RECONSTRUCTION OF OBSTACLES IMMERSED IN AN INCOMPRESSIBLE FLUID

HORST HECK, GUNTHER UHLMANN, AND JENN-NAN WANG

ABSTRACT. We consider the reconstruction of obstacles inside a bounded domain filled with an incompressible fluid. Our method relies on special complex geometrical optics solutions for the stationary Stokes equation with a variable viscosity.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with smooth boundary. Assume that D is a subset of Ω such that $\overline{D} \subset \Omega$ and $\Omega \setminus \overline{D}$ is connected. Let $u = (u_1, u_2, u_3)$ be a vector-valued function satisfying

$$\begin{cases} \operatorname{div}(\nu S(\nabla u)) - \nabla p = 0 \quad \text{in} \quad \Omega \setminus \overline{D}, \\ \operatorname{div} u = 0 \quad \text{in} \quad \Omega \setminus \overline{D}, \\ u = 0 \quad \text{on} \quad \partial D, \\ u = f \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where the parameter ν is called *viscosity* and $S(M) = M + M^\top$ for any matrix $M \in \mathbb{C}^{3\times 3}$. The boundary condition f satisfies the compatibility condition

$$\int_{\partial\Omega} f \cdot \mathbf{n} ds = 0, \tag{1.2}$$

where $\mathbf{n}$ is the unit outer normal to $\partial\Omega$. On the other hand, the condition $u|_{\partial D} = 0$ is the no-slip condition. If ν is a positive constant this is the classical stationary Stokes system describing the motion of a homogeneous, incompressible, viscous fluid in the domain $\Omega \setminus \overline{D}$. Here we will also treat the more general case $\nu(x) \in C^4(\overline{\Omega})$, $0 < \delta \le \nu$ in Ω for some constant $\delta > 0$. A physical description of (1.1) with a variable viscosity is given in [8]. The domain D is considered as an obstacle immersed in an incompressible fluid which fills $\Omega \setminus \overline{D}$. For a variable $\nu(x)$, the well-posedness of (1.1) is proved in [8]. More precisely, for any $f \in H^{1/2}(\partial\Omega)$ satisfying (1.2), there exist $u \in H^1(\Omega)$ and $p \in L^2(\Omega)$ (unique up to a constant) satisfying (1.1). Denote $\sigma(u,p) = \nu S(\nabla u) - p$. Then we have

$$\sigma(u,p)\mathbf{n}|_{\partial\Omega} \in H^{-1/2}(\partial\Omega),$$

which is called the *Cauchy forces.* We define

$$\Lambda_D(f) = \{\sigma(u,p)\mathbf{n}|_{\partial\Omega} : (u,p) \text{ is a solution of (1.1) with } u|_{\partial\Omega} = f\}.$$

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Note that Λ_D maps $H^{1/2}(\partial\Omega)$ to a "set" of $H^{-1/2}(\partial\Omega)$. Nonetheless, any two elements of $\Lambda_D(f)$ for a fixed f differ by a constant multiple of $\mathbf{n}$. In this problem, we are interested in the inverse problem of determining D from the boundary data $(f, \Lambda_D(f)) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$.

Our study is inspired by Alvarez, Conca, Fritz, Kavian and Ortega's paper [2] in which they considered the object determination problem for the Stokes as well as for the Navier-Stokes equations (with constant ν). They were able to prove uniqueness of this inverse problem for the stationary and the non-stationary case. Part of the work is also motivated by a recent paper of the last two authors [20] where we considered the reconstruction problem of the inclusion for the isotropic elasticity using the so-called *complex spherical waves*, which was first defined in [9] for the Schrödinger equation. The connection of the system (1.1) to the isotropic elasticity is described in [8].

On the other hand, for the 2D-Oseen equation Kress and Meyer gave in [12] a solution to the unique identification problem for an obstacle if the velocity of the fluid is measured on some arc outside the obstacle. They also study a Newton iteration scheme in order to approximate the solution. The object determination problem in the conductivity equation is much more well studied. We will not try to give a full account of these developments here. For detailed references, we refer to [9].

In this paper we focus on the reconstruction method for the proposed inverse problem. The method we will employ relies on the use of complex geometrical optics solutions. Such kind of solutions were first used in Sylvester and Uhlmann [17] for the inverse conductivity problem. Since then their method became the base for treating many inverse problems. For the elasticity equation such solutions were first used by Nakamura and Uhlmann in [14]. Recently, complex geometrical optics solutions with more general phase functions were constructed in [13] for the Schrödinger equation and in [7] for the Schrödinger equation with magnetic potential. The method used in [13] and [7] relies on Carleman type estimates, which is a more flexible tool in treating lower order perturbations. In this paper we apply the ideas of [7], [13], and [20] in order to construct complex geometric optics solutions for (1.1). The real part of the phase function here is a radial function. With the help of these special solutions we are able to detect unknown objects inside the fluid with known background viscosity.

2. Preliminaries

In this section we will derive important integral inequalities that play a key role in our inverse problem. The derivation here is motivated by Ikehata's work [10]. It turns out that the integral inequalities do not depend on the pressure p. We shall take advantage of this property in the construction of complex geometrical optics solutions. For the obstacle D, in addition to the conditions described above, we assume $\partial D \in C^2$. Let (u, p) be a solution of (1.1), then using Green's formula we obtain

$$\begin{aligned}\langle \sigma(\bar{u},\bar{p})\mathbf{n}, f\rangle &= \int_{\partial\Omega} \sigma(\bar{u},\bar{p})\mathbf{n}\cdot f ds \\ &= \int_{\Omega\setminus\overline{D}} (\nu S(\nabla\bar{u})\cdot S(\nabla u) - \bar{p}\,\mathrm{div}\, u) dx \\ &= \int_{\Omega\setminus\overline{D}} \nu S(\nabla\bar{u})\cdot S(\nabla u) dx. \end{aligned} \tag{2.1}$$

Now we let (u_0, p_0) be a solution of

$$\begin{cases} \mathrm{div}(\nu S(\nabla u_0)) - \nabla p_0 = 0 \quad \text{in} \quad \Omega, \\ \mathrm{div}\, u_0 = 0 \quad \text{in} \quad \Omega, \\ u_0 = f \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2.2}$$

