclusion or cavity, or the reconstruction of the convex hull of a general inclusion [4~7]. He called this method the enclosure method. It should be pointed out that so far exponentially growing solutions considered in [4~7] and [12] have been available only for operators and system of operators whose leading part are Laplacian and diagonal operators with Laplacian in the diagonal elements, respectively. For the details of the construction for these operators and system of operators see [1] and [11]. Notably, when the medium in the elasticity system is isotropic, exponentially growing solutions have been constructed in [2,9,10].

In order to extend Ikehata's idea to the elliptic equation or system with variable coefficients, it is quite natural to ask for a substitute of the exponentially growing solution. In this paper we look for the "oscillating-decaying solution" as a substitute of the exponentially growing solution. Roughly speaking, given a hyperplane or hypersurface, the oscillating-decaying solution is defined in the one side of this plane or surface including the plane or surface which is highly oscillating along this plane or surface and decaying exponentially in the direction orthogonal to the same plane or surface. Of course, by this substitute we may loose some of nice properties of the exponentially growing solution. However, it is still possible to preserve some other properties of it so that we can apply the oscillating-decaying solution to inverse problems. This type of solution was first described and applied for inverse problems for scalar elliptic equations in [8]. In this paper we mainly aim to present a framework of constructing the oscillating-decaying solution defined in the one side of hyperplane including this plane for the general anisotropic elasticity system. The main step of this construction is to suitably decompose the conjugate operator $M$ according to an appropriate definition of order (see Theorem 3.2). These oscillating-decaying solutions can be applied to some inverse problems concerning the identification of cavities or inclusions embedded in an elastic body. These results will be reported elsewhere.

## 2. DEFINITION OF OSCILLATING-DECAYING SOLUTIONS

Assume that $C(x) = (C_{ijkl}(x)) \in B^{\infty}(\mathbf{R}^3) = \{f \in C^{\infty}(\mathbf{R}^3): \partial^{\alpha} f \in L^{\infty}(\mathbf{R}^3), \forall\alpha \in \mathbf{Z}^3{}_{+}\}$ is the elasticity tensor satisfying the following conditions:
(Hyperelasticity)

$$C_{ijkl}(x) = C_{klij}(x) \qquad \forall\, x \in \mathbf{R}^3$$

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(Strong convexity) There exists $\delta > 0$ such that for any $x \in \mathbf{R}^3$ and real matrices $E = (\varepsilon_{ij})$, $G = (g_{ij})$

$$C(x)\, E \cdot E \geq \delta\, |E|^2 \tag{1}$$

where

$$C(x)\, E \sum_{k,l} C_{ijkl}(x)\, \varepsilon_{kl}$$

and

$$E \cdot G = \sum_{i,j} \varepsilon_{ij}\, g_{ij}\,, \qquad |E|^2 = E \cdot E$$

Here and below, all Latin indices are set to be from 1 to 3 unless otherwise indicated. Note that from the hyperelasticity we have

$$CE \cdot G = CG \cdot E$$

for any real matrices $E$ and $G$. Also, we denote

$$(\nabla u)_{kl} = \partial_l\, u_k$$

and

$$(\nabla \cdot G)_i = \sum_j \partial_j\, g_{ij} \text{ for any matrix function } G = (g_{ij})$$

Before going to the main theme of the section, we want to define several notations. Assume that $\Omega \subset \mathbf{R}^3$ is an open set and $\omega \in \mathbf{S}^2$ is given. Let $\eta \in \mathbf{S}^2$ and $\zeta \in \mathbf{S}^2$ be chosen so that $\{\eta, \zeta, \omega\}$ forms an orthonormal system of $\mathbf{R}^3$. We then denote $x' = (x\cdot\eta, x\cdot\zeta)$. Let $t \in \mathbf{R}$, $\Omega_t(\omega) = \Omega \cap \{x\cdot\omega > t\}$ and $\Sigma_t(\omega) = \{x\cdot\omega = t\}$. Let $S$ be a closed connected disk on $\Sigma_t$ on which $\Sigma_t \cap \overline{\Omega} \neq 0$. We consider a vector function $u_{\chi_t, b, t, N, \omega}(x, \tau) =: u(x, \tau) = [u_1(x, \tau), u_2(x, \tau), u_3(x, \tau)]^T \in C^\infty(\overline{\Omega_t(\omega)})$ with $\tau >> 1$ satisfying

$$\begin{cases} \pounds_C u = \nabla \cdot (C(x) \nabla u) = 0 & \text{in} \quad \Omega_t(\omega) \\ u\,|_{\Sigma_t(\omega)} = e^{i\tau x\cdot\theta} \chi_t(x')\, b \end{cases} \tag{2}$$

where $\theta \in \mathbf{S}^2$ lying in the span of $\eta$ and $\zeta$ is chosen and fixed, $\chi_t(x') \in C_0^\infty(\mathbf{R}^2)$ with $\mathrm{supp}\,(\chi_t) \subset S$ and $0 \neq b \in \mathbf{C}^3$. Furthermore, $u_{\chi_t, b, t, N, \omega}$ is written into $u_{\chi_t, b, t, N, \omega} = \upsilon_{\chi_t, b, t, N, \omega} + r_{\chi_t, b, t, N, \omega}$ with

$$\upsilon_{\chi_t, b, t, N, \omega} = \chi_t(x')\, e^{i\tau x\cdot\theta} e^{-\tau(x\cdot\omega - t) A_t(x')} b + \gamma_{\chi_t, b, t, N, \omega}(x, \tau)$$

