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質環上的可加性映射

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ABSTRACT

Let A be a prime ring with nonzero right ideal R and $f: R \to A$ an additive map. Next, let $k, n_1, n_2, \cdots, n_k$ be natural numbers. Suppose that $[\cdots[[f(x), x^{n_1}], x^{n_2}], \cdots, x^{n_k}] = 0$ for all $x \in R$. Then it is proved that $[f(x), x] = 0$ provided that either char $A = 0$ or char $A > n_1 + n_2 + \cdots + n_k$. This result is a simultaneous generalization of a number of results proved earlier.

Key words: prime ring, additive map, Engel condition

摘　　要

設 A 為一質環，R 為 A 中一非零右理想，$f: R \to A$ 為一可加性映射，又設 $k, n_1, n_2, \ldots, n_k$ 皆為自然數。假設對 R 中任一 x 恒有 $[\cdots[[f(x), x^{n_1}], x^{n_2}], \ldots, x^{n_k}] = 0$，則當 $\mathrm{char}\, A = 0$ 或 $\mathrm{char}\, A > n_1 + n_2 + \cdots + n_k$ 時，$[f(x), x] = 0$ 對 R 中任一 x 恒成立。這個定理同時推廣了先前已證過的許多相關結果。

關鍵詞：質環，可加性映射，Engel 條件

On Additive Maps of Prime Rings Satisfying Engel Condition

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Abstract

Let A be a prime ring with nonzero right ideal R and $f : R \to A$ an additive map. Next, let $k, n_1, n_2, \ldots, n_k$ be natural numbers. Suppose that $[\ldots[[f(x), x^{n_1}], x^{n_2}], \ldots, x^{n_k}] = 0$ for all $x \in R$. Then it is proved in Theorem 1.1 that $[f(x), x] = 0$ provided that either $\mathrm{char}(A) = 0$ or $\mathrm{char}(A) > n_1 + n_2 + \ldots + n_k$. Theorem 1.1 is a simultaneous generalization of a number of results proved earlier.

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1 Introduction

We refer the reader to the book of Beidar, Martindale and Mikhalev [3] for the basic terminology and results of Rings with Generalized Polynomial Identities Theory (i.e., Rings with GPI).

Let A be a ring, $x, y \in A$, $k > 1$ a natural number and $\overline{n} = (n_1, n_2, \ldots, n_k)$ a k-tuple of natural numbers. Setting $[x, y] = xy - yx$, we define $[x, y^{\overline{n}}]_k$ inductively as follows:

$$\begin{aligned} [x, y^{\overline{n}}]_0 &= x, \\ [x, y^{\overline{n}}]_1 &= xy^{n_1} - y^{n_1}x, \\ [x, y^{\overline{n}}]_{t+1} &= [[x, y^{\overline{n}}]_t, y^{n_{t+1}}], \quad t = 1, 2, \ldots, k-1. \end{aligned}$$

If $n_1 = n_2 = \ldots = n_k = 1$, then we shall write $[x, y]_k$ in place of $[x, y^{\overline{n}}]_k$. Note that $[x, y^n]_k = [x, y^{\overline{n}}]_k$ if $n_1 = n_2 = \ldots = n_k = n$. Next, we set $|\overline{n}| = n_1 + n_2 + \ldots + n_k$. Given $1 \le i < k$, we set $\overline{n}_{(i)} = (n_{i+1}, \ldots, n_k)$. For simplicity we shall write $[x, y^{\overline{n}\#}]_{k-i}$ in place of $[x, y^{\overline{n}(i)}]_{k-i}$. Let L be a subgroup of $(A, +)$. An additive map $f : L \to A$ is called k-commuting (k-centralizing) on L if $[f(x), x]_k = 0$ (respectively, $[f(x), x]_k \in Z(A)$ the center of A) for all $x \in L$.

¿From now on we assume that A is a prime ring with extended centroid C and Martindale right ring of quotients Q, $S = AC \subseteq Q$ is a C-subalgebra of Q generated by A, k is a natural number, and $\overline{n} = (n_1, n_2, \ldots, n_k)$ is a k-tuple of natural numbers. Recall that A is called *centrally closed* if $A = S$.

The study of commuting and centralizing maps goes back to 1957 when Posner [25] showed that the existence of a nonzero 1-centralizing derivation in a prime ring A implies that A is commutative. Mayne [23] proved the analogous result for centralizing automorphisms. A variety of results on commuting and centralizing maps have since been obtained by a number of authors (see [2], [4–20], [23–25]) Many of these isolated results were simultaneously generalized by Brešar [6] and [8] which proved that if $f : A \to A$ is an additive 2-commuting mapping and $\mathrm{char}(A) \neq 2$, then there exists $\lambda \in C$ and an additive map $\mu : A \to C$ such that $f(x) = \lambda x + \mu(x)$ for all $x \in A$. Further, on the one hand Lanski [16] studied k-commuting derivations and Brešar [10] described k-commuting additive maps, on the other hand Brešar and Hvala [11] considered a condition $[f(x), x^2] = 0$ for all $x \in A$ where f is an additive mapping (see also [4]). Also a number of papers were devoted to the study of commuting and centralizing maps $f : L \to A$ where L is either right ideal of R or some other subgroups [9], [12], [15], [18], [19], and [20] (for further references see [8] and [19]). Our goal is to prove the following results which are generalizations of many of the results mentioned above. Note that our approach is different from that in [10] and is based on the technique developed in [2].

