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is a generalized (σ, τ)–derivation.

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$\Delta(x,y)[\sigma(u),\sigma(v)]$ *for all* $x,y,u,v\in S$.

Proof. Since Δ is a (σ,σ)–derivation in each argument, we can prove the lemma by computing $\Delta(xu,yv)$ for $x,y,u,v\in S$ in two different ways as the argument given in the proof of [5, Lemma 3.1].

Theorem 1. *Let* R *be a semiprime ring,* I *an ideal of* R *and* σ *a ring endomorphism of* R *such that* $\sigma(I)$ *contains a large right ideal of* R. *Then every* σ*-commuting map* $f\colon I\to Q$, *which is additive modulo* C, *must be of the form* $f(x)=\lambda\sigma(x)+\zeta(x)$ *for all* $x\in I$, *where* $\lambda\in C$ *and* $\zeta\colon I\to C$.

Proof. Since $f\colon I\to Q$ is additive modulo C, linearizing $[f(x),\sigma(x)]=0$ we see that $[f(x),\sigma(y)]=-[f(y),\sigma(x)]$ for all $x,y\in I$. Then $\Delta\colon I\times I\to Q$, defined by $\Delta(x,y)=[f(x),\sigma(y)]$ for all $x,y\in I$, is a (σ,σ)–biderivation. In view of Lemma 1, we have $[\sigma(x),\sigma(y)]\Delta(u,v)=\Delta(x,y)[\sigma(u),\sigma(v)]$ for all $x,y,u,v\in I$. That is,

$$[\sigma(x),\sigma(y)][f(u),\sigma(v)]=[f(x),\sigma(y)][\sigma(u),\sigma(v)] \tag{1}$$

for all $x,y,u,v\in I$. By assumption, let ρ be a large right ideal of R such that $\rho\subseteq\sigma(I)$. Thus, by (1), we see that

$$[\sigma(x),y][f(u),v]=[f(x),y][\sigma(u),v] \tag{2}$$

for all $x,u\in I$ and all $y,v\in\rho$. Replacing y with yz in (2), where $y,z\in\rho$, we have

$$[\sigma(x),y]z[f(u),v]=[f(x),y]z[\sigma(u),v] \tag{3}$$

for all $x,u\in I$ and all $y,z\in\rho$. In view of [17, Theorem 1], ρ and Q satisfy the same GPIs with coefficients in Q. This implies that (3) holds for all $x,u\in I$ and all $y,z\in Q$. Note that C is the extended centroid of Q. In view of [3, Theorem 3.1], there exist

As an immediate consequence of Theorem 2 we have the following

Corollary 1. *Let R be a semiprime ring with σ an epimorphism of R. Then every σ-commuting map $f: R \to U$, which is additive modulo C, must be of the form $f(x) = \lambda\sigma(x) + \zeta(x)$ for all $x \in R$, where $\lambda \in C$ and $\zeta: R \to C$.*

Let σ and τ be endomorphisms of the ring R. A map $d: R \to R$ is called a *(σ,τ)–derivation* if $d(x+y) = d(x) + d(y)$ and $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ for all $x, y \in R$. The following result gives the semiprime case of Posner's theorem [25] for (σ,τ)–derivations.

Theorem 3. *Let R be a semiprime ring with d a (σ,τ)–derivation of R. Suppose that I is a nonzero ideal of R such that both $\sigma(I)$ and $\tau(I)$ contain some large right ideals of R. If $[d(x),\sigma(x)] = 0$ for all $x \in I$, then $d(R) \subseteq Z(R)$ and $[R,R]d(R) = 0$. In addition, if $d \neq 0$, then R contains a nonzero central ideal.*

Proof. In view of Theorem 1, there exist $\lambda \in C$ and a map $\zeta: I \to C$ such that $d(x) = \lambda\sigma(x) + \zeta(x)$ for all $x \in I$. Let $x, y \in I$. Then

(6) $$d(xy) = d(x)\sigma(y) + \tau(x)d(y) = \lambda\sigma(xy) + \zeta(x)\sigma(y) + \lambda\tau(x)\sigma(y) + \zeta(y)\tau(x).$$

On the other hand, $d(xy) = \lambda\sigma(xy) + \zeta(xy)$. Comparing this with (6) we see that