and $r_{\chi_t, b, t, N, \omega}$ satisfying

$$\| r_{\chi_t, b, t, N, \omega} \|_{H^1(\Omega_t(\omega))} \leq c\, \tau^{-N-1/2} \tag{3}$$

where $A_t(\cdot) \in B^\infty(\mathbf{R}^2)$ is a matrix function such that all eigenvalues of $A_t$, denoted by $\mathrm{spec}(A_t)$, satisfies $\mathrm{spec}(A_t) \subset \mathbf{C}_r = \{z \in \mathbf{C}\text{: Re } z > 0\}$ and $\gamma_{\chi_t, b, t, N, \omega}$ is a smooth vector function supported in $\mathrm{supp}\,(\chi_t)$ satisfying

$$\| \partial_x^\alpha\, \gamma_{\chi_t, b, t, N, \omega} \|_{L^2(\Omega_s(\omega))} \leq c\, \tau^{|\alpha| - 3/2}\, e^{-\tau(s-t)\lambda} \tag{4}$$

for $|\alpha| \leq 1$ and $s \geq t$, where $\lambda > 0$ is some constant depending on $\mathrm{spec}(A_t)$. Here and below, we use $c$ to denote a general positive constant whose value may vary from line to line.

## 3. CONSTRUCTION OF OSCILLATING-DECAYING SOLUTIONS

Without loss of generality, we consider the special case where $t = 0$, $\omega = e_3 = (0, 0, 1)$ and choose $\eta = (1, 0, 0)$, $\zeta = (0, 1, 0)$. The general case can be easily obtained from this special case by obvious change of coordinates. Define $\pounds = \pounds_C$ and $\widetilde{M}\, \cdot = e^{-i\tau x'\cdot\theta'} \pounds(e^{i\tau x'\cdot\theta'} \cdot)$, where $x' = (x_1, x_2)$ and $\theta' = (\theta_1, \theta_2)$ with $|\theta'| = 1$. Clearly, $\widetilde{M}$ is a matrix differential operator. To be precise, the component $\widetilde{M}_{ik}$ of $\widetilde{M}$ is given by

$$\widetilde{M}_{ik} = -\tau^2 \Sigma_{jl} C_{ijkl} \theta_j \theta_l + \tau\, \Sigma_{jl} C_{ijkl}(i\theta_l)\partial_j + \tau\, \Sigma_{jl} C_{ijkl}(i\theta_l)\partial_l$$
$$+ \Sigma_{jl}\, C_{ijkl} \partial_j\, \partial_l + \Sigma_{jl}(\partial_j\, C_{ijkl})(i\tau\theta_l) + \Sigma_{jl}(\partial_j C_{ijkl})\partial_l$$

$$= -\tau^2 \Sigma_{jl} C_{ijkl} \theta_j \theta_l + \tau\, \Sigma_l C_{i3kl}(i\theta_l)\partial_3 + \tau\, \Sigma_j C_{ijk3}(i\theta_j)\partial_3$$

$$+ C_{i3k3}\partial_3^2 + \tau\, \Sigma_{j\neq 3, l}\, C_{ijkl}(i\theta_l)\partial_j + \tau\, \Sigma_{l\neq 3, j}\, C_{ijkl}(i\theta_j)\partial_l$$
$$+ \Sigma'_{jl}\, C_{ijkl}\, \partial_j\, \partial_l + \Sigma_{jl}(\partial_j C_{ijkl})(i\tau\theta_l) + \Sigma_{jl}(\partial_j C_{ijkl})\, \partial_l$$

with $\theta_3 = 0$, where $\Sigma'_{jl} = \Sigma_{j,l\backslash\{3,3\}}$. Our task is now reduced to solve

$$\widetilde{M}\upsilon = 0 \tag{5}$$

Obviously, we observe that (5) is equivalent to

$$M\upsilon = 0 \tag{6}$$

where $M = -C_3^{-1}\widetilde{M}$ and the $(i, k)$ entry of $C_3$ is $C_{i3k3}$. Define $< a, b > = (< a, b >_{ik})$ for $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$, where $< a, b >_{ik} = \Sigma_{jl} C_{ijkl}\, a_j b_l$. Also, we denote $< a, b >_0 = < a, b > |_{x_3 = 0}$. We define the order of $P$, denoted by $\mathrm{ord}(P)$, in the following sense:

$$\| P(e^{-\tau x_3 A(x')} \widehat{\varphi}(x')) \|_{L^2(\mathbf{R}^3_+)} \leq c\, \tau^{\mathrm{ord}(P) - 1/2}$$

where $\mathbf{R}^3_+ = \{x_3 > 0\}$, $A(x')$ is a smooth matrix function of $x'$ with $\mathrm{spec}(A) \subset \mathbf{C}_r$ and $\widehat{\varphi}(x') \in C_0^\infty(\mathbf{R}^2)$. In this sense, we can see that $\tau$, $\partial_3$ are of order 1, $\partial_1$, $\partial_2$, are of order 0 and $x_3$ is of order $-1$. Note that the order of $x_3$ is a simple consequence of the integration by parts. To verify the order of $\partial_j$, $j = 1$ or 2, we observe that

$$\partial_j(e^{-\tau x_3 A(x')}\widehat{\varphi}(x')) = e^{-\tau x_3 A(x')}\partial_j\widehat{\varphi}(x') - \tau\int_0^{x3} e^{-\tau(x_3-s)A(x')}\partial_j A(x')e^{-\tau s A(x')}\widehat{\varphi}(x')\,ds$$

and therefore

$$\| \partial_j(e^{-\tau x_3 A(x')}\widehat{\varphi}(x'))\|_{L^2(\mathbf{R}^3_+)} \le c\,\tau^{-1/2}$$