Let us define $(\tilde{u},\tilde{p}) = (u,p)$ for $x \in \Omega\setminus\overline{D}$ and $(\tilde{u},\tilde{p}) = (0,0)$ for $x \in \overline{D}$. Since $u = 0$ on ∂D, $\tilde{u} \in H^1(\Omega)$. So we can see that

$$\langle \sigma(\bar{u}_0,\bar{p}_0)\mathbf{n}, f\rangle = \int_{\Omega} \nu S(\nabla\bar{u}_0)\cdot S(\nabla\tilde{u}) dx = \int_{\Omega\setminus\overline{D}} \nu S(\nabla\bar{u}_0)\cdot S(\nabla u) dx. \tag{2.3}$$

Next, subtracting (2.3) from (2.1) leads to

$$\begin{aligned} &\langle \sigma(\bar{u},\bar{p})\mathbf{n} - \sigma(\bar{u}_0,\bar{p}_0)\mathbf{n}, f\rangle \\ = &\int_{\Omega\setminus\overline{D}} \nu S(\nabla\bar{u})\cdot S(\nabla u) dx - \int_{\Omega\setminus\overline{D}} \nu S(\nabla\bar{u}_0)\cdot S(\nabla u) dx \\ = &\int_{\Omega\setminus\overline{D}} \nu S(\nabla(\bar{u}-\bar{u}_0))\cdot S(\nabla u) dx. \end{aligned} \tag{2.4}$$

On the other hand, we can deduce that

$$\begin{aligned} &\int_{\Omega\setminus\overline{D}} \nu S(\nabla u_0)\cdot S(\nabla(\bar{u}-\bar{u}_0)) dx \\ = &\int_{\Omega\setminus\overline{D}} \nu S(\nabla u_0)\cdot S(\nabla\bar{u}) dx - \int_{\Omega} \nu S(\nabla u_0)\cdot S(\nabla\bar{u}_0) dx \\ &+ \int_D \nu |S(\nabla u_0)|^2 dx \\ = &\int_{\Omega\setminus\overline{D}} [\nu S(\nabla u_0)\cdot S(\nabla\bar{u}) - p_0\,\mathrm{div}\,\bar{u}] dx - \int_{\Omega} [\nu S(\nabla u_0)\cdot S(\nabla\bar{u}_0) - p_0\,\mathrm{div}\, u_0] dx \\ &+ \int_D \nu |S(\nabla u_0)|^2 dx \\ = &\langle \sigma(u_0,p_0)\mathbf{n}, \bar{f}\rangle - \langle \sigma(u_0,p_0)\mathbf{n}, \bar{f}\rangle + \int_D \nu |S(\nabla u_0)|^2 dx \\ = &\int_D \nu |S(\nabla u_0)|^2 dx. \end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5) yields

$$\langle \sigma(\bar{u},\bar{p})\mathbf{n} - \sigma(\bar{u}_0,\bar{p}_0)\mathbf{n}, f\rangle = \int_{\Omega\setminus\overline{D}} \nu |S(\nabla(u-u_0))|^2 dx + \int_D \nu |S(\nabla u_0)|^2 dx. \tag{2.6}$$

Now we observe that $(u-u_0, p-p_0)$ satisfies

$$\begin{cases} \operatorname{div}(\nu S(\nabla(u-u_0))) - (p-p_0) = 0 \quad \text{in} \quad \Omega\setminus\overline{D}, \\ \operatorname{div}(u-u_0) = 0 \quad \text{in} \quad \Omega\setminus\overline{D}, \\ u-u_0 = -u_0 \quad \text{on} \quad \partial D, \\ u-u_0 = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

The regularity theorem implies

$$\|u-u_0\|_{H^1(\Omega\setminus\overline{D})} \le C\|u_0\|_{H^{1/2}(\partial D)} \le C\|u_0\|_{H^1(D)} \tag{2.7}$$

for some constant $C>0$. In view of Korn's inequality, putting together (2.6) and (2.7) gives

$$\int_D |S(\nabla u_0)|^2 dx \le \langle \sigma(\bar{u},\bar{p})\mathbf{n} - \sigma(\bar{u}_0,\bar{p}_0)\mathbf{n}, f\rangle \le C\big(\int_D |S(\nabla u_0)|^2 dx + \int_D |u_0|^2 dx\big),$$

that is,

$$\int_D |S(\nabla u_0)|^2 dx \le \langle (\Lambda_D - \Lambda_0)(\bar{f}), f\rangle \le C\big(\int_D |S(\nabla u_0)|^2 dx + \int_D |u_0|^2 dx\big), \tag{2.8}$$

where

$$\Lambda_0(f) = \{\{\sigma(u_0,p_0)\mathbf{n}|_{\partial\Omega}\} : (u_0,p_0) \text{ is a solution of (2.2) with } u_0|_{\partial\Omega} = f\}.$$

Now we would like to construct special solutions (u_0, p_0) satisfying

$$\begin{cases} \operatorname{div}(\nu S(\nabla u_0)) - \nabla p_0 = 0 \quad \text{in} \quad \Omega, \\ \operatorname{div} u_0 = 0 \quad \text{in} \quad \Omega. \end{cases} \tag{2.9}$$

As in [8] (see the related result for the elasticity in [3], [5], [19]) , we set $u_0 = \nu^{-1/2}w + \nu^{-1}\nabla g - (\nabla\nu^{-1})g$ and

$$p_0 = \nabla\nu^{1/2}\cdot w + \nu^{1/2}\operatorname{div} w + 2\Delta g = \operatorname{div}(\nu^{1/2}w) + 2\Delta g, \tag{2.10}$$

then (u_0,p_0) is a solution of (2.9) provided $\begin{pmatrix} w \\ g \end{pmatrix}$ satisfies

$$P\begin{pmatrix} w \\ g \end{pmatrix} := \Delta\begin{pmatrix} w \\ g \end{pmatrix} + A_1(x)\begin{pmatrix} \nabla g \\ \operatorname{div} w \end{pmatrix} + A_0(x)\begin{pmatrix} w \\ g \end{pmatrix} = 0 \tag{2.11}$$

with

$$A_1(x) = \begin{pmatrix} -2\nu^{1/2}\nabla^2\nu^{-1} & -\nu^{-1}\nabla\nu \\ 0 & \nu^{1/2} \end{pmatrix},$$

where $\nabla^2 g$ denotes the Hessian of the function g. The exact formula of $A_0(x)$ is not important here. We only need to note that $A_0(x)$ contains at most three derivatives of ν.