(7) $$(\lambda\tau(x) + \zeta(x))\sigma(y) + \zeta(y)\tau(x) \in C.$$

Commuting (7) with $\tau(x)$ gives

(8) $$(\lambda\tau(x) + \zeta(x))[\sigma(y), \tau(x)] = 0$$

for all $x, y \in I$. Let ρ be a large right ideal of R contained in $\sigma(I)$. Therefore, by (8), we have

(9) $$(\lambda\tau(x) + \zeta(x))[y, \tau(x)] = 0$$

idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$ and an invertible element $\mu \in C$ such that $\varepsilon_i\varepsilon_j = 0$ for $i \neq j$, $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$, and

$$\varepsilon_1[f(x), y] = \mu\varepsilon_1[\sigma(x), y], \quad [\varepsilon_2 f(x), y] = 0, \quad [\varepsilon_3\sigma(x), y] = 0 \tag{4}$$

for all $x \in I$ and all $y \in Q$. This implies that

$$\varepsilon_1 f(x) - \mu\varepsilon_1\sigma(x) \in C, \quad \varepsilon_2 f(x) \in C, \quad \varepsilon_3\sigma(x) \in C \tag{5}$$

for all $x \in I$. Since, by (5), $\varepsilon_3\sigma(I) \subseteq C$ and $\rho \subseteq \sigma(I)$, we have $\varepsilon_3 Q \subseteq C$ [17, Theorem 1]. In particular, $\varepsilon_3 f(x) \in C$ for all $x \in I$. This fact together with (5) implies that $f(x) - \mu\varepsilon_1\sigma(x) \in C$ for all $x \in I$. Set $\lambda = \mu\varepsilon_1 \in C$ and $\zeta(x) = f(x) - \lambda\sigma(x) \in C$ for $x \in I$. The theorem is thus proved.

We note that R and each large ideal of R satisfy the same GPIs with coefficients in U (see, for instance, [17, Main Theorem] or [16, Theorem 3]). Applying this fact together with the same argument given in Theorem 1, we have the following result.

Theorem 2. *Let R be a semiprime ring, I an ideal of R and σ an endomorphism of the ring R such that $\sigma(I)$ contains a large ideal of R. Then every σ-commuting map $f: I \to U$, which is additive modulo C, must be of the form $f(x) = \lambda\sigma(x) + \zeta(x)$ for all $x \in I$, where $\lambda \in C$ and $\zeta: I \to C$.*

Remark. In Theorems 1 and 2, the conclusions do not remain true if $\sigma(I)$ does not contain some large right ideal of R. For instance let $R = \mathrm{M}_n(C) \oplus \mathrm{M}_n(C)$, where $n >$ and $\mathrm{M}_n(C)$ is the $n \times n$ matrix ring over a field C. Let $I = \{(x, 0) \mid x \in \mathrm{M}_n(C)\}$ and σ be an automorphism of R defined by $\sigma((x, y)) = (y, x)$ for $(x, y) \in R$. Let $f: I \to R$ be defined by $f((x, 0)) = (0, x)$ for $x \in \mathrm{M}_n(C)$. Then it is clear that f is an additive σ-commuting map but is not of the form concluded in Theorem 1 or Theorem 2. Of course, in this case $\sigma(I)$ contains no large right ideals.

for all $x \in I$ and all $y \in \rho$. In view of [17, Theorem 1], (9) holds for all $x \in I$ and all $y \in Q$. By the semiprimeness of Q, a direct computation proves that $\lambda\tau(x)+\zeta(x) \in C$ and hence $\lambda\tau(x) \in C$ for all $x \in I$. Thus $\lambda\rho_0 \subseteq C$, where ρ_0 is a large right ideal of R contained in $\tau(I)$. In view of [17, Theorem 1] again, $\lambda Q \subseteq C$ follows. For $x \in I$ we have $d(x) = \lambda\sigma(x) + \zeta(x) \in C$. This proves that $d(I) \subseteq Z(R)$.

Let $x \in I, y \in R$; then $xy \in I$. Thus $d(xy) = d(x)\sigma(y) + \tau(x)d(y) \in Z(R)$, implying that $[\sigma(y), \tau(I)d(y)] = 0$. Since $\tau(I)$ contains a large right ideal of R, it follows from [17, Theorem 1] that $[\sigma(y), Qd(y)] = 0$. In particular, $[\sigma(y), d(y)] = 0$ for all $y \in R$. Applying the case above with $I = R$, we see that $d(R) \subseteq Z(R)$.