According to the definition of order, the principal part $M_2$ (order 2) of $M$ is given by

$$M_2 = \{D_3^2 + \tau < e_3, e_3 >_0^{-1} (< e_3, \vartheta >_0 + < \vartheta, e_3 >_0) D_3 + \tau^2 < e_3, e_3 >_0^{-1} < \vartheta, \vartheta > 0\} \quad (7)$$

with $D_3 = -i\partial_3$ and $\vartheta = (\theta_1, \theta_2, 0) = (\theta', 0)$. Notice that $M_2$ is obtained by the Taylor's expansion of $M$ at $x_3 = 0$, i.e.,

$$\begin{aligned} M(x', x_3) &= M(x', 0) + x_3\,\partial_3 M(x', 0) + \cdots \\ &\quad + \frac{x_3^{N-1}}{(N-1)!}\partial_3^{N-1} M(x', 0) + \cdots \\ &= M_2 + M_1 + M_0 + \cdots + \cdots + M_{-j} + \cdots \\ &= \Sigma_{j=-2}^{\infty} M_{-j} \end{aligned} \quad (8)$$

where $\mathrm{ord}(M_{-j}) = -j$ for $j \ge -2$. We can find that the symbol of $M_2$ is

$$\begin{aligned} \tau^2 L(x', \theta', \xi_3/\tau) &= \tau^2\{(\xi_3/\tau)^2 \\ &\quad + < e_3, e_3 >_0^{-1} (< e_3, \vartheta >_0 + < \vartheta, e_3 >_0)(\xi_3/\tau) \\ &\quad + < e_3, e_3 >_0^{-1} < \vartheta, \vartheta >_0\} \end{aligned}$$

It follows from the strong convexity condition (1) that for any fixed $(x', \theta')$, $\det(L)(x', \theta', \zeta) = 0$ in $\zeta$ admits roots $\lambda_j^{\pm}$ with $\pm \operatorname{Im} \lambda_j^{\pm} > 0$, $1 \le j \le 3$. Now we can factorize the symbol of $M_2$ by the following result.

**Theorem 3.1** [3] Let

$$K(x',\theta') := (\oint_\Gamma \zeta\,(< e_3, e_3 >_0 L(x',\theta',\zeta))^{-1} d\zeta)\,(\oint_\Gamma (< e_3, e_3 >_0 L(x',\theta',\zeta))^{-1} d\zeta)^{-1} \quad (9)$$

where $\Gamma \subset \mathbf{C}_+ := \{\zeta \in \mathbf{C} : \operatorname{Im}\zeta > 0$ is a closed contour enclosing all $\lambda_j^{\pm}$ for $1 \le j \le 3$. Then we have

$$L(x',\theta',\zeta) = (\zeta - K'(x',\theta'))(\zeta - K(x',\theta'))$$

with $\mathrm{spec}(K(x', \theta')) \subset \mathbf{C}_+$ and $K' = <e_3, e_3>_0^{-1} K^* <e_3, e_3>_0$, where $K^*$ is the transpose of $\overline{K}$.

Next we observe that the operator $M_{-j}$ can be explicitly written as

$$M_{-j} = x_3^{j+2}\,\tau^2 H_{-j,1} + x_3^{j+2}\,\tau\, H_{-j,2}\, D_3 + x_3^{j+1}\,\tau\, H_{-j,3} + x_3^{j+1}\, H_{-j,4}\, D_3 + (j_+) x_3^j\, H_{-j,5} \quad (j \ge -1) \quad (10)$$

where

$$j_+ = \begin{cases} 1 & \text{if } j \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

and $H_{-j,l} = H_{-j,l}(x', \theta')$ for $j \ge -1$, $l = 1, 2$, $H_{-j,l} = H_{-j,l}(x', \theta', D')$ with $D' = (-i\partial_1, -i\partial_2)$, $j \ge -1$, $l = 3, 4, 5$ are operators of zeroth order. In other words, $H_{-j,l}$ for $j \ge -1$, $l = 3, 4, 5$ contain $x'$ derivatives with coefficients independent of $x_3$. We now want to show that

**Theorem 3.2** There exist $K_{-j}$, $\widetilde{K}_{-j}$ $(j \ge -1)$ of order $-j$ containing no $D_3$ elements such that for any $N \in \mathbf{N}$

$$R_{(N)} := M - (D_3 - \widetilde{K}_{(N)})(D_3 - K_{(N)}) \quad (11)$$

is of order $-N$, where $K_{(N)}$ and $\widetilde{K}_{(N)}$ are defined by

$$K_{(N)} = \sum_{j=-1}^{N} K_{-j}\,, \qquad \widetilde{K}_{(N)} = \sum_{j=-1}^{N} \widetilde{K}_{-j}$$

with $K_1 = \tau\, K(x', \theta')$, $\widetilde{K}_1 = \tau\, K'(x', \theta')$ and $K(x', \theta')$ being given by (9). Moreover, $K_{-j}$ and $\widetilde{K}_{-j}$ $(j \ge 0)$ can be written as

$$\begin{cases} K_{-j} = \tau^{-j}\, \Sigma_{k=0}^{j+1}\, (x_3\tau)^k\, G_{-j,k}(x', \theta', D') \\ \widetilde{K}_{-j} = \tau^{-j}\, \Sigma_{k=0}^{j+1}\, (x_3\tau)^k\, \widetilde{G}_{-j,k}(x', \theta', D') \end{cases} \quad (12)$$