3. Construction of complex spherical waves

Our aim here is to construct special solutions u_0, called complex spherical waves, from (2.11). The method here is completely analogue to the elasticity system treated in [20], which is based on the Carleman method introduced in [7] and [13]. We set the matrix operator $P_h = -h^2P$. More precisely, we have

$$P_h = (hD)^2 + ihA_1(x)\begin{pmatrix} hD \\ hD\cdot \end{pmatrix} + h^2A_0$$

where $D = -i\nabla$. Later on we shall denote the matrix operator

$$iA_1(x)\begin{pmatrix} hD \\ hD\cdot \end{pmatrix} = A_1(x, hD).$$

The construction here is simpler than the one given in [14] and [16] where the technique of intertwining operators were first introduced. Furthermore, we do not need to work with C^∞ coefficients here. Our first goal is to derive a Carleman estimate with semiclassical H^{-2} norm for P_h.

The conjugation of P_h with $e^{\varphi/h}$ is given by

$$e^{\varphi/h} \circ P_h \circ e^{-\varphi/h} = (hD + i\nabla\varphi)^2 + hA_1(x, hD + i\nabla\varphi) + h^2 A_0(x).$$

We first consider the leading operator $(hD + i\nabla\varphi)^2$ and denote

$$(hD + i\nabla\varphi)^2 = A + iB,$$

where $A = (hD)^2 - (\nabla\varphi)^2$ and $B = \nabla\varphi \circ hD + hD \circ \nabla\varphi$. The Weyl symbols of A and B are given as

$$a(x,\xi) = \xi^2 - (\nabla\varphi)^2 \quad \text{and} \quad b(x,\xi) = 2\nabla\varphi\cdot\xi,$$

respectively. Let Ω_0 be an open bounded domain such that $\overline{\Omega} \subset \Omega_0$. Accordingly, we extend ν to Ω_0 by preserving its smoothness. We now let φ have nonvanishing gradient in Ω_0 and be a limiting Carleman weight in Ω_0:

$$\{a, b\} = 0 \quad \text{when} \quad a = b = 0,$$

i.e.,

$$\langle \nabla^2\varphi, \nabla\varphi\otimes\nabla\varphi + \xi\otimes\xi\rangle = 0 \quad \text{when} \quad \xi^2 = (\nabla\varphi)^2 \text{ and } \nabla\varphi\cdot\xi = 0.$$

Let us denote $\varphi_\varepsilon = \varphi + h\varphi^2/(2\varepsilon)$, where $\varepsilon > 0$ will be chosen later. Also, we denote a_ε and b_ε the corresponding symbols as φ is replaced by φ_ε. Then one can easily check that

$$\{a_\varepsilon, b_\varepsilon\} = \frac{4h}{\varepsilon}(1 + \frac{h}{\varepsilon}\varphi)^2(\nabla\varphi)^4 > 0 \quad \text{when} \quad a_\varepsilon = b_\varepsilon = 0.$$

Arguing as in [13], we get

$$\{a_\varepsilon, b_\varepsilon\} = \frac{4h}{\varepsilon}(1 + \frac{h}{\varepsilon}\varphi)^2(\nabla\varphi)^4 + \alpha(x)a_\varepsilon + \beta(x,\xi)b_\varepsilon,$$

where $\beta(x,\xi)$ is linear in ξ. Therefore, at the operator level, we have

$$i[A_\varepsilon, B_\varepsilon] = \frac{4h^2}{\varepsilon}(1+\frac{h}{\varepsilon}\varphi)^2(\nabla\varphi)^4 + \frac{h}{2}(\alpha\circ A_\varepsilon + A_\varepsilon\circ\alpha) + \frac{h}{2}(\beta^w\circ B_\varepsilon + B_\varepsilon\circ\beta^w) + h^3 c(x), \tag{3.1}$$

where β^w denotes the Weyl quantization of β.

With the help (3.1), we can now estimate

$$\|(A_\varepsilon + iB_\varepsilon)V\|^2 = \|A_\varepsilon V\|^2 + \|B_\varepsilon V\|^2 + i\langle B_\varepsilon V|A_\varepsilon V\rangle - i\langle A_\varepsilon V|B_\varepsilon V\rangle$$

for $V \in C_0^\infty(\Omega)$. Here and below, we define the norm $\|\cdot\|$ and the inner $\langle\cdot\mid\cdot\rangle$ in terms of $L^2(\Omega)$. Integrating by parts, we conclude

$$\langle B_\varepsilon V|A_\varepsilon V\rangle = \langle A_\varepsilon B_\varepsilon V|V\rangle \quad \text{and} \quad \langle A_\varepsilon V|B_\varepsilon V\rangle = \langle B_\varepsilon A_\varepsilon V|V\rangle. \tag{3.2}$$

On the other hand, we observe that

$$\|h\nabla V\|^2 = \langle A_\varepsilon V|V\rangle + \|\sqrt{(\nabla\varphi)}V\|^2 \le C(\|A_\varepsilon V\|^2 + \|V\|^2) \tag{3.3}$$

and the obvious estimate

$$\|(h\nabla)^2 V\|^2 \le C(\|A_\varepsilon V\|^2 + \|V\|^2). \tag{3.4}$$

Using (3.1), (3.2), (3.3), and (3.4) gives

$$\|(A_\varepsilon + iB_\varepsilon)V\|^2 \ge C\big((1 - O(\frac{h^2}{\varepsilon}))\|A_\varepsilon V\|^2 + \frac{h^2}{\varepsilon}(\|A_\varepsilon V\|^2 + \|V\|^2)\big).$$

Thus, taking h and ε ($h \ll \varepsilon$) sufficiently small, we arrive at

$$\|(A_\varepsilon + iB_\varepsilon)V\|^2 \ge \frac{Ch^2}{\varepsilon}(\|V\|^2 + \|h\nabla V\|^2 + \|(h\nabla)^2 V\|^2),$$

namely,

$$\|(A_\varepsilon + iB_\varepsilon)V\|^2 \ge \frac{Ch^2}{\varepsilon}\|V\|^2_{H^2_h(\Omega)}. \tag{3.5}$$

Here we define the semiclassical Sobolev norms by

$$\|v\|^2_{H^m_h(\Omega)} = \sum_{|\alpha|\le m} \|(h\nabla)^\alpha v\|^2 \quad \forall\, m \in \mathbb{N}$$

and

$$\|v\|^2_{H^s_h(\mathbb{R}^3)} = \int (1 + |h\xi|^2)^s |\hat{v}(\xi)|^2 d\xi = \|\langle hD\rangle^s v\|^2 \quad \forall\, s \in \mathbb{R}$$

where $\hat{v}$ denotes the Fourier transform of v.