Theorem 1.1 *Let A be a prime ring with right ideal R and additive map $f : R \to S$ such that $[f(x), x^{\overline{n}}]_k = 0$ for all $x \in R$. Suppose that either* $\mathrm{char}(A) = 0$ *or* $\mathrm{char}(A) > |\overline{n}|$. *Then $[f(x), x] = 0$ for all $x \in R$.*

Corollary 1.2 *Let A be a prime ring with nonzero right ideal R and additive map $f : R \to S$ such that $[f(x), x^{\overline{n}}]_k = 0$ for all $x \in R$. Suppose that $[R, R]R \neq 0$ and either* $\mathrm{char}(A) = 0$ *or* $\mathrm{char}(A) > |\overline{n}|$. *Then there exist $\lambda \in C$ and an additive map $\mu : R \to C$ such that $f(x) = \lambda x + \mu(x)$ for all $x \in R$.*

2 Preliminary Results

Let t be a natural number and $y \in A$, we define maps $J_{t,y} : A \to A$ and $\mathrm{ad}(y) : A \to A$ by the rules $J_{t,y}x = \sum_{i=0}^{t} y^i x y^{t-i}$ and $\mathrm{ad}(y)x = [x, y]$ for all $x \in A$. Given a natural number s one can easily show that

$$\begin{aligned} J_{t,y} J_{s,y} &= J_{s,y} J_{t,y}, \\ \mathrm{ad}(y^s) J_{t,y} &= J_{t,y} \mathrm{ad}(y^s), \end{aligned}$$

$$\begin{aligned}
\mathrm{ad}(y^s) &= J_{s-1,y}\mathrm{ad}(y),\\
\mathrm{ad}(y^{st}) &= J_{s-1,y^t}\mathrm{ad}(y^t),\\
[x, y^{\overline{n}}]_k &= \mathrm{ad}(y^{n_k})\mathrm{ad}(y^{n_{k-1}})\ldots\mathrm{ad}(y^{n_1})x \quad \text{and}\\
[x, y^{\overline{n}}]_k &= J_{n_k-1,y}J_{n_{k-1}-1,y}\cdot\ \cdot J_{n_1-1,y}\mathrm{ad}(y)^k x
\end{aligned} \tag{1}$$

for all $x, y \in A$.

In what follows we shall assume that A is a prime ring such that either $\mathrm{char}(A) = 0$ or $\mathrm{char}(A) > |\overline{n}|$, R is an additive subgroup of A and $f : R \to S$ is an additive map. We consider the following conditions:

$$\text{If } q \in Q \text{ and } [q, z^{\overline{n}}]_k = 0 \text{ for all } z \in R, \text{ then } q \in C \tag{2}$$

$$[f(x), x^{\overline{n}}]_k = 0 \tag{3}$$

$$[f(y), x^{\overline{n}}]_k + \sum_{t=0}^{k-1}[[[f(x), x^{\overline{n}}]_t, J_{n_{t+}-1,x}y], x^{\overline{n}\#}]_{k-t-1} = 0 \tag{4}$$

where $x, y \in R$. We shall assume that f satisfies (3) for all $x \in R$. If $\mathrm{char}(A) = 2$, then $|\overline{n}| = 1$, $k = 1$ and so $[f(x), x] = 0$ for all $x \in R$. Therefore we may assume without loss of generality that $\mathrm{char}(A) \neq 2$. Now our goal is to prove the following result.

Theorem 2.1 *Let A be a prime ring with a nonzero additive subgroup R, extended centroid C, central closure S and additive map $f : R \to S$. Suppose that the condition (2) holds in A and f satisfies (3) for all $x \in R$. Denote by W the C-subspace of S generated by R. Next, let $m = lcm(n_1, n_2, \ldots, n_k)$ be the least common multiple of $n_1, n_2, \ldots, n_k$, F an infinite extension of C, $D = S \otimes_C F$, and $G = W \otimes_C F$. Then D is a prime ring. Further, there exists a C-linear map $h : G \to D$ such that $f(x) - h(x) \in C$ for all $x \in R$ and*

$$[h(x), x^m]_k = 0 \tag{5}$$

$$[h(y), x^m]_k + \sum_{t=0}^{k-1}[[[h(x), x^m]_t, J_{m-1,x}y], x^m]_{k-t-1} = 0 \tag{6}$$

for all $x, y \in G$. Finally, assume that there exists a natural number s such that $x^s \in R$ for all $x \in R$ and set $J_0 = J^k_{s-1,x^m}$, $J_1 = J_{m-1,x}$, $J_2 = J_{m-1,x^s}$. Then

$$\sum_{t=0}^{k-1}\{[[[h(x^s), x^{sm}]_t, J_2 y], x^{sm}]_{k-t-1} - J_0[[[h(x), x^m]_t, J_1 y], x^m]_{k-t-1}\} = 0 \tag{7}$$

for all $x, y \in G$.

The proof of the theorem rests on the following lemmas.