From the last paragraph, $[\sigma(y), R]d(y) = 0$ for all $y \in R$. Let P be a prime ideal of R. Passing to the prime ring R/P, we have that either $[\sigma(y), R] \subseteq P$ or $d(y) \in P$ for each $y \in R$. Thus R is the union of its two additive subgroups: $\{y \in R \mid [\sigma(y), R] \subseteq P\}$ and $\{y \in R \mid d(y) \in P\}$. Thus either $[\sigma(R), R] \subseteq P$ or $d(R) \subseteq P$, implying that $[\sigma(R), R]d(R) \subseteq P$. The semiprimeness of R implies that $[\sigma(R), R]d(R) = 0$. Since $\sigma(R)$ contains a large right ideal of R, in view of [17, Theorem 1] we have $[R, R]d(R) = 0$, as desired. We remark that $\ell_R([R, R]) = r_R([R, R])$ and that an ideal J of R is central if and only if J is contained in $\ell_R([R, R])$. Hence, $\ell_R([R, R])$ is the unique largest central ideal of R. Thus, if $d \neq 0$, then $0 \neq d(R) \subseteq \ell_R([R, R]) \subseteq Z(R)$. So $\ell_R([R, R])$ is the largest nonzero central ideal of R. This proves the theorem.

As a special case of Theorem 3, we have the following

Corollary 2. *Let R be a semiprime ring with d a (σ, τ)-derivation of R, where σ and τ are epimorphisms of R. Suppose that $[d(x), \sigma(x)] = 0$ for all $x \in R$, then $d(R) \subseteq Z(R)$ and $[R, R]d(R) = 0$.*

$\lambda \in C$ *and* $\zeta : I \to C$ *unless* char $R = 2$ *and* $\dim_C RC = 4$.

Proof. Suppose that either char $R \neq 2$ or $\dim_C RC > 4$. The key step to the proof is implicit in the proof of [5, Lemma 6.3]. Let $x, a \in L$. Note that $[f(x), \sigma(a)] + [f(a), \sigma(x)] \in C$. Repeating an analogous argument as that of [5, Lemma 6.3] or [19, Theorem 4] to compute $\big[f([x,a]), \sigma([x,a])\big] \in C$, we have

$$\begin{aligned}
C \ni &\big[f([x,a]), \sigma([x,a])\big] \\
&= \big[[f([x,a]), \sigma(x)], \sigma(a)\big] + \big[\sigma(x), [f([x,a]), \sigma(a)]\big] \\
&= \big[[\sigma([x,a]), f(x)], \sigma(a)\big] + \big[\sigma(x), [\sigma([x,a]), f(a)]\big] \\
&= \big[[\sigma(x), [\sigma(a), f(x)]], \sigma(a)\big] + \big[\sigma(x), [\sigma([x,a]), f(a)]\big] \\
&= \big[[\sigma(x), \sigma(a)], [f(a), \sigma(x)]\big] + \big[\sigma(x), [[f(a), \sigma(x)], \sigma(a)]\big] \\
&\quad + \big[\sigma(x), [[\sigma(x), \sigma(a)], f(a)]\big] \\
&= \big[[\sigma(x), \sigma(a)], [f(a), \sigma(x)]\big] + \big[\sigma(x), [[f(a), \sigma(x)], \sigma(a)]\big] \\
&\quad + \big[\sigma(x), [[\sigma(x), f(a)], \sigma(a)]\big] \\
&= \big[[\sigma(x), \sigma(a)], [f(a), \sigma(x)]\big]
\end{aligned} \tag{10}$$

for all $a, x \in L$. Since $\sigma(L)$ is a noncentral Lie ideal of R, applying [13, Theorem 4] to (10) we see that $1, \sigma(a)$ and $f(a)$ are C-dependent for each $a \in L$. Thus either $\sigma(a) \in Z(R)$ or $f(a) - \lambda_a \sigma(a) \in Z(R)$ for each $a \in L$.

We can replace u, v, L with $\sigma(u), \sigma(v), \sigma(L)$, respectively, in the second paragraph of [19, p.265] to proceed the proof. We omit its details.

For $L = R$ in Theorem 4 we have the following:

Corollary 5. *Let R be a prime ring with σ an epimorphism of R. Then every σ-centralizing map $f: R \to U$, which is additive modulo C, must be of the form $f(x) = \lambda\sigma(x) + \zeta(x)$ for all $x \in R$, where $\lambda \in C$ and $\zeta: R \to C$, unless* char $R = 2$ *and*

For the prime case we have the following two results.