Now let Ω_1 be open and $\overline{\Omega} \subset \Omega_1 \subset \Omega_0$. The estimate (3.5) also holds for $V \in C_0^\infty(\Omega_1)$. Then as done in [7], we can obtain that

$$\frac{h^2}{\varepsilon}\|V\|^2_{H^2_h(\mathbb{R}^3)} \le C\|(A_\varepsilon + iB_\varepsilon)\langle hD\rangle^2 V\|^2_{H^{-2}_h(\mathbb{R}^3)}. \tag{3.6}$$

To add the first order perturbation $hA_{1,\varepsilon}V + h^2 A_0 V = hA_1(x, hD + i\nabla\varphi_\varepsilon)V + h^2 A_0 V$ into (3.6), we note that

$$\|(hA_{1,\varepsilon} + h^2 A_0)\langle hD\rangle^2 V\|^2_{H^{-2}_h(\mathbb{R}^3)} \le Ch^2\|V\|^2_{H^1_h(\mathbb{R}^3)}. \tag{3.7}$$

In view of (3.7), we get from (3.6) that

$$\|(A_\varepsilon + iB_\varepsilon + hA_{1,\varepsilon} + h^2 A_0)\langle hD\rangle^2 V\|^2_{H^{-2}_h(\mathbb{R}^3)} \ge Ch^2\|\langle hD\rangle^2 V\|^2 \tag{3.8}$$

provided $\varepsilon \ll 1$. Transforming back to the original operator, (3.8) is equivalent to

$$\|\langle hD\rangle^2 V\| \le Ch\|e^{\phi_\varepsilon/h} P e^{-\varphi_\varepsilon/h}\langle hD\rangle^2 V\|_{H^{-2}_h(\mathbb{R}^3)} \tag{3.9}$$

for $V \in C_0^\infty(\Omega_1)$.

Let $\chi \in C_0^\infty(\Omega_1)$ with $\chi = 1$ on Ω and $W \in C_0^\infty(\Omega)$. Substituting $V = \chi\langle hD\rangle^{-2}W$ into (3.9) and using the property that

$$\|(1-\chi)\langle hD\rangle^{-2}W\|_{H^s_h} = O(h^\infty)\|W\|$$

for any $s \in \mathbb{R}$, we get that

$$\|W\| \le Ch\|e^{\phi_\varepsilon/h} P e^{-\varphi_\varepsilon/h} W\|_{H^{-2}_h(\mathbb{R}^3)}. \tag{3.10}$$

Now since $e^{\varphi_\varepsilon/h} = e^{\varphi^2/\varepsilon}e^{\varphi/h}$ and $e^{\varphi^2/\varepsilon} = O(1)$, (3.10) becomes

$$\|W\| \le Ch\|e^{\phi/h} P e^{-\varphi/h} W\|_{H^{-2}_h(\mathbb{R}^3)}. \tag{3.11}$$

Note that (3.11) also holds when φ is replaced by $-\varphi$. Therefore, by the Hahn-Banach theorem, we have the following existence theorem.

Theorem 3.1. *For h sufficiently small, for any $F \in L^2(\Omega)$, there exists $V \in H_h^2(\Omega)$ such that*
$$e^{\varphi/h} P_h(e^{-\varphi/h} V) = F$$
and $h\|V\|_{H_h^2(\Omega)} \lesssim \|F\|_{L^2(\Omega)}$.

Next, we let ψ be a solution of the eikonal equation
$$\begin{cases} (\nabla\psi)^2 = (\nabla\varphi)^2 \\ \nabla\varphi \cdot \nabla\psi = 0, \end{cases} \quad \forall\ x \in \Omega. \tag{3.12}$$
Since $\{a, b\} = 0$ on $a = b = 0$, there exists a solution to (3.12). To construct complex spherical waves, we choose the limiting Carleman weight
$$\varphi(x) = \log|x - x_0| \quad \text{for} \quad x_0 \notin \overline{\mathrm{ch}(\Omega)},$$
then a solution of (3.12) is
$$\psi(x) = \frac{\pi}{2} - \arctan \frac{\omega \cdot (x - x_0)}{\sqrt{(x - x_0)^2 - (\omega \cdot (x - x_0))^2}} = d_{\mathbb{S}^2}\left(\frac{x - x_0}{|x - x_0|}, \omega\right)$$
where $\mathrm{ch}(\Omega) :=$ convex hull of Ω and $\omega \in \mathbb{S}^2$ such that $\omega \neq (x - x_0)/|x - x_0|$ for all $x \in \overline{\Omega}$ [7]. Having found ψ, we look for $U = e^{-(\varphi+i\psi)/h}(L + R)$ satisfying
$$(-h^2\Delta + h^2 A_1(x, D) + h^2 A_0(x))U = 0 \quad \text{in} \quad \Omega.$$
Equivalently, we need to solve
$$e^{(\varphi+i\psi)/h} P_h(e^{-(\varphi+i\psi)/h}(L + R)) = 0 \quad \text{in} \quad \Omega.$$
We can compute that
$$e^{(\varphi+i\psi)/h} P_h e^{-(\varphi+i\psi)/h} = hQ + P_h$$
where $Q = -\nabla\psi \cdot D - D \cdot \nabla\psi + i\nabla\varphi \cdot D + iD \cdot \nabla\varphi + A_1(x, i\nabla\varphi - \nabla\psi)$. Hence we want to find L, independent of h, so that
$$QL = 0 \quad \text{in} \quad \Omega. \tag{3.13}$$
The equation (3.13) is a system of Cauchy-Riemann type. Using the results in [4], [6], or [15], one can find an invertible 4×4 matrix $G(x) \in C^2(\overline{\Omega})$ satisfying (3.13). So L can be chosen from columns of G. Then, R is required to satisfy
$$e^{\varphi/h} P_h(e^{-(\varphi+i\psi)/h} R) = -e^{-i\psi/h} P_h L. \tag{3.14}$$
Note that $\|e^{-i\psi/h} P_h L\| \lesssim h^2$. Thus Theorem 3.1 implies that
$$\|e^{-i\psi/h} R\|_{H_h^2(\Omega)} \lesssim h, \tag{3.15}$$
which leads to
$$\|\partial^\alpha R\|_{L^2(\Omega)} \lesssim h^{1-|\alpha|} \quad \text{for} \quad |\alpha| \leq 2. \tag{3.16}$$
So if we write $L = \begin{pmatrix} \ell \\ d \end{pmatrix}$ and $R = \begin{pmatrix} r \\ s \end{pmatrix}$ with $\ell, r \in \mathbb{C}^3$, then
$$w = e^{-(\varphi+i\psi)/h}(\ell + r) \quad \text{and} \quad g = e^{-(\varphi+i\psi)/h}(d + s) \tag{3.17}$$
where r and s satisfy the estimate (3.16). Therefore, $u_0 = \nu^{-1/2} w + \nu^{-1}\nabla g - g\nabla\nu^{-1}$ is the needed complex spherical wave, i.e., the real part of the phase function of u_0 is a radial function.