Here $G_{-j,j+1}(x', \theta', D') = G_{-j,j+1}(x', \theta')$ and $\widetilde{G}_{-j,j+1}(x',\theta',D') = \widetilde{G}_{-j,j+1}(x',\theta')$ and $C^\infty$ matrix functions of $x'$, $\theta'$ containing no differential operator $D'$, which are uniformly bounded in any compact set of $x'$. Also, $G_{-j,k}(x', \theta', D')$ and $\widetilde{G}_{-j,k}(x', \theta', D')$ are matrix differential operator in $D'$ of order at most two with coefficients uniformly bounded $C^\infty$ in any compact set of $x'$, $0 \le k \le j$.

**Proof:** We first compute

$$\begin{aligned} (D_3 - \widetilde{K}_{(N)})\,(D_3 - K_{(N)}) &= D_3^2 - (\widetilde{K}_{(N)} + K_{(N)})D_3 - (D_3 K_{(N)}) + \widetilde{K}_{(N)} K_{(N)} \\ &= D_3^2 - (\widetilde{K}_1 + K_1)D_3 - \Sigma_{j=0}^{N}(\widetilde{K}_{-j} + K_{-j})D_3 \\ &\quad - \Sigma_{j=0}^{N}(D_3\, K_{-j}) + \widetilde{K}_{(N)} K_{(N)} \end{aligned}$$

Comparing this with $M$, we hope to have

$$M_2 = D_3^2 - (\widetilde{K}_1 + K_1)D_3 - \widetilde{K}_1 K_1 \quad (13)$$

and

$$M_{-j} = -(\widetilde{K}_{-j-1} + K_{-j-1})D_3 - (D_3 K_{-j-1}) + \sum_j \widetilde{K}_r\, K_s \qquad -1 \le j \le N-1 \quad (14)$$

where $\Sigma_j$ is the summation with respect to $r, s \in \mathbf{Z}$ satisfying $r, s \le 1, r+s=-j$.

From (13) and Theorem 3.1, we immediately get that

$$K_1 = \tau K(x', \theta') \quad \text{and} \quad \widetilde{K}_1 = \tau K'(x', \theta')$$

For $j=-1$, (14) becomes

$$\begin{aligned} M_1 &= -(\widetilde{K}_0 + K_0)D_3 - (D_3 K_0) + \Sigma_{j=1}\widetilde{K}_r K_s \\ &= -(\widetilde{K}_0 + K_0)D_3 - (D_3 K_0) + \widetilde{K}_1 K_0 + \widetilde{K}_0 K_1 \end{aligned} \tag{15}$$

which is equivalent to

$$\begin{aligned} & x_3\tau^2 H_{1,1} + x_3\tau H_{1,2}D_3 + \tau H_{1,3} + H_{1,4}D_3 \\ &\quad = -(\widetilde{K}_0 + K_0)D_3 - (D_3 K_0) + \tau K'K_0 + \tau \widetilde{K}_0 K \end{aligned} \tag{16}$$

Now let $K_0$ and $\widetilde{K}_0$ be defined by

$$\begin{aligned} K_0 &= x_3\tau G_{0,1} + G_{0,0} = x_3\tau K_0^0(x', \theta') \\ &\quad + (\hat{K}_0^0(x', \theta') + K_0^1(x', \theta', D')) \end{aligned}$$

and

$$\begin{aligned} \widetilde{K}_0 &= x_3\tau \widetilde{G}_{0,1} + \widetilde{G}_{0,0} = x_3\tau \widetilde{K}_0^0(x', \theta') \\ &\quad + (\hat{\widetilde{K}}_0^0(x', \theta') + \widetilde{K}_0^1(x', \theta', D')) \end{aligned}$$

where $K_0^1$ and $\widetilde{K}_0^1$ are pure first order in $D'$. Comparing both sides (16), we find that $K_{0,1}$ and $\widetilde{K}_{0,1}$ are required to satisfy

$$\begin{cases} K_0^0 + \widetilde{K}_0^0 = -H_{1,2} \\ K' K_0^0 + \widetilde{K}_0^0 K = H_{1,1} \end{cases} \tag{17}$$

We get from the first equation of (17) that $\widetilde{K}_0^0 = -H_{1,2} - K_0^0$. Substituting this into the second equation of (17) yields

$$K' K_0^0 - K_0^0 K = H_{1,1} + H_{1,2}K \tag{18}$$

Since spec(K) $\cap$ spec($K'$) = 0, the equation (18) is uniquely solvable. Therefore, we can find suitable $K_0^0$ and $\widetilde{K}_0^0$ satisfying (17).