Corollary 3. *Let R be a prime ring and I an ideal of R. Suppose that σ is an epimorphism of R such that $\sigma(I) \neq 0$. Then every σ-commuting map $f: I \to U$, which is additive modulo C, must be of the form $f(x) = \lambda\sigma(x) + \zeta(x)$ for all $x \in I$, where $\lambda \in C$ and $\zeta: I \to C$.*

Proof. Since σ is an epimorphism of R, we have $0 \neq R\sigma(I)R = \sigma(RIR) \subseteq \sigma(I)$. Therefore, $\sigma(I)$ contains a nonzero ideal of R. Note that, by the primeness of R, each nonzero ideal of R is large. In view of Theorem 2, the corollary is proved.

The following result gives a slight generalization of [10, Lemma 2(3)].

Corollary 4. *Let R be a prime ring and I a nonzero ideal of R. Let $d: R \to R$ be a nonzero (σ, τ)-derivation of R such that both $\sigma(I)$ and $\tau(I)$ contain some large right ideals of R. If $[d(x), \sigma(x)] = 0$ for all $x \in I$, then R is commutative.*

Proof. In view of Theorem 3, we have $[R, R]d(R) = 0$. Since $d \neq 0$, the primeness of R implies that $[R, R] = 0$. Thus R is commutative, proving the corollary.

§3. σ–Centralizing Maps

In this section R always denotes a prime ring. A Lie ideal L of R is called *noncommutative* if $[L, L] \neq 0$ and is called *noncentral* if $L \not\subseteq Z(R)$. By [14, Theorem 4], L is noncentral if and only if it is noncommutative unless char $R = 2$ and $\dim_C RC = 4$. The following theorem gives a generalization of [19, Theorem 4].

Theorem 4. *Let R be a prime ring, L a noncentral Lie ideal of R and σ an epimorphism of R with $\sigma(L) \not\subseteq Z(R)$. Then every σ-centralizing map $f: L \to U$, which is additive modulo C, must be of the form $f(x) = \lambda\sigma(x) + \zeta(x)$ for all $x \in I$, where*

$\dim_C RC = 4$.

The following result is a generalization of [19, Theorem 5]. Since its proof is analogous to that of [19, Theorem 5] by replacing [19, Theorem 4] with Theorem 4, we only give its statement.

Theorem 5. *Let R be a prime ring with an epimorphism σ and L be a noncentral Lie ideal of R such that $\sigma(L) \not\subseteq Z(R)$. If T is an endomorphism of R which is σ-centralizing on L, then either $T(L) \subseteq Z(R)$ or $T = \sigma$ unless* $\operatorname{char} R = 2$ *and* $\dim_C RC = 4$.

§4. Posner's Theorem for (σ, τ)–Derivations on Lie Ideals

We end this paper by generalizing [1, Theorem 2] to the Lie ideal case. The following result gives Posner's theorem for (σ, τ)–derivations on Lie ideals. See §5. for generalized (σ, τ)–derivations.

Theorem 6. *Let R be a prime ring, L a noncentral Lie ideal of R and d a (σ, τ)-derivation. Suppose that σ and τ are epimorphisms such that $\sigma(L) \not\subseteq Z(R)$, $\tau(L) \not\subseteq Z(R)$ and $d(L) \not\subseteq Z(R)$. If $[d(x), \sigma(x)] \in Z(R)$ for all $x \in L$, then* $\operatorname{char} R = 2$ *and* $\dim_C RC = 4$.

To show the theorem we need some preliminary lemmas. The first useful lemma is due to Martindale [21] and is given in the following form by applying [6, Theorem 2].

Lemma 2. *Let R be a prime ring and $a_i, b_i \in U$ for $1 \leq i \leq n$. If $\sum_{i=1}^{n} a_i x b_i = 0$ for all $x \in R$ and $b_i \neq 0$ for some i, then $a_1, \cdots, a_n$ are C-dependent.*

Lemma 3. *Let R be noncommutative prime ring and $a, b \in R$. If $[a, [R, R]b] = 0$,* *then either $a \in Z(R)$ or $b = 0$.*

Proof. Since R is not commutative, $[R, R]$ is a noncommutative Lie ideal of R. If R is not a PI–ring, then $[R, R]$ and R satisfy the same GPIs [18, Lemma 2]. Thus, by assumption, we have $[a, Rb] = 0$, implying that either $a \in Z(R)$ or $b = 0$, as desired.