Remark 3.2. *Even though the four-vector* $\begin{pmatrix} \ell \\ d \end{pmatrix}$ *is nonzero in* Ω, *we cannot conclude that* d *never vanishes in* Ω. *However, for any point* $y \in \Omega$, *it is easy to show that there exists a small ball* $B_\delta(y)$ *of* y *with* $B_\delta(y) \subset \Omega$ *such that* d *does not vanish in* $B_\delta(y)$. *We will use this fact in studying our inverse problem in the next section.*

4. Reconstruction of the Obstacle

In this section we shall use complex spherical waves constructed in the previous section and the integral inequality (2.8) to determine some part of D by boundary measurements $(u|_{\partial\Omega}, \sigma(u,p)\mathbf{n}|_{\partial\Omega})$. To this end, we set $u^0_{h,t} = e^{\log t/h} u_0$, with parameters $t > 0$ and $h > 0$. We choose the function u_0 to be of the form $u_0 = \nu^{-1/2} w + \nu^{-1}\nabla g - g\nabla\nu^{-1}$ where $w = e^{-(\varphi+i\psi)/h}(\ell + r)$ and $g = e^{-(\varphi+i\psi)/h}(d + s)$ are the functions given in (3.17). With $u^0_{h,t}$, we can define $p^0_{h,t} = e^{\log t/h} p_0$ with p_0 given by (2.10). Then $(u^0_{h,t}, p^0_{h,t})$ satisfies (2.9) for all h, t and since ν is known, we can determine $\sigma(u^0_{h,t}, p^0_{h,t})\mathbf{n}|_{\partial\Omega} = \Lambda_0(f_{h,t})$ with $f_{h,t} = u^0_{h,t}|_{\partial\Omega}$. Now let $(u_{h,t}, p_{h,t})$ be a solution of (1.1) with the boundary value $f_{h,t}$. Then $(f_{h,t}, \sigma(u_{h,t}, p_{h,t})\mathbf{n}|_{\partial\Omega}) = (f_{h,t}, \Lambda_D(f_{h,t}))$ is the observable data. Therefore, we can determine

$$E_t(h) := \langle (\Lambda_D - \Lambda_0)(\bar{f}_{h,t}), f_{h,t}\rangle \geq 0$$

by boundary measurements. We now use the behavior of $E_t(h)$ as $h \to 0$ at different t's to reconstruct some part of ∂D. Our method shares the same spirit as Ikehata's *enclosure method* (see Ikehata's survey article [11]). Furthermore, since we probe the region by complex spherical waves, it is possible to recover some concave parts of inclusions. For other related results, we refer to [20] and references therein.

The behavior of $E_t(h)$ will be determined by (2.8). In order to use (2.8), we need to compute $S(\nabla u_0)$. First note that

$$S(\nabla u_0) = 2S(\nabla\nu^{-1/2} \otimes w) + \nu^{-1/2} S(\nabla w) + \nu^{-1}\nabla^2 g - g\nabla^2\nu^{-1}$$

and therefore

$$\begin{aligned} S(\nabla u_0) \quad = \quad & e^{-(\varphi+i\psi)/h}\Big(S(\nabla\nu^{-1/2} \otimes (\ell + r)) - \frac{1}{h}\nu^{-1/2} S((\nabla\varphi + i\nabla\psi) \otimes (\ell + r)) \\ & + \nu^{-1/2} S(\nabla(\ell + r)) + \nu^{-1}\nabla^2(d + s) - \nu^{-1}\frac{1}{h}(d + s)\nabla^2(\varphi + i\psi) \\ & - \nu^{-1}\frac{2}{h} S(\nabla(\varphi + i\psi) \otimes \nabla(d + s)) - (d + s)\nabla^2\nu^{-1} \\ & + \nu^{-1}\frac{1}{h^2}\nabla(\varphi + i\psi) \otimes \nabla(\varphi + i\psi)(d + s)\Big). \end{aligned}$$

Now we are in the position to formulate the main result of this work.

Theorem 4.1. *Let* $x_0 \notin \mathrm{ch}(\Omega)$. *We set* $f_{h,t} = u^0_{h,t}|_{\partial\Omega}$. *Then there exists* $h_0 > 0$ *such that for* $t > 0$ *and* $h \leq h_0$ *the following assertions are true:*

(a) *If* $\mathrm{dist}(D, x_0) > t$, *then* $E_t(h) \leq C a^{\frac{1}{h}}$ *for some constants* $C > 0$ *and* $a < 1$.
(b) *If* $\mathrm{dist}(D, x_0) < t$, *then* $E_t(h) \geq C b^{\frac{1}{h}}$ *for some constants* $C > 0$ *and* $b > 1$ *and an appropriate choice of* $f_{h,t}$.