To find other terms in $K_0$ and $\widetilde{K}_0$, we also need to compare related terms on both sides of (16). However, here we have to work on the full symbol of the composition $\widetilde{K}_0 K$. Before going further, we first set

$$H_{1,3}(x', \theta', D') = H_{1,3}^1(x', \theta', D') + H_{1,3}^0(x', \theta')$$

where $H_{1,3}^1$ contains terms with first derivatives in $x'$ variables and $H_{1,3}^0(x', \theta')$ is the usual zeroth order term. Let the (full) symbol of $H_{1,3}^1(x', \theta', D')$ be $H_{1,3}^1(x', \theta', \xi')$. Here and below, for simplicity, we use the same notation for the operator and the associated symbol. Similarly, we set

$$H_{1,4}(x', \theta', D') = H_{1,4}^1(x', \theta', D') + H_{1,4}^0(x', \theta')$$

We now look at the principal symbols of $K'K_0$ and $\widetilde{K}_0 K$ and deduce from (16) that

$$\begin{cases} K'(x', \theta')K_0^1(x', \theta', \xi') + \widetilde{K}_0^1(x', \theta', \xi')K(x', \theta') \\ = H_{1,3}^1(x', \theta', \xi') - K_0^1(x', \theta', \xi') - \widetilde{K}_0^1(x', \theta', \xi') \\ = K_{1,4}^1(x', \theta', \xi') \end{cases} \tag{19}$$

It is clear that the system of equations (19) is similar to (17) and we can find a unique pair of $K_0^1(x', \theta', \xi')$ and $\widetilde{K}_0^1(x', \theta', \xi')$ because of the condition spec($K'$) $\cap$ spec($K$) = 0. Having determined $K_0^1$ and $\widetilde{K}_0^1$, we use (16) again and obtain that

$$\begin{cases} K'(x', \theta')\hat{K}_0^0(x', \theta') + \hat{\widetilde{K}}_0^0(x', \theta')K(x', \theta') \\ \quad = H_{1,3}^0(x', \theta') - i K_0^0(x', \theta') \\ \qquad - \{\sigma(\widetilde{K}_0^1 K)(x', \theta', \xi') - \widetilde{K}_0^1(x', \theta', \xi')K(x', \theta')\} \\ \qquad - \hat{K}_0^0(x', \theta') - \hat{\widetilde{K}}_0^0(x', \theta') = H_{1,4}^0(x', \theta') \end{cases} \tag{20}$$

where $\sigma(\widetilde{K}_0^1 K)$ denote the full symbol of the composition operator $(\widetilde{K}_0^1 K)$. Notice that the last term on the right hand side of the first equation of (20) stands for the zeroth order symbol of $\widetilde{K}_0^1 K$. Now the system of equation (20) can be solved in a similar way since spec($K'$) $\cap$ spec($K$) = 0. Hence, $\hat{K}_0^0$ and $\hat{\widetilde{K}}_0^0$ are determined. In other words, we have found $K_0$ and $\widetilde{K}_0$ such that (15) is satisfied.

For clarity, we would like to work out the case $j=0$. The rest of results ($j \ge 1$) are then derived by induction. Putting $j=0$ in (14) gives

$$M_0 = -(\widetilde{K}_{-1} + K_{-1})D_3 - (D_3 K_{-1}) + \sum_{j=0} \widetilde{K}_r K_s \tag{21}$$

which can be written as

$$\begin{aligned} & x_3^2\tau^2 H_{0,1} + x_3^2\tau\, H_{0,2}D_3 + x_3\tau\, H_{0,3} + x_3 H_{0,4}D_3 + H_{0,5} \\ &= -(\widetilde{K}_{-1} + K_{-1})D_3 - (D_3 K_{-1}) + \tau\, K'K_{-1} + \widetilde{K}_0 K_0 + \tau\, \widetilde{K}_{-1}K \end{aligned} \tag{22}$$

Note that here $H_{0,5}$ has the following form

$$H_{0,5}(x', \theta', D') = H_{0,5}^1(x', \theta', D') + H_{0,5}^2(x', \theta', D')$$

where $H_{0,5}^1$ and $H_{0,5}^2$ are first and second operators in

$D'$, respectively. Therefore, we anticipate that

$$K_{-1} = x_3^2 \tau G_{-1,2} + x_3 G_{-1,1} + \tau^{-1} G_{-1,0}$$
$$= x_3^2 \tau K_{-1}^0(x', \theta') + x_3 (\hat{K}_{-1}^0(x', \theta') + \hat{K}_{-1}^1(x', \theta', D'))$$
$$+ \tau^{-1}(\overline{K}_{-1}^0(x', \theta') + \overline{K}_{-1}^1(x', \theta', D') + \overline{K}_{-1}^2(x', \theta', D'))$$

and

$$\widetilde{K}_{-1} = x_3^2 \tau \widetilde{G}_{-1,2} + x_3 \widetilde{G}_{-1,1} + \tau^{-1} \widetilde{G}_{-1,0}$$
$$= x_3^2 \tau \widetilde{K}_{-1}^0(x', \theta') + x_3 (\widetilde{\hat{K}}_{-1}^0(x', \theta') + \widetilde{\hat{K}}_{-1}^1(x', \theta', D'))$$
$$+ \tau^{-1}(\widetilde{\overline{K}}_{-1}^0(x', \theta') + \widetilde{\overline{K}}_{-1}^1(x', \theta', D') + \widetilde{\overline{K}}_{-1}^2(x', \theta', D'))$$