Suppose next that R is a PI–ring. Denote by F the algebraic closure of C and set $S = RC \otimes_C F$. Then $S \cong \mathrm{M}_n(F)$, where $n > 1$. Moreover, $[a, [S, S]b] = 0$. In particular, $[a, [S, S]b[S, S]b] = 0$ and, hence, $[a, ([S, S] + [S, S]b[S, S])b] = 0$. If $[S, S]b[S, S] \not\subseteq [S, S]$, then $[S, S] + [S, S]b[S, S] = S$ as $\dim_F S = 1 + \dim_F [S, S]$. Thus $[a, Sb] = 0$, implying that either $a \in Z(R)$ or $b = 0$. Suppose that $[S, S]b[S, S] \subseteq [S, S]$. In view of [20, Lemma 1], $b[S, S] \subseteq F$, implying that $b = 0$. This proves the lemma.

Lemma 4. *Let R be a prime ring and $a, b \in R$. If there exist $u, v \in R$, not both zero, such that $u[a, x] + v[b, x] = 0$ for all $x \in R$, then $1, a$ and b are C-dependent.*

Proof. Let $x \in R$. By assumption, $u[a, x] + v[b, x] = 0$, implying that $(ua + vb)x - uxa - vxb = 0$. Since either $a \neq 0$ or $b \neq 0$, Lemma 2 implies that $1, a$ and b are C–dependent, as desired.

Lemma 5. *Let R be a prime ring and $a, b \in R$. If $a[a, [b, x]] = 0$ for all $x \in R$ and $1, a, b$ are C–independent, then there exist $\alpha, \beta \in C$ such that $a^2 = \alpha a$, $a(b - \beta) = 0$ and $(b - \beta)(a - \alpha) = 0$.*

Proof. Define $\delta(x) = [a, x]$ and $d(x) = [b, x]$ for $x \in R$. Then, by assumption, we have $a\delta d(x) = 0$ for all $x \in R$. Let $x, y \in R$. Expanding $a\delta d(xy) = 0$, we have

$$a\delta(x)d(y) + ad(x)\delta(y) + ax\delta d(y) = 0. \tag{11}$$

Replacing x by xa in (11), we see that $a\delta(xa)d(y) + ad(xa)\delta(y) = 0$. By Lemma 4, $a\delta(xa) = 0$ and $ad(xa) = 0$ for all $x \in R$. That is, $Ra\delta(Ra) = 0$ and $Rad(Ra) = 0$.

In view of Herstein's theorem [9], there exist $\alpha, \beta \in C$ such that $Ra(a-\alpha) = 0$ and $Ra(b-\beta) = 0$, implying that $a^2 = \alpha a$ and $a(b-\beta) = 0$. Using the two relations to expand $a[a, [b-\beta, x]] = 0$, we have $(b-\beta)(a-\alpha) = 0$, as desired.

Using these lemmas, we are ready to deal with Theorem 6 for $(1, \tau)$–derivations.

Lemma 6. *Let R be a prime ring, L a noncentral Lie ideal of R and τ an endomorphism of R such that $\tau(L) \not\subseteq Z(R)$. If d is a $(1, \tau)$-derivation of R such that $[d(x), x] \in Z(R)$ for all $x \in L$, then $d(L) \subseteq Z(R)$ unless* $\operatorname{char} R = 2$ *and* $\dim_C RC = 4$.

Proof. Suppose that either $\operatorname{char} R \neq 2$ or $\dim_C RC > 4$. The aim is to prove that $d(L) \subseteq Z(R)$. Suppose not. By Theorem 4, there exist $\lambda \in C$ and a map $\zeta: L \to C$ such that $d(u) = \lambda u + \zeta(u)$ for all $u \in L$. Since $d(L) \not\subseteq Z(R)$, we have $\lambda \neq 0$. Note that $\lambda^{-1}d$ is still a $(1, \tau)$–derivation of R. Replacing d by $\lambda^{-1}d$, we may assume that $\lambda = 1$. Let $u \in L$ and $x \in R$; then $u[u, x] = [u, ux] \in L$. Thus $d(u[u, x]) = u[u, x] + \zeta(u[u, x])$. On the other hand,

$$\begin{aligned} d(u[u,x]) &= d(u)[u,x] + \tau(u)d([u,x]) \\ &= (u+\zeta(u))[u,x] + \tau(u)([u,x] + \zeta([u,x])) \\ &= u[u,x] + (\tau(u) + \zeta(u))[u,x] + \zeta([u,x])\tau(u) \end{aligned}$$