(c) *If* $\mathrm{dist}(D, x_0) = t$ *with* y *such that* $\|y - x_0\| = \mathrm{dist}(D, x_0)$ *then there are constants* $C_1, C_2 > 0$ *such that*

$$C_1 h^{-1} \le E_t(h) \le C_2 h^{-3},$$

provided the function d *does not vanish in a neighborhood of* y.

Proof. The key to obtain estimates of the terms $E_t(h)$ is to plug in the special solutions $u^0_{h,t}$ and to look at the leading terms in view of small h. This leading terms come from the expression $S(\nabla u^0_{h,t})$. In the case $\mathrm{dist}(D, x_0) > t$ these are the terms which behave like h^{-4} and are given by

$$((\nabla\varphi)^2 + (\nabla\psi)^2)^2 |d|^2 \frac{1}{h^4} \left(\frac{t}{|x - x_0|}\right)^{2/h} = 4(\nabla\varphi)^4 |d|^2 \frac{1}{h^4} \left(\frac{t}{|x - x_0|}\right)^{2/h}.$$

So from the second inequality of (2.8) we get

$$E_t(h) \le Ch^{-4} \left(\frac{t}{\mathrm{dist}(D, x_0)}\right)^{2/h}$$

for $h \le h_0$ and therefore the first assertion is proved.

In order to treat the second assertion we choose a small ball $B_\epsilon \Subset B_t(x_0) \cap D$ and the special solution $u^0_{h,t}$ such that the corresponding function d in the definition of $u^0_{h,t}$ never vanishes in B_ϵ. Then the boundary value of $u_{h,t}$ on $\partial\Omega$ is given by

$$f_{h,t} = u^0_{h,t}|_{\partial\Omega} = e^{(\log t - \varphi - i\psi)/h} (\ell + d)|_{\partial\Omega}.$$

Since for $h \le h_0$ small enough and $j = 1, 2, 3$ the inequality

$$h^{-4} |d|^2 - h^{-j} C \ge C' h^{-4}$$

holds in B_ϵ we get

$$E_t(h) \ge Ch^{-4} \left|\frac{t}{\mathrm{dist}(D, x_0)}\right|^{\frac{2}{h}}$$

and therefore assertion (b).

Analyzing the behavior of $E_t(h)$ when $y \in \partial D \cap \partial B_t(x_0)$ we choose $u^0_{h,t}$, as before, such that d does not vanish in the ball $B_\epsilon(y)$. Pick a small cone with vertex at y, say Γ, so that there exists an $\eta > 0$ satisfying

$$\Gamma_\eta := \Gamma \cap \{0 < |x - y| < \eta\} \subset B_\epsilon(y) \cap D.$$

We observe that if $x \in \Gamma_\eta$ with $|x - y| = \rho < \eta$ then $|x - x_0| \le \rho + t$, i.e.

$$\frac{1}{|x - x_0|} \ge \frac{1}{\rho + t}.$$

Thus, we get from the first inequality of (2.8) that

$$\begin{aligned} E_t(h) &\ge C\frac{1}{h^4} \int_D (\nabla\varphi)^4 |d|^2 \left(\frac{t}{|x - x_0|}\right)^{2/h} dx && (4.1)\\ &\ge C\epsilon \frac{1}{h^4} \int_0^\eta \left(\frac{t}{\rho + t}\right)^{2/h} \rho^2 d\rho && (4.2)\\ &\ge C\epsilon h^{-1}. && (4.3) \end{aligned}$$

On the other hand, we can choose a cone $\widetilde{\Gamma}$ with vertex at x_0 such that $\overline{D} \subset \widetilde{\Gamma} \cap \{|x-x_0| > t\}$. Hence, we can estimate

$$\begin{aligned} E_t(h) &\leq C\frac{1}{h^4}\int_{\widetilde{\Gamma}\cap\{t<|x-x_0|<t+\eta\}}(\frac{t}{|x-x_0|})^{2/h}dx \qquad (4.4) \\ &\quad +C\frac{1}{h^4}\int_{\widetilde{\Gamma}\cap\{t+\eta\leq|x-x_0|\}}(\frac{t}{|x-x_0|})^{2/h}dx \qquad (4.5) \\ &\leq C\frac{1}{h^4}\int_t^{t+\eta}(\frac{t}{r})^{2/h}r^2dr + O((\frac{t}{t+\eta})^{2/h}) \qquad (4.6) \\ &\leq Ch^{-3}. \qquad (4.7) \end{aligned}$$

Combining (4.1) and (4.4) yields the assertion (c). □

Remark 4.2. *In the proofs of (b) and (c) we need to choose d which is nonvanishing in small subdomains of Ω. Since d depends only on the known background medium, they can be chosen to be nonvanishing near any point in Ω at our disposal. If fact, it suffices to take d which is nonvanishing near the probe front $\{|x-x_0| = t\}$. Different choices of d will give rise to different Dirichlet data $f_{h,t}$ and therefore different measurements. In real applications, we believe that the concerns in (b) and (c) can be ignored.*

Next we would like to discuss the possibility of localizing boundary measurements. It turns out that we cannot produce a similar result as in [9] and [20]. This is due to the compatibility condition for the Dirichlet data (1.2) or, equivalently, the divergence-free condition. Those restrictions are global ones. They cannot be localized. Even though we cannot exactly localize the measurements as we wish, we can still do it *approximately*. We now describe the method.