To determine all terms in $K_{-1}$ and $\widetilde{K}_{-1}$, we simply compare related terms on both sides of (22). For example, we can deduce that

$$\begin{cases} K'(x', \theta')K_{-1}^0(x', \theta') + \widetilde{K}_{-1}^0(x', \theta')K(x', \theta') \\ \quad = H_{0,1}(x', \theta') - \widetilde{K}_0^0(x', \theta')K_0^0(x', \theta') \\ \quad -\widetilde{K}_{-1}^0(x', \theta') - K_{-1}^0(x', \theta') = H_{0,2}(x', \theta') \end{cases} \quad (23)$$

The system of equations (23) can be solved in the same way as before by taking into account of spec($K'$) $\cap$ spec($K$) = 0. Similar equations can be found for the rest of terms in $K_{-1}$ and $\widetilde{K}_{-1}$, i.e.,

$$\begin{cases} K'\hat{K}_{-1}^j + \widetilde{\hat{K}}_{-1}^j K = \text{konwn quantity} \\ \hat{K}_{-1}^j + \widetilde{\hat{K}}_{-1}^j = \text{konwn quantity} \end{cases}$$

for $j = 0, 1$ and

$$\begin{cases} K'\overline{K}_{-1}^j + \widetilde{\overline{K}}_{-1}^j K = \text{konwn quantity} \\ \overline{K}_{-1}^j + \widetilde{\overline{K}}_{-1}^j = 0 \end{cases}$$

for $j = 0, 1, 2$. Here we note that $\overline{K}_{-1}^j + \widetilde{\overline{K}}_{-1}^j = 0$ for $j = 0, 1, 2$ since the left side of (22) does not contain any term with $\tau^{-1}D_3$. Thus, we can find suitable $K_{-1}$ and $\widetilde{K}_{-1}$ satisfying (21).

Now we are going to do the induction. Assume that we have proved (12) up to $j$ with $1 \le j < N$, i.e., for $1 \le j < N$

$$\begin{cases} K_{-j} = \tau^{-j} \Sigma_{k=0}^{j+1} (x_3\tau)^k G_{-j,k}(x', \theta', D') \\ \widetilde{K}_{-j} = \tau^{-j} \Sigma_{k=0}^{j+1} (x_3\tau)^k \widetilde{G}_{-j,k}(x', \theta', D') \end{cases}$$

where $G_{-j,j+1}(x',\theta',D') = G_{-j,j+1}(x',\theta')$ and $\widetilde{G}_{-j,j+1}(x',\theta',D') = \widetilde{G}_{-j,j+1}(x',\theta')$. Therefore, using (14) and the form of $M_{-j}$ we have that

$$x_3^{j+2}\tau^2 H_{-j,1} + x_3^{j+2}\tau H_{-j,2}D_3 + x_3^{j+1}\tau H_{-j,3} + x_3^{j+1} H_{-j,4}D_3 + x_3^j H_{-j,5}$$
$$= -(\widetilde{K}_{-j-1} + K_{-j-1})D_3 - (D_3K_{-j-1}) + \Sigma_j \widetilde{K}_r K_s$$
$$= -(\widetilde{K}_{-j-1} + K_{-j-1})D_3 - (D_3K_{-j-1})$$
$$+ \tau K'K_{-j-1} + \tau \widetilde{K}_{-j-1}K + \Sigma'_j \widetilde{K}_r K_s \quad (25)$$

where $\Sigma'_j$ is the summation with respect to $r$, $s$ satisfying $r, s \le 0$ and $r + s = -j$. We now plug the ansatz

$$\begin{cases} K_{-j-1} = \tau^{-j-1} \Sigma_{k=0}^{j+2} (x_3\tau)^k G_{-j-1,k}(x', \theta', D') \\ \widetilde{K}_{-j-1} = \tau^{-j-1} \Sigma_{k=0}^{j+2} (x_3\tau)^k \widetilde{G}_{-j-1,k}(x', \theta', D') \end{cases}$$

into (25) and get that

$$x_3^{j+2}\tau^2 H_{-j,1} + x_3^{j+2}\tau H_{-j,2}D_3 + x_3^{j+1}\tau H_{-j,3} + x_3^{j+1} H_{-j,4}D_3 + x_3^j H_{-j,5}$$
$$= -\Sigma_{k=0}^{j+2}\{\tau^{-j-1}(x_3\tau)^k G_{-j-1,k} + \tau^{-j-1}(x_3\tau)^k \widetilde{G}_{-j-1,k}\}D_3$$
$$- \tau^{-j-1}D_3\{\Sigma_{k=0}^{j+2}(x_3\tau)^k G_{-j-1,k}\}$$
$$+ \tau^{-j}\, \Sigma_{k=0}^{j+2}(x_3\tau)^k \{K'G_{-j-1,k} + G_{-j-1,k}K\}$$
$$+ \Sigma'_j \tau^{-r-s}\{\Sigma_{k=0}^{j+1}(x_3\tau)^k G_{-r,k}\}\{\Sigma_{k=0}^{s+1}(x_3\tau)^k G_{-s,k}\} \quad (26)$$

Clearly, we can see that

$$\begin{cases} H_{-j,1} = H_{-j,1}(x', \theta') \\ H_{-j,2} = H_{-j,2}(x', \theta') \\ H_{-j,3} = H_{-j,3}^0(x', \theta') + H_{-j,3}^1(x', \theta', D') \\ H_{-j,4} = H_{-j,4}^0(x', \theta') + H_{-j,4}^1(x', \theta', D') \\ H_{-j,5} = H_{-j,5}^1(x', \theta', D') + H_{-j,5}^2(x', \theta', D') \end{cases}$$