Lemma 2.2 *The map f satisfies (4) for all $x, y \in R$.*

Proof. Let $\mathcal{Z}$ be the ring of integers. We set

$$t = \begin{cases} \infty & \text{if } \operatorname{char}(A) = 0, \\ \operatorname{char}(A) & \text{if } \operatorname{char}(A) > 0. \end{cases}$$

By assumption we can choose distinct elements $r_1, r_2, \ldots, r_{|\overline{n}|} \in \mathcal{Z}$ such that $0 < r_i < t$ for all $i = 1, 2, \ldots, |\overline{n}|$. Clearly $f(nx) = nf(x)$ for all $n \in \mathcal{Z}$. Now we substitute $x + r_i y$ for x in (3), $i = 1, 2, \ldots, |\overline{n}|$. Using equalities $[f(x), x^{\overline{n}}]_k = 0 = [f(y), y^{\overline{n}}]_k$ and a van der Monde determinant argument we complete the proof.

Let B be an algebra over a field F, X an infinite set, $F\langle X\rangle$ the free F-algebra on X, and $B\langle X\rangle$ the free product of F-algebras B and $F\langle X\rangle$.

Lemma 2.3 *Let K be a subring of F, $g(x_1, x_2, \ldots, x_n) \in B\langle X\rangle$ and U a submodule of the K-module B. Denote by V the F-subspace of B generated by U. Next, let $h_i : V \to B$ be an F-linear map, $i = 1, 2, \ldots, n$. Suppose that $g(h_1(u_1), \ldots, h_n(u_n)) = 0$ for all $u_1, u_2, \ldots, u_n \in U$ and $\deg_{x_i}(g) < |K|$ for all $i = 1, 2, \ldots, n$. Then $g(h_1(v_1), \ldots, h_n(v_n)) = 0$ for all $v_1, v_2, \ldots, v_n \in V$. Next, let $f : V \to B$ be an F-linear map satisfying (3) and (4) for all $x, y \in U$. Then f satisfies (3) and (4) for all $x, y \in V$.*

Proof. Choose an F-basis $\{v_s \mid s \in S\} \subseteq U$ of V and extend it to an F-basis $\{v_t \mid t \in T\}$ of B (i.e., $S \subseteq T$). Let M be any nonempty finite subset of S. It is enough to show that $g(h_1(y_1), \ldots, h_n(y_n)) = 0$ for all $y_1, y_2, \ldots, y_n \in \sum_{m \in M} F v_m$. Clearly there exist a finite subset $L \subseteq T$ and polynomials

$$P_l(z_{im} \mid 1 \le i \le n,\ m \in M) \in F[z_{im} \mid 1 \le i \le n,\ m \in M],\quad l \in L,$$

such that

$$g(h_1(y_1), \ldots, h_n(y_n)) = \sum_{l \in L} P_l(\lambda_{im} \mid 1 \le i \le n,\ m \in M) v_l$$

for all $y_i = \sum_{m \in M} \lambda_{im} v_m$, $i = 1, 2, \ldots, n$. Since $\deg_{x_i}(g) < |K|$ for all i, we conclude that $\deg_{z_{im}}(P_l) < |K|$ for all i, m as well. If all λ_{im}'s are in K, then y_i's are in U and so $g(h_1(y_1), \ldots, h_n(y_n)) = 0$ which implies that $P_l(\lambda_{im}) = 0$. Recalling that $\deg_{z_{im}}(P_l) < |K|$, we infer that $P_l(z_{im}) = 0$ for all l and hence $g(h_1(y_1), \ldots, h_n(y_n)) = 0$ for all $y_1, y_2, \ldots, y_n \in V$. The last statement follows from the first one. The proof is complete.

Proof of Theorem 2.1. It follows directly from [3, Theorem 2.3.5] that D is a prime ring. By Lemma 2.2 the map f satisfies both conditions (3) and (4). Choose a C-subspace V of S such that $S + C = V \oplus C$. Let π be the canonical projection of S onto V. Setting $g = \pi \circ f$, we note that $f(x) - g(x) \in C$ for all $x \in R$. Therefore g satisfies (3) and (4) for all $x, y \in R$.

Every element $z \in W$ is representable in the form $z = \sum_{i=1}^{n} \lambda_i z_i$ where λ_i's are in C and z_i's are in R. We set

$$h(z) = \sum_{i=1}^{n} \lambda_i g(z_i).$$

We show that h is well-defined. We have that

$$[\lambda_i g(z_i), x^{\overline{n}}]_k + \sum_{t=0}^{k-1} [[[g(x), x^{\overline{n}}]_t, J_{n_{t+1}-1,x}(\lambda_i z_i)], x^{\overline{n}\#}]_{k-t-1} = 0$$

for all $i = 1, 2, \ldots, n$, $x \in R$. Therefore

$$[h(z), x^{\overline{n}}]_k + \sum_{t=0}^{k-1} [[[g(x), x^{\overline{n}}]_t, J_{n_{t+1}-1,x} z], x^{\overline{n}\#}]_{k-t-1} =$$

$$\sum_{i=1}^{n} \left([\lambda_i g(z_i), x^{\overline{n}}]_k + \sum_{t=0}^{k-1} [[[g(x), x^{\overline{n}}]_t, J_{n_{t+1}-1,x}(\lambda_i z_i)], x^{\overline{n}\#}]_{k-t-1} \right) = 0$$

for all $z \in W$, $x \in R$. Suppose that $z = \sum_{i=1}^{n} \lambda_i z_i = 0$. Then $[h(z), x^{\overline{n}}]_k = 0$ for all $x \in R$. According to (2), $\sum_{i=1}^{n} \lambda_i g(z_i) = h(z) \in C$. On the other hand $g(z_i)$'s are in V and so $h(z) \in V$. Therefore $h(z) = 0$. Thus h is a well-defined C-linear map $W \to S$. Clearly $h(x) = g(x)$ and so $f(x) - h(x) \in C$ for all $x \in R$.