Let $\phi_{\delta,t}(x) \in C_0^\infty(\mathbb{R}^3)$ satisfy

$$\phi_{\delta,t}(x) = \begin{cases} 1 & \text{on} \quad B_{t+\delta/2}(x_0) \\ 0 & \text{on} \quad \mathbb{R}^3 \setminus \overline{B_{t+\delta}(x_0)} \end{cases}$$

where $\delta > 0$ is sufficiently small. We are going to use the measurements $f_{\delta,h,t} = \phi_{\delta,t} f_{h,t} - c_{\delta,h,t}\mathbf{n} = \phi_{\delta,t} u^0_{h,t}|_{\partial\Omega} - c_{\delta,h,t}\mathbf{n}$, where

$$c_{\delta,h,t} = \int_{\partial\Omega} \phi_{\delta,t} f_{h,t}\cdot\mathbf{n}ds = \int_{\partial\Omega\cap B_{t+\delta}(x_0)} \phi_{\delta,t} f_{h,t}\cdot\mathbf{n}ds.$$

With such constant $c_{\delta,h,t}$, the boundary value $f_{\delta,h,t}$ satisfies the compatibility condition (1.2). We can see that the constant $c_{\delta,h,t}$ tends to zero as $h \to 0$. Indeed, we can estimate

$$\begin{aligned} |c_{\delta,h,t}| &\leq |\int_{\partial\Omega}\phi_{\delta,t} f_{h,t}\cdot\mathbf{n}ds| \\ &= |\int_{\partial\Omega}(1-\phi_{\delta,t}) f_{h,t}\cdot\mathbf{n}ds| \\ &\leq \sqrt{\text{Area}(\partial\Omega)}\ \|(1-\phi_{\delta,t})f_{h,t}\|_{L^2(\partial\Omega)} \\ &\leq \sqrt{\text{Area}(\partial\Omega)}\ \|(1-\phi_{\delta,t})u^0_{h,t}|_{\partial\Omega}\|_{H^{1/2}(\partial\Omega)} \\ &\leq C\sqrt{\text{Area}(\partial\Omega)}\ \|(1-\phi_{\delta,t})u^0_{h,t}\|_{H^1(\Omega)} \\ &\leq \tilde{C}a_0^{1/h} \end{aligned}$$

for some $0 < a_0 < 1$. The last estimate follows from the decaying property of $u^0_{h,t}$ in $\Omega \setminus \overline{B_{t+\delta/2}(x_0)}$. We argue that the use of $f_{\delta,h,t}$ is better than $f_{h,t}$ because $f_{\delta,h,t}$ can be kept constant on most part of $\partial\Omega$ and this constant is as small as we want. Let us now define

$$E_{\delta,t}(h) = \langle (\Lambda_D - \Lambda_0)(\bar{f}_{\delta,h,t}), f_{\delta,h,t}\rangle.$$

Theorem 4.3. *The statements of Theorem 4.1 are valid for $E_{\delta,t}(h)$.*

Proof. The main idea is to prove that the error caused by the remaining part of the measurement $g_{\delta,h,t} := (1-\phi_{\delta,t})f_{h,t}|_{\partial\Omega} + c_{\delta,h,t}\mathbf{n}$ is as small as any given polynomial order. Let $(w_{\delta,h,t}, p_{\delta,h,t})$ be a solution of (2.2) with boundary value $g_{\delta,h,t}$. We now want to compare $w_{\delta,h,t}$ with $(1-\phi_{\delta,t})u^0_{h,t}$. Note that a pressure function associated with $u^0_{h,t}$ is given by

$$p^0_{h,t} = e^{\log t/h}(\mathrm{div}(\nu^{1/2} w) + 2\Delta g).$$

For simplicity, we denote $v := (1-\phi_{\delta,t})u^0_{h,t} - w_{\delta,h,t}$ and $p := (1-\phi_{\delta,t})p^0_{h,t} - p_{\delta,h,t}$. Then we have

$$\begin{cases} \mathrm{div}(\nu S(\nabla v)) - \nabla p = \mathrm{div}(\nu S(\nabla((1-\phi_{\delta,t})u^0_{h,t}))) - \nabla((1-\phi_{\delta,t})p^0_{h,t}) := F \quad \text{in} \quad \Omega, \\ \mathrm{div}\, v = \mathrm{div}((1-\phi_{\delta,t})u^0_{h,t}) = \nabla(1-\phi_{h,t})\cdot u^0_{h,t} := G \quad \text{in} \quad \Omega, \\ v = -c_{\delta,h,t}\mathbf{n} \quad \text{on} \quad \partial\Omega. \end{cases} \tag{4.8}$$

We can check that

$$\mathrm{supp}\ (F),\ \mathrm{supp}\ (G) \subset \overline{\Omega} \cap \{t+\delta/2 \le |x-x_0| \le t+\delta\}$$

and therefore

$$\|F\|_{L^2(\Omega)} \le C a_1^{1/h} \quad \text{and} \quad \|G\|_{H^1(\Omega)} \le C a_1^{1/h}$$

for some $0 < a_1 < 1$. Furthermore, we observe that

$$\mathrm{div}(\nu S(\nabla v)) - \nabla p = \nu\Delta v + \nu\nabla(\mathrm{div}\, v) + S(\nabla v)\nabla\nu - \nabla p = F.$$

Thus, we can rewrite (4.8) as

$$\begin{cases} \Delta v + \nu^{-1}S(\nabla v)\nabla\nu - \nu^{-1}\nabla p = -\nabla G + \nu^{-1}F := \tilde{F} \quad \text{in} \quad \Omega, \\ \mathrm{div}\, v = G \quad \text{in} \quad \Omega, \\ v = -c_{\delta,h,t}\mathbf{n} \quad \text{on} \quad \partial\Omega. \end{cases} \tag{4.9}$$

Also, we have

$$\|\tilde{F}\|_{L^2(\Omega)} \le C a_1^{1/h} \quad \text{and} \quad \|c_{\delta,h,t}\mathbf{n}\|_{H^{3/2}(\partial\Omega)} \le C a_0^{1/h}.$$

The system (4.9) is a nonhomogeneous Stokes system, which is an elliptic system in the sense of Agmon, Douglis, and Nirenberg [1]. Here we are concerned with the regularity estimate of (4.9):

$$\|v\|_{H^2(\Omega)} \le C\big(\|\tilde{F}\|_{L^2(\Omega)} + \|G\|_{H^1(\Omega)} + \|c_{\delta,h,t}\mathbf{n}\|_{H^{3/2}(\partial\Omega)}\big). \tag{4.10}$$

In order to derive (4.10), we can follow the approach used in [18, Proposition 2.2], which is based on [1]. We want to remark that ν considered in [18, Proposition 2.2] is a constant. Nevertheless, the same proof of Proposition 2.2 in [18] can be adapted to the case where ν is variable with only minor modifications. For brevity, we will not repeat the arguments here.