Comparing the two expressions, we see that

$$(\tau(u) + \zeta(u))[u,x] + \zeta([u,x])\tau(u) \in C. \tag{12}$$

Commuting (12) with $\tau(u)$ gives that $\big[\tau(u), (\tau(u) + \zeta(u))[u,x]\big] = 0$, implying that

$$(\tau(u) + \zeta(u))\big[\tau(u) + \zeta(u), [u,x]\big] = 0 \tag{13}$$

for all $u \in L$ and all $x \in R$.

Suppose for the moment that $\ker(\tau) \neq 0$. Let $x \in \ker(\tau)$ and $y \in R$. Then $d(xy) = d(x)y + \tau(x)d(y) = d(x)y$. Thus there exists $a \in U$ such that $d(x) = ax$

for all $x \in \ker(\tau)$. For $r \in R$ and $x \in \ker(\tau)$ we have $d(rx) = arx$. On the other hand, $d(rx) = d(r)x + \tau(r)ax$, implying that $(d(r) - ar + \tau(r)a)x = 0$. By the primeness of R, $d(r) = ar - \tau(r)a$ for all $r \in R$. Let $u \in L$ and $x \in \ker(\tau)$. Then $d([u,x]) = [u,x] + \zeta([u,x])$. On the other hand, $d([u,x]) = a[u,x]$ as $[u,x] \in \ker(\tau)$. Thus we have $(a-1)[L, \ker(\tau)] \subseteq C$, implying that $a = 1$ as $[L, \ker(\tau)]$ is a noncentral Lie ideal of R. Now, for $u \in L$, we have $d(u) = u + \zeta(u) = u - \tau(u)$ and, hence, $\tau(u) \in C$. That is, $\tau(L) \subseteq C$, contrary to the fact that $\tau(L)$ is a noncentral Lie ideal of R. Thus $\ker(\tau) = 0$ and so τ is an automorphism of R. It is well–known that τ can be uniquely extended to an automorphism of U.

We claim that R is a PI–ring. Let $u \in L$. Applying Lemma 5 to (13) gives that either $\tau(u) \in Cu + C$ or $\tau(u)^2 \in C\tau(u) + C$. The latter case proves that u is algebraic over C of degree ≤ 2. Consider next the first case. Write $\tau(u) = \alpha u + \beta$, where $\alpha, \beta \in C$. If $\alpha = 0$, then $u \in C$. Suppose that $\alpha \neq 0$. Commuting u with (12), we have $(\alpha u + \beta + \zeta(u))\big[u, [u,x]\big] = 0$ for all $x \in R$. Lemma 2 implies that u is algebraic over C of degree ≤ 3. Up to now, we have proved that u is algebraic over C of degree ≤ 3 for each $u \in L$. In particular, L satisfies a PI with coefficients ± 1. Since L is a noncetral Lie ideal of R, R is also a PI–ring, as desired. So $U = RC$ follows from Posner's theorem [11, Theorem p.57] for prime PI–rings.

In view of [26, Theorem 1.5.33], $Z(R) \neq 0$. We claim that $d(Z(R)) = 0$ and $\tau(\beta) = \beta$ for all $\beta \in Z(R)$. Let $I = R[L,L]R$. Then I is a nonzero ideal of R and $[I,R] \subseteq L$ (see the proof of [8, Lemma 1.3]). Let $\beta \in Z(R)$ and $u \in [I,R]$; then $\beta u \in L$. Therefore, we have $d(\beta u) = \beta u + \zeta(\beta u)$. On the other hand, $d(\beta u) = d(u\beta) = d(u)\beta + \tau(u)d(\beta) = \beta u + \beta\zeta(u) + \tau(u)d(\beta)$. Thus $\tau(u)d(\beta) \in C$. That is, $[\tau(I), R]d(\beta) \subseteq C$. As $[\tau(I), R]$ is a noncentral Lie ideal of R, we have $d(\beta) = 0$.