As before, comparing related terms on both sides of (26), assuming appropriate forms for $G_{-j-1,k}$ and $\widetilde{G}_{-j-1,k}$, we are able to find all $G_{-j-1,k}$'s and $\widetilde{G}_{-j-1,k}$'s recursively under the condition spec($K'$) $\cap$ spec($K$) = 0. For example, for $G_{-j-1,j+2}$ and $\widetilde{G}_{-j-1,j+2}$, we have that

$$\begin{cases} K'G_{-j-1,j+2} + \widetilde{G}_{-j-1,j+2}K = H_{-j,1} - \Sigma'_j \widetilde{G}_{r,-r-1}G_{s,-s-1} \\ -G_{-j-1,j+2} - G_{-j-1,j+2} = H_{-j,2} \end{cases} \quad (27)$$

Notice that by the induction assumption $\widetilde{G}_{r,-r+1}$ and $G_{s,-s+1}$ in $\Sigma'_j$ above satisfy $\widetilde{G}_{r,-r+1} = \widetilde{G}_{r,-r+1}(x', \theta')$ and $G_{s,-s+1} = G_{s,-s+1}(x', \theta')$ for all $r, s \le 0$ and $r + s = -j$. Therefore, we can solve for $G_{-j-1,j+2}$ and $\widetilde{G}_{-j-1,j+2}$ from (27) provided spec($K'$) $\cap$ spec($K$) = 0. Also, we can get that $G_{-j-1,j+2} = G_{-j-1,j+2}(x', \theta')$ and $\widetilde{G}_{-j-1,j+2} = \widetilde{G}_{-j-1,j+2}(x', \theta')$. It is complicated but straightforward to derive similar equations for $G_{-j-1,k}$ and $\widetilde{G}_{-j-1,k}$ with $0 \le k \le j+1$. Solving those derived equations will give explicitly forms of $G_{-j-1,k}$ and $\widetilde{G}_{-j-1,k}$ for $0 \le k \le j+1$. Thus, we have proved (12) for $j + 1$.

We now want to solve the following initial value problem

$$\begin{cases} (D_3 - K_{(N)})\, z = 0 & \text{in } x_3 > 0 \\ z\,|_{x_3=0} = \chi_t(x')b & b \in \mathbf{C}^3 \end{cases} \tag{28}$$

We will not intend to solve (28) exactly. Instead, we would like to find an approximate solution of (28). Namely, let $w_N^{(0)}, w_N^{(-1)}, \cdots, w_N^{(-N-1)}$ satisfy

$$\begin{cases} (D_3 - \tau K(x', \theta'))w_N^{(0)} = 0, \quad w_N^{(0)}\,|_{x_3=0} = \chi_t(x')b \\ (D_3 - \tau K(x', \theta'))w_N^{(-1)} = K_0 w_N^{(0)}, \quad w_N^{(-1)}\,|_{x_3=0} = 0 \\ \quad\vdots \\ (D_3 - \tau K(x', \theta'))w_N^{(-N-1)} = \Sigma_{j=0}^{N} K_{-j}\, w_N^{(-N+j)}, \quad w_N^{(-N-1)}\,|_{x_3=0} = 0 \end{cases} \tag{29}$$

The system of equations (29) can be solved recursively. For example, it is easy to see that

$$w_N^{(0)} = \exp(i\tau x_3 K)\chi_t(x')b$$

and

$$\begin{aligned} w_N^{(-1)} &= \exp(i\tau x_3 K)\int_0^{x_3} \exp(-i\tau y_3 K)K_0 w_N^{(0)}(x', y_3, \tau, \theta')dy_3 \\ &= \exp(i\tau x_3 K)\int_0^{x_3} \exp(-i\tau y_3 K)\{(\tau y_3)\, G_{0,1}(x', \theta', D') \\ &\quad + G_{0,0}(x', \theta', D')\}\, w_N^{(0)}(x', \theta')dy_3 \end{aligned}$$

Furthermore, we can obtain that

$$\begin{aligned} &\| w_N^{(-1)} \|^2_{L^2(\mathbf{R}^3_+)} \\ &= \int_{\mathbf{R}^2} dx' \int_0^\infty |\exp(i\tau x_3 K) \int_0^{x_3} \exp(-i y_3 \tau K) K_0 w_N^{(0)} dy_3\,|^2\, dx_3 \\ &\le \int_{\mathbf{R}^2} dx' \int_0^\infty e^{-2\tau x_3 \lambda}\left(\int_0^{x_3} e^{\tau y_3 \lambda}\,|K_0 w_N^{(0)}|\, dy_3\right)^2 dx_3 \\ &\le \int_{\mathbf{R}^2} dx' \int_0^\infty e^{-2\tau x_3 \lambda} x_3 \left(\int_0^{x_3} e^{2\tau y_3 \lambda}\,|K_0 w_N^{(0)}|^2 dy_3\right) dx_3 \\ &= (2\tau\lambda)^{-1} \int_{\mathbf{R}^2} dx' \int_0^\infty e^{-2\tau x_3 \lambda}\left\{\int_0^{x_3} e^{2\tau x_3 \lambda}\,|K_0 w_N^{(0)}|^2 dy_3 \right. \\ &\quad \left. + x_3 e^{-2\tau x_3 \lambda}\,|K_0 w_N^{(0)}|^2\right\} dx_3 \\ &= (2\tau\lambda)^{-2} \int_{\mathbf{R}^2} dx' \int_0^\infty |K_0 w_N^{(0)}|^2\, dx_3 \\ &\quad + (2\tau\lambda)^{-1} \int_{\mathbf{R}^2} dx' \int_0^\infty x_3\,|K_0 w_N^{(0)}|^2\, dx_3 \\ &\le c\tau^{-3} \end{aligned} \tag{30}$$