Subsequently, we get from (4.10) that

$$\|(1-\phi_{\delta,t})u^0_{h,t} - w_{\delta,h,t}\|_{H^1(\Omega)} \le Ca_2^{1/h},$$

in particular,

$$\|(1-\phi_{\delta,t})u^0_{h,t} - w_{\delta,h,t}\|_{H^1(D)} \le Ca_2^{1/h} \tag{4.11}$$

for some $0 < a_2 < 1$. Using the second inequality of (2.8) for $\langle(\Lambda_D - \Lambda_0)(\bar{g}_{\delta,h,t}), g_{\delta,h,t}\rangle$ with u_0 being replaced by $w_{\delta,h,t}$, we get from (4.11) and the decaying property of $u^0_{h,t}$ that

$$\langle(\Lambda_D - \Lambda_0)(\bar{g}_{\delta,h,t}), g_{\delta,h,t}\rangle \le Ca_3^{1/h}$$

for some $0 < a_3 < 1$.

First we consider (a) of Theorem 4.1 for $E_{\delta,t}(h)$. We shall use a trick given in [9]. From the proof of (2.8), we see that

$$0 \le \langle(\Lambda_D - \Lambda_0)(\zeta \bar{f}_{\delta,h,t} \pm \zeta^{-1}\bar{g}_{\delta,h,t}), \zeta f_{\delta,h,t} \pm \zeta^{-1} g_{\delta,h,t}\rangle$$

for any $\zeta > 0$, which implies

$$\begin{aligned} & |\langle(\Lambda_D - \Lambda_0)(\bar{f}_{\delta,h,t}), g_{\delta,h,t}\rangle + \langle(\Lambda_D - \Lambda_0)(\bar{g}_{\delta,h,t}), f_{\delta,h,t}\rangle| \\ \le \; & \zeta^2\langle(\Lambda_D - \Lambda_0)(\bar{f}_{\delta,h,t}), f_{\delta,h,t}\rangle + \zeta^{-2}\langle(\Lambda_D - \Lambda_0)(\bar{g}_{\delta,h,t}), g_{\delta,h,t}\rangle. \end{aligned} \tag{4.12}$$

Using $f_{h,t} = f_{\delta,h,t} + g_{\delta,h,t}$ and (4.12) with $\zeta = 1/\sqrt{2}$, we obtain that

$$\begin{aligned} & \frac{1}{2}\langle(\Lambda_D - \Lambda_0)(\bar{f}_{\delta,h,t}), f_{\delta,h,t}\rangle \\ \le \; & \langle(\Lambda_D - \Lambda_0)(\bar{g}_{\delta,h,t}), g_{\delta,h,t}\rangle + \langle(\Lambda_D - \Lambda_0)(\bar{f}_{h,t}), f_{h,t}\rangle \\ \le \; & Ca_3^{1/h} + \langle(\Lambda_D - \Lambda_0)(\bar{f}_{h,t}, f_{h,t}\rangle. \end{aligned} \tag{4.13}$$

So from (a) of Theorem 4.1, the same statement holds for $E_{\delta,t}(h)$. Similarly, the second inequality of (c) in Theorem 4.1 is also true for $E_{\delta,t}(h)$.

Next we consider (b) and the first inequality of (c) in Theorem 4.1 for $E_{\delta,t}(h)$. Choosing $\zeta = 1$ in (4.12) we get that

$$\begin{aligned} & \frac{1}{2}\langle(\Lambda_D - \Lambda_0)(\bar{f}_{h,t}), f_{h,t}\rangle \\ \le \; & \langle(\Lambda_D - \Lambda_0)(\bar{g}_{\delta,h,t}), g_{\delta,h,t}\rangle + \langle(\Lambda_D - \Lambda_0)(\bar{f}_{\delta,h,t}), f_{\delta,h,t}\rangle \\ \le \; & Ca_3^{1/h} + \langle(\Lambda_D - \Lambda_0)(\bar{f}_{\delta,h,t}), f_{\delta,h,t}\rangle. \end{aligned} \tag{4.14}$$

Thus, combining (b) of Theorem 4.1 with (4.14) implies that the same fact holds for $E_{\delta,t}(h)$. Likewise, the first inequality of (c) in Theorem 4.1 holds true for $E_{\delta,t}(h)$. □

To end this section, we give an algorithmic description of our method.

Step 1 Pick a point x_0 near ch(Ω). Construct complex spherical waves $u^0_{h,t}$ (with associated $p^0_{h,t}$).

Step 2 Draw two balls $B_t(x_0)$ and $B_{t+\delta}(x_0)$. Set the velocity field on the boundary $f_{\delta,h,t} = \phi_{\delta,t}u^0_{h,t}|_{\partial\Omega} - c_{\delta,h,t}\mathbf{n}$. Measure the Cauchy forces $\Lambda_D(f_{\delta,h,t})$.

Step 3 Calculate $E_{\delta,t}(h) = \langle(\Lambda_D - \Lambda_0)(\bar{f}_{\delta,h,t}), f_{\delta,h,t}\rangle$. If $E_{\delta,t}(h)$ tends to zero as $h \to 0$, then the probing front $\{|x - x_0| = t\}$ does not intersect the obstacle. Increase t and compute $E_{\delta,t}(h)$ again.

Step 4 If $E_{\delta,t}(h)$ increases to ∞ as $h \to 0$, then the front $\{|x - x_0| = t\}$ intersects the obstacle. Decrease t to make more accurate estimate of ∂D.

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Technische Universität Darmstadt, FB Mathematik, AG 4, Schlossgartenstr. 7, D-64289 Darmstadt, Germany

E-mail address: heck@mathematik.tu-darmstadt.de

Department of Mathematics, University of Washington, Seattle, WA 98195-4350, USA

E-mail address: gunther@math.washington.edu

Department of Mathematics, National Taiwan University, Taipei 106, Taiwan

E-mail address: jnwang@math.ntu.edu.tw