Next, we compute $d(\beta u) = d(\beta)u + \tau(\beta)(u + \zeta(u) = \tau(\beta)(u + \zeta(u))$, implying that $(\tau(\beta) - \beta)u \in C$. That is, $(\tau(\beta) - \beta)[I, R] \subseteq Z(R)$ and so $\tau(\beta) = \beta$ follows.

Since $\tau(\beta) = \beta$ for all $\beta \in Z(R)$, we have $\tau(\beta) = \beta$ for all $\beta \in C$. In view of [7, Corollary p.100], τ is an inner derivation of U. There exists an invertible element $b \in U$ such that $\tau(x) = bxb^{-1}$ for all $x \in U$. In this case, $b^{-1}d$ is an ordinary derivation from R into U. It is well–known that $b^{-1}d$ is uniquely extended to a derivation of U [16, Lemma 2]. Since $b^{-1}d(Z(R)) = 0$ and C is the quotient field of $Z(R)$, $b^{-1}d(C) = 0$ follows. So $b^{-1}d$ is inner [7, Proposition p.100]. Thus there exists $a \in U$ such that $b^{-1}d(x) = ax - xa$ for all $x \in U$. That is, $d(x) = bax - bxa$ for all $x \in R$.

Let $u \in L$. Then $u + \zeta(u) = d(u) = bau - bua$, implying that $(ba - 1)u - bua \in C$. Since $[I, R] \subseteq L$, it follows from [6, Theorem 2] that $(ba - 1)x - bxa \in C$ for all $x \in [RC, RC]$. Denote by F the algebraic closure of C. Then $(ba - 1)x - bxa \in F$ for all $x \in [RC \otimes_C F, RC \otimes_C F]$. Note that $RC \otimes_C F \cong \mathrm{M}_n(F)$, where $n > 1$. Hence, we may assume that $R \cong \mathrm{M}_n(C)$ and that $(ba - 1)x - bxa \in C$ for all $x \in [R, R]$. Suppose for the moment that $(ba - 1)x = bxa$ for all $x \in [R, R]$. Then, for $r, s \in R$, we have $b^{-1}(ba - 1)[r, s] = [r, s]a$ and so $[r, s]as = b^{-1}(ba - 1)[r, s]s = b^{-1}(ba - 1)[rs, s] = [rs, s]a = [r, s]sa$, implying that $[r, s][a, s] = 0$. That is, $[R, s][a, s] = 0$ and so either $s \in Z(R)$ or $[a, s] = 0$. In either case, we have $[a, s] = 0$ for all $s \in R$, implying $a \in Z(R)$. This implies $d = 0$, a contradiction. Hence, $(ba - 1)x - bxa \neq 0$ for some $x \in [R, R]$. Since $[R, R]$ is generated over C by all elements of rank 1, there exists $u \in [R, R]$ of rank 1 such that $0 \neq (ba - 1)u - bua \in C$, implying $n = 2$. If $\mathrm{char}\, R \neq 2$, then $[R, R] + C = R$ and, hence, $(ba - 1)x - bxa \in C$ for all $x \in R$. Let $x, y \in R$. Then $(ba - 1)(xy) - b(xy)a \in C$ and $(ba - 1)xy - bxay \in Cy$, implying that $bx[a, y] \in Cy + C$. Since b is invertible in R, we have $x[a, y] \in Cy + C$. Commuting with y gives $[y, R[a,y]] = 0$, implying that either $y \in Z(R)$ or $[a,y] = 0$ for each $y \in R$. But R is not commutative; so $a \in Z(R)$ and, hence, $d = 0$ follows, a contradiction. This completes the proof.

We are now ready to give the proof of Theorem 6.

Proof of Theorem 6. By Theorem 4, there exist $\lambda \in C$ and a map $\zeta: L \to C$ such that $d(u) = \lambda\sigma(u) + \zeta(u)$ for all $u \in L$. Since $d(L) \not\subseteq Z(R)$, we have $\lambda \neq 0$. Replacing d by $\lambda^{-1}d$, we may assume that $\lambda = 1$.

Suppose on the contrary that either $\operatorname{char} R \neq 2$ or $\dim_C RC > 4$. We will reduce d to a nonzero $(1,\theta)$–derivation centralizing on L and, hence, give a contradiction by Lemma 6. If σ is an automorphism, then $\sigma^{-1}d$ is a nonzero $(1, \sigma^{-1}\tau)$–derivation centralizing on L. So we are done in this case. Therefore we may assume that $\ker(\sigma) \neq 0$.