Denote by K the subring of C generated by 1. Since either $\mathrm{char}(A) = 0$ or $\mathrm{char}(A) > |\overline{n}|$, we have that $|K| > |\overline{n}|$. Clearly R is a K-submodule of A. Applying Lemma 2.3 (with $U = R$) we conclude that $h(x)$ satisfies (3) and (4) for all $x, y \in W$. Again applying Lemma 2.3 (with $U = W$) we infer that $h(x)$ satisfies (3) and (4) for all $x, y \in G$.

Write $m = l_i n_i$, $i = 1, 2, \ldots, k$. It follows from (1) that

$$[h(x), x^m]_k = J_{l_1-1,x^{n_1}} J_{l_2-1,x^{n_2}} \cdots J_{l_k-1,x^{n_k}} [h(x), x^{\overline{n}}]_k = 0$$

for all $x \in G$. Since $|F| = \infty$, substitutions of $x + \lambda y$ for x with $\lambda \in F$ yield that (6) holds for all $x, y \in G$.

Finally, substituting x^s for x in (6) and using (1), we obtain that

$$J_0([h(y), x^m]_k) + \sum_{t=0}^{k-1} [[[h(x^s), x^{sm}]_t, J_2 y], x^{sm}]_{k-t-1} = 0 \qquad (8)$$

for all $x, y \in R$. Applying J_0 to (6) and subtracting the result from (8), we see that

$$\sum_{t=0}^{k-1} \{[[[h(x^s), x^{sm}]_t, J_2 y], x^{sm}]_{k-t-1} - J_0[[[h(x), x^m]_t, J_1 y], x^m]_{k-t-1}\} = 0$$

for all $x, y \in G$, which completes the proof.

As there is no convenient reference known to us, we include a proof of the following simple lemma for the sake of completeness.

Lemma 2.4 *Let D be a ring, $x, e \in D$ and r a natural number. Suppose $e^2 = e$. Then:*

(i) $[x, e]_3 = [x, e]$;
(ii) $[x, e]_{2r+1} = [x, e]$;
(iii) *if* $[x, e]_k = 0$, *then* $[x, e] = 0$.

Proof. (i) We have

$$[x, e]_3 = xe^3 - 3exe^2 + 3e^2xe - e^3x = xe - 3exe + 3exe - ex = [x, e].$$

(ii) The second statement follows from the first one by easy induction.
(iii) If k is odd, then there is nothing to prove. If k is even, then $[x, e] = [x, e]_{k+1} = [[x, e]_k, e] = 0$.

Corollary 2.5 *Suppose that f satisfies (3) for all $x \in R$. Then $[f(e), e] = 0$ for all $e^2 = e \in R$.*

Corollary 2.6 *Let $y, e \in R$. Suppose that $e^2 = e$, $[y, e] = 0$ and f satisfies (3) for all $x \in R$. Then $[f(y), e] = 0$.*

Proof. By Lemma 2.2 the map f satisfies (4) for all $x, y \in R$. Substituting e for x in (4), we get

$$[f(y), e]_k + \sum_{t=0}^{k-1} [[[f(e), e]_t, J_{n_{t+1}-1,e}y], e]_{k-t-1} = 0.$$

Since $[y, e] = 0$, $[J_{n_{t+1}-1,e}y, e] = 0$ and so

$$[[[f(e), e]_t, J_{n_{t+1}-1,e}y], e]_{k-t-1} = [[f(e), e]_{k-1}, J_{n_{t+1}-1,e}y] = 0$$

for all t because $k - 1 \geq 1$ and $[f(e), e] = 0$ by Corollary 2.5. Therefore $[f(y), e]_k = 0$ and thus $[f(y), e] = 0$ by Lemma 2.4.

Let $n \geq 2$ be a natural number. We set

$$g_n(x, y_0, y_1, \ldots, y_{n-1}) = \sum_{\sigma \in S_{n+1}} \mathrm{sgn}(\sigma) x^{\sigma(0)} y_0 x^{\sigma(1)} y_1 \ldots y_{n-1} x^{\sigma(n)}.$$

Lemma 2.7 ([3, Corollary 2.3.8]) *Let $a \in A$. Then the following conditions are equivalent:*

(i) *a is an algebraic element over C of degree $\leq n$;*
(ii) $g_n(a, r_0, r_1, \ldots, r_{n-1}) = 0$ *for all* $r_i \in A$, $i = 0, 1, \ldots, n-1$.