where the parameter $\lambda > 0$ depending on $K$. Notice that the last two equalities of (30) are obtained from the integration by parts. To get the last inequality of (30), we make use of $\text{ord}(K_0) = 0$ and $\text{ord}((x_3^{1/2}) = -1/2$. Similar arguments can be carried out and we deduce that

$$\| \partial^\beta_{x_3} \partial^\alpha_{x'}\, w_N^{(-1)} \|^2_{L^2(\mathbf{R}^3_+)} \le c\tau^{2\beta-3} \tag{31}$$

for any $0 \le \beta \le 1$ and any multi-index $\alpha \in \mathbf{Z}^2_+$. Also, by induction, it is not difficult to show that

$$\| \partial^\beta_{x_3} \partial^\alpha_{x'}\, w_N^{(-1)} \|^2_{L^2(\mathbf{R}^3_+)} \le c\tau^{2\beta-1-2j} \quad (2 \le j \le N+1) \tag{32}$$

for $0 \le \beta \le 1$ and $\alpha \in \mathbf{Z}^2_+$. For the later purpose, it is useful to compute that

$$\begin{aligned} &(D_3 - K_{(N)})\, \Sigma_{j=0}^{N+1}\, w_N^{(-j)} \\ &= (D_3 - \tau K - K_0 - \cdots - K_{-N})(w_N^{(0)} + w_N^{(-1)} + \cdots + w_N^{(-N-1)}) \\ &= \{(D_3 - \tau K)\, w_N^{(0)}\} + \{(D_3 - \tau K)\, w_N^{(-1)} - K_0\, w_N^{(0)}\} + \cdots \\ &\quad + \{(D_3 - \tau K)\, w_N^{(-N-1)} - \Sigma_{j=0}^{N} K_{-j}\, w_N^{(-N+j)}\} \\ &\quad + \sum_{k=0}^{N} \sum_{\substack{r+s=k+N+1 \\ 0\le r\le N, 1\le s\le N+1}} K_{-r}\, w_N^{-s} \\ &= \sum_{k=0}^{N} \sum_{\substack{r+s=k+N+1 \\ 0\le r\le N, 1\le s\le N+1}} K_{-r}\, w_N^{-s} \\ &:= Q_{-N-1} \end{aligned}$$

Now let $\upsilon_N = \exp(i\tau x' \cdot \theta')\, \Sigma_{j=0}^{N+1}\, w_N^{(-j)}$ then we have that

$$\begin{aligned} \upsilon_N &= \exp(i\tau x' \cdot \theta')\exp(i\tau x_3 K)\chi_t(x')b \\ &\quad + \exp(i\tau x' \cdot \theta')\Sigma_{j=1}^{N+1}\, w_N^{(-j)} \\ &= \exp(i\tau x' \cdot \theta')\exp(i\tau x_3 K)\chi_t(x')b + \gamma_N \end{aligned}$$

and $\gamma_N$ satisfies the estimate (4). In addition, we observe that

$$\begin{aligned} \pounds\, \upsilon_N &= -e^{i\tau x'\cdot\theta'}\, C_3 M\, \Sigma_{j=0}^{N+1}\, w_N^{(-j)} \\ &= -e^{i\tau x'\cdot\theta'}\, C_3\, [(D_3 - \widetilde{K}_{(N)})(D_3 - K_{(N)}) + R_{(N)}]\, \Sigma_{j=0}^{N+1}\, w_N^{(-j)} \\ &= -e^{i\tau x'\cdot\theta'}\, C_3\, [(D_3 - \widetilde{K}_{(N)})\, Q_{-N-1} + R_{(N)}\, \Sigma_{j=0}^{N+1}\, w_N^{(-j)} \\ &:= S_N \end{aligned}$$

It is readily seen that $S_N$ satisfies

$$\| S_N \|_{L^2(\mathbf{R}^3_+)} \le c\tau^{-N-1/2} \tag{33}$$

By (33) and the Lax-Milgram theorem, there exists a unique solution $r_N$ to the boundary value problem

$$\begin{cases} \pounds\, r_N = -S_N & \text{in } \Omega_0 := \Omega_0(e_3) \\ r_N\,|_{\partial\Omega_0} = 0 \end{cases}$$

and the following estimate holds

$$\| r_N \|_{H^1(\Omega_0)} \le c\tau^{-N-1/2}$$

which is the estimate (3). Therefore, let $u_N = \upsilon_N + r_N$ then $u_N$ is the desired oscillating-decaying solution for the case $t = 0$ and $\omega = e_3$. Now the oscillating-decaying solution in the general case can be constructed by reducing it to this case with the help of change of coordinates. On the other hand, since the construction of the oscillating-decaying solution is local near any point on the hyperplane $\Sigma_t(\omega)$ and the strong convexity condition is invariant under change of coordinates, we can construct the oscillating-decaying solution with respect to any curved hypersurface as well. The only extra work needed to do is to flatten the boundary.


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