We claim that $d(\ker(\sigma)) = 0$ and $\tau(\ker(\sigma)) = 0$. Let $x, y \in \ker(\sigma)$ and $u \in L$. Then we have $d([x,[y,u]]) = \sigma([x,[y,u]]) + \zeta([x,[y,u]]) = \zeta([x,[y,u]]) \in C$. Thus we have

$$\begin{aligned} d([x,[y,u]]) &= d(x[y,u]) - d([y,u]x) \\ &= \tau(x)d([y,u]) - \tau([y,u])d(x) \qquad (14) \\ &= \tau(x)\zeta([y,u]) - \tau([y,u])d(x) \in C. \end{aligned}$$

Commuting (14) with $\tau(x)$ gives that $\big[\tau(x), \tau([y,u])d(x)\big] = 0$. That is,

$$\big[\tau(x), [\tau(\ker(\sigma)), \tau(L)]d(x)\big] = 0 \text{ for all } x \in \ker(\sigma).$$

Note that $[\tau(\ker(\sigma)), \tau(L)]$ is a noncentral Lie ideal of R. Set $N = [\tau(\ker(\sigma)), \tau(L)]$. Then $0 \neq [R[N,N]R, R] \subseteq N$ and so $[[\tau(x), [R,R]d(x)] = 0$ for all $x \in \ker(\sigma))$ (see [6, Theorem 2]). By Lemma 3, either $\tau(x) \in Z(R)$ or $d(x) = 0$ for each $x \in \ker(\sigma)$. Thus we have either $\tau(\ker(\sigma)) \subseteq Z(R)$ or $d(\ker(\sigma)) = 0$.

If $\tau(\ker(\sigma)) \subseteq Z(R)$, then $\tau(\ker(\sigma)) = 0$ as R is not commutative. Suppose the latter case that $d(\ker(\sigma)) = 0$. For $x \in \ker(\sigma)$ and $u \in L$, $d([x, u]) = d(x)\sigma(u) + \tau(x)d(u) - d(u)\sigma(x) - \tau(u)d(x) = \tau(x)d(u) \in C$, implying that $\tau(\ker(\sigma))d(L) \subseteq C$. But $d(L) \neq 0$, we have $\tau(\ker(\sigma)) = 0$. Therefore, in either case, we $\tau(\ker(\sigma)) = 0$.

Next, we want to prove $d(\ker(\sigma)) = 0$. Let $u, v \in L$ and $x \in \ker(\sigma)$. Then $d([v, x]) \in C$ and $d([u, [v, x]]) \in C$. Thus we have

$$d([u, [v, x]]) = d(u[v, x]) - d([v, x]u) = d([v, x])(\tau(u) - \sigma(u)) \in C,$$

implying that

$$d\big([\ker(\sigma), L]\big)(\tau(u) - \sigma(u)) \subseteq C. \tag{15}$$

If $d\big([\ker(\sigma), L]\big) \neq 0$, then (15) implies that $\tau(u) = \sigma(u)$ for all $u \in L$. It is easy to prove that $\sigma = \tau$. Then, for $x \in \ker(\sigma)$ and $u \in L$, we have $d([u, x]) = [\sigma(u), d(x)] \in C$. That is, $[d(\ker(\sigma)), \sigma(L)] \subseteq C$. As $\sigma(L)$ is a noncentral Lie ideal of R, we have $d(\ker(\sigma)) \subseteq C$. So $d([u, x]) = [d(u), d(x)] = 0$, implying that $d\big([\ker(\sigma), L]\big) = 0$, a contradiction. This proves that $d\big([\ker(\sigma), L]\big) = 0$.

Let $x \in \ker(\sigma)$, $u \in L$ and $y \in R$. Then $0 = d([u, x] = \tau(u)d(x) - d(x)\sigma(u)$. That is,

$$d(x)\sigma(u) = \tau(u)d(x). \tag{16}$$

Since $yx \in \ker(\sigma)$, by (16) we have $d(xy)\sigma(u) = \tau(u)d(xy)$, implying $\tau(y)d(x)\sigma(u) = \tau(u)\tau(y)d(x)$, and so, by (16) again, $[\tau(y), \tau(u)]d(x) = 0$. But τ is surjective, we have $[R, \tau(L)]d(\ker(\sigma)) = 0$. Then $d(\ker(\sigma)) = 0$ follows from the fact that $[R, \tau(L)]$ is a noncentral Lie ideal of R.