Lemma 2.8 (**[3, Corollary 6.5.16]**) *Let A be a primitive algebra with identity over an infinite field C having a nonzero idempotent w such that wAw is a finite dimensional division C-algebra with center wC. Further let $f, h \in A_C\langle X\rangle$ and let $x_k \in X$ be a variable which is not involved in f and h. Suppose that fx_kh is a GPI on A. Then either f or h is a GPI on A.*

Lemma 2.9 *Let A be a centrally closed prime ring with infinite extended centroid C. Suppose that A is not algebraic of bounded degree $\leq n$ over C. Choose $\lambda \in C \setminus \{0, 1\}$. Then for every $x \in A$ there exists $y \in A$ such that neither y, nor $x + y$, nor $x + \lambda y$ is algebraic of degree $\leq n$ over C.*

Proof. If not, then

$$h = g_n(y, y_0, \ldots, y_{n-1})ug_n(x + y, y_0, \ldots, y_{n-1})vg_n(x + \lambda y, y_0, \ldots, y_{n-1})$$

is a generalized polynomial identity in $y, y_0, \ldots y_{n-1}, u, v$ on A by Lemma 2.7. According to [3, Theorem 6.4.4] every GPI on A is a GPI on Q. Therefore h is a GPI on Q. By [22, Theorem 3], Q is a primitive ring with nonzero idempotent e such that eQe is a division algebra finite dimensional over its center eC (see also [3, Theorem 6.1.6]). Since $|C| = \infty$, we conclude that either $g_n(y, y_0, \ldots, y_{n-1})$ or $g_n(x + y, y_0, \ldots, y_{n-1})$ or $g_n(x + \lambda y, y_0, \ldots, y_{n-1})$ is a GPI on Q (see Lemma 2.8). In all cases A is algebraic of bounded degree $\leq n$ over C by Lemma 2.7, a contradiction.

The following result is a corollary to the proof of Lemma 2.9 which will be used in the forthcoming paper.

Corollary 2.10 *Let A be a prime ring with additive subgroup R and let n be a natural number. Suppose that R does not satisfies GPI. Choose $\lambda \in C \setminus \{0, 1\}$. Then for every $x \in R$ there exists $y \in R$ such that neither y, nor $x + y$, nor $x + \lambda y$ is algebraic of degree $\leq n$ over C.*

Given a subset $S \subseteq A$ we set

$$l(A; S) = \{a \in A \mid aS = 0\}.$$

Lemma 2.11 *Let A be a centrally closed prime ring. Next, let R be a nonzero right ideal of the C-algebra A. Then $R/l(R; R)$ is a centrally closed prime C-algebra.*

Proof. Clearly $l(R; R)$ is an ideal of R and $\overline{R} = R/l(R; R)$ is a prime C-algebra. We shall write $\overline{a}$ for $a + l(R; R) \in \overline{R}$, $a \in R$. Denote by F the extended centroid of $\overline{R}$. Obviously $F \supseteq C$. Choose any $\alpha \in F$. Next, choose $0 \neq \overline{a} \in \overline{R}$ such that $\overline{b} = \alpha\overline{a} \in \overline{R}$. Finally, choose $\overline{r} \in \overline{R}$ such that $\overline{a}\overline{r} \neq 0$. Clearly $\overline{a}x\overline{b} = \overline{b}x\overline{a}$ for $x \in \overline{R}$. Therefore $axb - bxa \in l(R; R)$ for all $x \in R$ and so $arybr - bryar = 0$ for all $y \in A$. It follows from [22, Theorem 1] that $br = \lambda ar$ for some $\lambda \in C$ (see also [3, Theorem 2.3.4]). Therefore $\alpha\overline{a}\overline{r} = \lambda\overline{a}\overline{r}$ and so $\alpha = \lambda$. Thus $F = C$ and the proof is complete.

Corollary 2.12 *Let A be a centrally closed prime ring with infinite extended centroid C and R a nonzero right ideal of the C-algebra A. Suppose that R is not algebraic of bounded degree $\leq n$ over C. Choose $\lambda \in C \setminus \{0, 1\}$. Then for every $x \in R$ there exists $y \in R$ such that neither y, nor $x + y$, nor $x + \lambda y$ is algebraic of degree $n - 1$ over C.*

Proof. Set $\overline{R} = R/l(R; R)$. If $\sum_{i=0}^{n-1} \lambda_i \overline{r}^i = 0$ for some $\overline{r} \in \overline{R}$ and λ_i's in C, then $\sum_{i=0}^{n-1} \lambda_i r^{i+1} = 0$. Therefore $\overline{R}$ is not algebraic of bounded degree $\leq n - 1$ over C. Applying Lemma 2.11 and Lemma 2.9, we complete the proof.

Lemma 2.13 *Let A be a prime ring with nonzero right ideal R and $q \in Q$. Suppose that $[q, x^{\overline{n}}]_k = 0$ for all $x \in R$. Then $q \in C$.*

Proof. Let X be an infinite set and $Q\langle X\rangle$ the free product of C-algebras Q and $C\langle X\rangle$. Suppose that $q \notin C$. Fix any nonzero $r \in R$ and choose $y \in X$. Clearly

$$[q, (ry)^{\overline{n}}]_k = p(q, ry)ry + (-1)^k (ry)^{|\overline{n}|} q$$

for some polynomial $p(u, v)$ in noncommuting variables u, v with integral coefficients. Since $q \notin C$, $[q, (ry)^{\overline{n}}]_k$ is a nonzero element of $Q\langle X\rangle$. Therefore $[q, (ry)^{\overline{n}}]_k$ is a nonzero generalized polynomial identity in y on A. According to [1, Theorem 2] it is also a generalized polynomial identity in y on Q (see also [3, Theorem 6.4.4]). By [3, Corollary 6.1.7], Q is a primitive ring with nonzero socle $\mathrm{Soc}(Q)$ and nonzero idempotent e such that eQe is a division algebra finite dimensional over its center eC (see also [3, Theorem 6.1.6]). If $|C| < \infty$, then $eQe = eC$ by Wedderburn's theorem on finite division rings [3, Theorem 4.2.3]. Suppose now that $|C| = \infty$. Let $\overline{C}$ be the algebraic closure of C. Then $\overline{Q} = Q \otimes_C \overline{C}$ is a prime algebra by [3, Theorem 2.3.5]. Clearly $\overline{Q}$ contains an idempotent $\overline{e}$ such that $\overline{e}\overline{Q}\overline{e} = \overline{e}\overline{C}$. Since $|C| = \infty$, $[q, (ry)^{\overline{n}}]_k$ is a generalized polynomial identity on $\overline{Q}$ by Lemma 2.3. Thus, without loss of generality we can assume that $eQe = eC$ and $[q, x^{\overline{n}}]_k = 0$ for all $x \in rQ$. As Q is a prime ring, $rQ \cap \mathrm{Soc}(Q) \neq 0$. Choose any idempotent $e' \in rQ \cap \mathrm{Soc}(Q)$ such that $e'Qe' = e'C$. Since $e'Q \subseteq rQ \cap \mathrm{Soc}(Q)$, we conclude that $[q, x^{\overline{n}}]_k = 0$ for all $x \in e'Q$. Let $w \in e'Q$ be any idempotent. Then $[q, w] = 0$ by Lemma 2.4. Since $e'Qe' = e'C$, $e'Q$ is a C-linear span of idempotents $w \in e'Q$. Therefore $[q, x] = 0$ for all $x \in e'Q$. It follows that $qe'y = e'yq$ for all $y \in Q$. Since the centralizer of a nonzero right ideal of a prime ring is just the center, we conclude that $q \in C$, a contradiction.

Let A be a prime ring with a nonzero right ideal R and $f : R \to S$ an additive map satisfying (3). It follows from Lemma 2.13 that (2) holds in A. By Theorem 2.1 we may assume without loss of generality that A is a centrally closed prime ring with infinite centroid, R is a nonzero right ideal of C-algebra A, f is a C-linear map satisfying (5), (6), and (7) for all $x, y \in R$.

✗ $e'Q = e'Qe' + e'Q(1-e') = e'Qe' + \sum_i e'Qe'a_i(1-e') = e'Qe' + \sum_i e'Qe'[e' + e'a_i(1-e')]$

3 The Case of Matrix Algebras

In this section F is a field, r a natural number, $M_r(F)$ the F-algebra of $r \times r$ matrices over F, and $f : M_r(F) \to M_r(F)$ is an F-linear map satisfying (3) and (4) for all $x, y \in M_r(F)$. We fix a set $\{e_{ij} \mid 1 \leq i, j \leq r\}$ of matrix units of $M_r(F)$ and identify F with Fe where $e = e_{11} + e_{22} + \ldots + e_{rr}$.

Lemma 3.1 *For each $i = 1, 2, \ldots, r$ there exist $\lambda_{ii}, \mu_{ii} \in F$ such that $f(e_{ii}) = \lambda_{ii} e_{ii} + \mu_{ii}$.*

Proof. By Corollary 2.6, $[f(e_{ii}), e_{jj}] = 0$ for all j. Hence $f(e_{ii})$ is a diagonal matrix. Write $f(e_{ii}) = \sum_{t=1}^{r} \alpha_t e_{tt}$ where α_t's are in F. For $j, s \neq i$, $j \neq s$, we have $[f(e_{ii}), e_{jj} + e_{js}] = 0$ since $e_{jj} + e_{js}$ is an idempotent commuting with e_{ii} (see Corollary 2.6). Therefore $[f(e_{ii}), e_{js}] = 0$ which yields $\alpha_j = \alpha_s$. Setting $\mu_{ii} = \alpha_j$ and $\lambda_{ii} = \alpha_i - \mu_{ii}$ we complete the proof.

Lemma 3.2 *If $1 \leq i \neq j \leq r$, then $[f(e_{ij}), e_{ij}] = 0$, $[f(e_{ii}), e_{ij}] + [f(e_{ij}), e_{ii}] = 0$ and $[f(e_{jj}), e_{ij}] + [f(e_{ij}), e_{jj}] = 0$.*

Proof. Since $e_{ii} + e_{ij}$ is an idempotent, we have that $[f(e_{ii} + e_{ij}), e_{ii} + e_{ij}] = 0$ by Corollary 2.5. Expanding and using $[f(e_{ii}), e_{ii}] = 0$, we get

$$[f(e_{ii}), e_{ij}] + [f(e_{ij}), e_{ii}] + [f(e_{ij}), e_{ij}] = 0.$$

Similarly, it follows from $[f(e_{ii} - e_{ij}), e_{ii} - e_{ij}] = 0$ that

$$[f(e_{ii}), e_{ij}] + [f(e_{ij}), e_{ii}] - [f(e_{ij}), e_{ij}] = 0.$$

Therefore, $[f(e_{ij}), e_{ij}] = 0$ and $[f(e_{ii}), e_{ij}] + [f(e_{ij}), e_{ii}] = 0$. The last identity is proved analogously.

Lemma 3.3 *Let $1 \leq i \neq j \leq r$. Then there exist $\lambda_{ij}, \mu_{ij} \in F$ such that $f(e_{ij}) = \lambda_{ij} e_{ij} + \mu_{ij}$.*

Proof. Write $f(e_{ij}) = \sum_{t,s} \alpha_{ts} e_{ts}$ where α_{ts}'s are in F. By Corollary 2.6, $[f(e_{ij}), e_{pp}] = 0$ for $p \neq i, j$. Hence $\alpha_{sp} = 0 = \alpha_{ps}$ for $s \neq p$. That is, $f(e_{ij}) = \sum_s \alpha_{ss} e_{ss} + \alpha_{ij} e_{ij} + \alpha_{ji} e_{ji}$. According to Lemma 3.2, $[f(e_{ij}), e_{ij}] = 0$ and so $\alpha_{ii} = \alpha_{jj}$ and $\alpha_{ji} = 0$. As $[f(e_{ij}), e_{is} + e_{ss}] = 0$ for $s \neq i, j$ by Corollary 2.6, $[f(e_{ij}), e_{is}] = 0$ and hence $\alpha_{ss} = \alpha_{ii}$. Thus $f(e_{ij}) = \lambda_{ij} e_{ij} + \mu_{ij}$ where $\lambda_{ij} = \alpha_{ij}$ and μ_{ij} is the common value for α_{ss}.

Lemma 3.4 *There exists $\lambda \in F$ such that $f(e_{ij}) - \lambda e_{ij} \in F$ for all $i, j = 1, 2, \ldots, r$.*

Proof. In view of Lemmas 3.1 and 3.3, we write $f(e_{ij}) = \lambda_{ij} e_{ij} + \mu_{ij}$ for all i, j where $\lambda_{ij}, \mu_{ij} \in F$. By Lemma 3.2, $[f(e_{ii}), e_{ij}] + [f(e_{ij}), e_{ii}] = 0$ which yields $\lambda_{ij} = \lambda_{ii}$ for all $i \neq j$. Similarly, $\lambda_{ij} = \lambda_{jj}$ follows from $[f(e_{jj}), e_{ij}] +$

$[f(e_{ij}), e_{jj}] = 0$. Let λ be the common value of the λ_{ij}'s. Then $f(e_{ij}) - \lambda e_{ij}$ is a scalar for all i, j.

Since f is an F-linear map and $\{e_{ij} \mid 1 \leq i\ j \leq r\}$ forms a basis of $M_r(F)$, Lemma 3.4 implies the following main result of this section.

Proposition 3.5 *Let F be a field, r a natural number and $f : M_r(F) \to M_r(F)$ an F-linear map satisfying (3) and (4) for all $x, y \in M_r(F)$. Then there exist $\lambda \in F$ and an F-linear map $\mu : M_r(F) \to F$ such that $f(x) = \lambda x + \mu(x)$ for all $x \in M_r(F)$. In particular, $[f(x), x] = 0$ for all $x \in M_r(F)$.*

4 Proof of Theorem 1.1

Lemma 4.1 *Suppose that $r \in R$ is not algebraic of degree $\leq 2km - 1$. Then $[f(r), r] = 0$.*

Proof. Since $x^2 \in R$ for all $x \in R$, we infer from (7) that

$$\sum_{t=0}^{k-1}\{[[[f(x^2), x^{2m}]_t, J_2 y], x^{2m}]_{k-t-1} - J_0[[[f(x), x^m]_t, J_1 y], x^m]_{k-t-1}\} = 0 \quad (9)$$

for all $x, y \in R$. Clearly $J_i(xz) = xJ_iz$ for all $i = 0, 1, 2$, $x \in R$, $z \in A$ and

$$\begin{aligned}
[[[f(x^2), x^{2m}]_t, J_2(xz)], x^{2m}]_{k-t-1} &= [[[f(x^2), x^{2m}]_t, x]J_2 z, x^{2m}]_{k-t-1} \\
&\quad + x[[[f(x^2), x^{2m}]_t, J_2 z], x^{2m}]_{k-t-1}, \\
[[[f(x), x^m]_t, J_1(xz)], x^m]_{k-t-1} &= [[[f(x), x^m]_t, x]J_1 z, x^m]_{k-t-1} \\
&\quad + x[[[f(x), x^m]_t, J_1 z], x^m]_{k-t-1}
\end{aligned}$$

Substituting xz for y in (9), we obtain that

$$\sum_{t=0}^{k-1}\{[[[f(x^2), x^{2m}]_t, x]J_2 z, x^{2m}]_{k-t-1} - J_0[[[f(x), x^m]_t, x]J_1 z, x^m]_{k-t-1}\} = 0 \quad (10)$$

for all $x \in R$ and $z \in A$. By Leibnitz formula we have that

$$\begin{aligned}
&[[[f(x^2), x^{2m}]_t, x]J_2 z, x^{2m}]_{k-t-1} = \\
&\quad \sum_{i=0}^{k-t-1} \binom{k-t-1}{i} [[[f(x^2), x^{2m}]_t, x], x^{2m}]_i [J_2 z, x^{2m}]_{k-t-i-1}, \\
&[[[f(x), x^m]_t, x]J_1 z, x^m]_{k-t-1} = \\
&\quad \sum_{i=0}^{k-t-1} \binom{k-t-1}{i} [[[f(x), x^m]_t, x], x^m]_i [J_1 z, x^m]_{k-t-i-1}
\end{aligned}$$

Since

$$\operatorname{ad}(x)\operatorname{ad}(x^{2m})^t = \operatorname{ad}(x^{2m})^t\operatorname{ad}(x) \quad \text{and} \quad \operatorname{ad}(x)\operatorname{ad}(x^m)^t = \operatorname{ad}(x^m)^t\operatorname{ad}(x)$$

for all $t \geq 0$, we can rewrite (10) as follows

$$\begin{aligned} & [f(x^2), x][J_2 z, x^{2m}]_{k-1} - J_0([f(x), x][J_1 z, x^m]_{k-1}) \\ + & \sum_{t=1}^{k-1} N_t[[f(x^2), x], x^{2m}]_t[J_2 z, x^{2m}]_{k-1-t} \\ - & J_0\left(\sum_{t=1}^{k-1} N_t[[f(x), x], x^m]_t[J_1 z, x^m]_{k-1-t}\right) \\ = & \ 0 \end{aligned} \tag{11}$$

for some natural numbers N_t's. Writing (11) as $\sum_{i=0}^{2km-1} p_i(x)zx^i = 0$, we see that $p_{2km-1}(x) = -[f(x), x]$. Substituting r for x, we get $\sum_{i=0}^{2km-1} p_i(r)zr^i = 0$ for all $z \in A$. Since r is not algebraic of degree $\leq 2km - 1$ over C, we conclude that $1, r, \ldots, r^{2km-1}$ are linearly independent over C and so $[f(r), r] = 0$ by [22, Theorem 2].

Lemma 4.2 *Suppose that R is not algebraic algebra of degree $\leq 2km$ over C. Then $[f(x), x] = 0$ for all $x \in R$.*

Proof. Let $x \in R$ and $\lambda \in C \setminus \{0, 1\}$. By Corollary 2.12 there exists $y \in R$ such that neither y nor $x + y$ nor $x + \lambda y$ is an algebraic element of degree $\leq 2km - 1$ over C. Then by Lemma 4.1

$$[f(y), y] = [f(x + y), x + y] = [f(x + \lambda y), x + \lambda y] = 0.$$

Therefore

$$\begin{aligned} 0 &= [f(x + y), x + y] = [f(x), x] + [f(x), y] + [f(y), x], \\ 0 &= [f(x + \lambda y), x + \lambda y] = [f(x), x] + \lambda[f(x), y] + \lambda[f(y), x] \end{aligned}$$

and so $[f(x), x] = 0$ for all $x \in R$.

Proof of Theorem 1.1. If R is not algebraic of degree $\leq 2km$ over C, then $[f(x), x] = 0$ for all $x \in R$ by Lemma 4.2. Suppose now that R is algebraic of degree $\leq 2km$ over C. Then, R is a *PI*-ring. According to [3, Theorem 6.3.20], A is GPI. Since A is centrally closed, A contains a nonzero idempotent p such that pAp is a finite dimensional central division algebra over C (see [3, Theorem 6.1.6]). It follows from [21, Theorem 1] that $R = e\mathrm{Soc}(A)$ for some idempotent $e \in \mathrm{Soc}(A)$. As $e \in \mathrm{Soc}(A)$, $R = eA$. We obtain from Litoff's Theorem [3, Theorem 4.3.11] that eAe is a finite dimensional central simple C-algebra. Tensoring by the algebraic closure F of C, we reduce the problem to the case when eAe is a matrix ring over C and $|C| = \infty$. Let now $0 \neq w^2 = w \in eA$. Then wAw is a matrix ring over C. Given any $x \in wAw$, write

$$f(x) = wf(x)w + wf(x)(1-w) + (1-w)f(x)w + (1-w)f(x)(1-w).$$

As $[f(x), w] = 0$ by Corollary 2.6, we conclude that

$$f(x) = wf(x)w + (1-w)f(x)(1-w) \text{ for all } x \in wAw \tag{12}$$

Applying Proposition 3.5 to $g : wAw \to wAw$ $g(x) = wf(x)w$, $x \in wAw$, we infer that $[wf(x)w, x] = 0$ for all $x \in wAw$. It follows from (12) that

$$[f(x), x] = 0 \text{ for all } x \in wAw \tag{13}$$

Let now $y \in eA$. Clearly $yA = vA$ for some $v^2 = v \in eA$. Note that $vy = y$. Since $|C| = \infty$, there exists an infinite subset $T \subseteq C$ such that $\tau v + yv$ is an invertible element of the matrix ring vAv for all $\tau \in T$. Fix any $\tau \in T$ and set $z = \tau v + y \in vA$. Then there exists $x \in vAv$ such that $zvx = v$. As $zx = zvx$, we conclude that $zx = v$. Consider $w = v + xz(1-v) \in vA$. As $vx = x$, $w^2 = w$. Next, $wz = z$ and

$$zw = zv + (zx)z(1-v) = zv + vz(1-v) = zv + z(1-v) = z$$

and so $z \in wAw$. According to (13), $[f(z), z] = 0$. That is to say $[f(\tau v + y), \tau v + y] = 0$ for all $\tau \in T$. As $|T| = \infty$, $[f(y), y] = 0$ for all $y \in eA$ and the proof is completed.

Finally we note that Corollary 1.2 follows from Theorem 1.1 and [9, Theorem 5.2] (see also [20, Theorem 1]).

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