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ALGEBRAIC DERIVATIONS WITH CONSTANTS SATISFYING A POLYNOMIAL IDENTITY

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Abstract. We prove that if a semiprime ring R possesses a derivation which is integral over its extended centroid C and whose constants satisfy a polynomial identity, then R itself is a PI–ring. This answers affirmatively a problem raised by M. Smith and generalizes all known results in this line.

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§1. Introduction

Rings considered here are always associative. Given a ring R and $b \in R$, we define the centralizer of b in R to be $C_R(b) = \{r \in R \mid br = rb\}$. If b is algebraic, then $C_R(b)$ is considered to be large in the sense that nice properties of $C_R(b)$ can be usually extended to the whole ring R. In [9], Herstein and Neumann initiated this line of research by proving the theorem: Let R be semiprime and let $b \in R$ be algebraic over the center Z of R. If $C_R(b)$ is simple, then R must also be simple. Let R be a semiprime algebra over a field F and let $b \in R$ be algebraic over F. Cohen [5] proved that if $C_R(b)$ is semiprime Artinian (Goldie resp.), then so is R. As polynomial identities are powerful tools, we naturally ask whether R is PI if $C_R(b)$ is PI. Montgomery [15] proved the result under the assumption that R is a simple ring with unit and b is power central. Smith [18] improved the result by assuming that R is prime and $b \in R$ is integral over the centroid of R. Rowen [17] also proved the result under the different assumption that R is prime, that $b \in R$ is algebraic over the extended centroid of R and that the minimum polynomial $\mu(X)$ of b over C satisfies that $\mu'(b)$ is invertible. Although Smith's result seems more general, neither actually implies the other. This shows that the last word on this problem has not yet been uttered even in the prime case. This type of results are very useful and have many applications. See, for instance, [8, Chapter 1]. Smith [18, p.149] also raised the question whether her result can be extended to semiprime rings. Our result answers Smith's problem affirmatively and generalizes all known results in the prime case. We actually work in a more general setting: Assume throughout that R is a semiprime ring with extended centroid C. By a derivation of R, we mean an additive map $\delta : R \to R$ such that $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in R$. A derivation δ of R is said to be integral over C if there exist $\alpha_1, \ldots, \alpha_{m-1} \in C$ such that $\delta^m(x) + \alpha_1\delta^{m-1}(x) + \cdots + \alpha_{m-1}\delta(x) = 0$ for all $x \in R$. The minimal such integer m is called the *integral degree* of δ over C and is denoted by $\deg_C \delta$. We define $R^{(\delta)} = \{x \in R \mid \delta(x) = 0\}$, the subring of constants of δ. Assume that δ is a derivation of R integral over C. Our main result is as follows: If $R^{(\delta)}$ is a PI–ring, then R is a PI–ring. Furthermore, if $R^{(\delta)}$ satisfies a nonzero PI with coefficients ± 1 of degree t, then R satisfies $S_{st}(X_1, \cdots, X_{st})$, the standard identity of degree st, where s is the integral degree of δ over C (see Theorem 6).

For $b \in R$, the map $\mathrm{ad}(b)\colon x \in R \mapsto [b, x] = bx - xb$ defines a derivation of R, called the inner derivation induced by b. If $\delta = \mathrm{ad}(b)$, then $R^{(\delta)}$ is just $C_R(b)$. Moreover, $\mathrm{ad}(b)$ is integral over C if and only if the element b is integral over C. In this sense, our main theorem generalizes all previously known results.

§2. The Prime Case

We refer the reader to [3] and [10] for the basic terminology and results of the theory of generalized polynomial identities (GPIs) or polynomial identities (PIs). Throughout this section, R always denotes a prime ring with extended centroid C and two–sided Martindale quotient ring Q. In this case, C is a field. Let δ be a derivation of R algebraic over C. In view of [11, Corollary 2] and [11, Corollary 3], there exists

$b \in Q$, which is algebraic over C, such that

$$(1) \qquad \begin{cases} \text{either } \delta = \mathrm{ad}(b) \text{ if } \mathrm{char}\, R = 0, \text{ or} \\ \delta^{p^m} + \alpha_1 \delta^{p^{m-1}} + \cdots + \alpha_m \delta = \mathrm{ad}(b), \\ \text{where } \alpha_i \in C, \text{ and } m \geq 0 \text{ is minimal possible if } \mathrm{char}\, R = p > 0. \end{cases}$$

The element b is unique up to within the addition of a central element and the algebraic degree of b over C is thus uniquely determined. For brevity, refer b above as *the element associated with* δ and the algebraic degree of b as *the reduced degree of* δ. For the case that $\mathrm{char}\, R = p > 0$, we always define

$$(1)' \qquad \begin{cases} H(X) = \delta^{p^m - 1}(X) + \alpha_1 \delta^{p^{m-1}-1}(X) + \cdots + \alpha_m X, \\ \ell(X) = X^{p^m} + \alpha_1 X^{p^{m-1}} + \cdots + \alpha_m X \text{ and} \\ d_i = \delta^{p^i} \text{ for } 0 \leq i \leq m-1. \end{cases}$$

Note that $\ell(\delta)(x) = \delta(H(x))$ for $x \in Q$, since $\delta(\alpha_i) = 0$ for each i and $\delta(b) \in C$. We see that $d_0, d_1, \cdots, d_{m-1}$ are C–independent modulo Q–inner derivations by the minimality of m. Fix the linear order $d_0 < d_1 < \cdots < d_{m-1}$. Then every δ^j $(1 \leq j < p^m)$ can be written as a regular word of the form $d_{m-1}^{s_{m-1}} \cdots d_i^{s_i} \cdots d_0^{s_0}$ with each $0 \leq s_i < p$. Therefore, these $\delta, \delta^2, \cdots, \delta^{p^m - 1}$ are distinct regular words in $d_0, d_1, \cdots, d_{m-1}$. We will retain these notations throughout this section.

Recall that a derivation d of R is said to be X–inner if $d = \mathrm{ad}(b)$ for some $b \in Q$ and X–outer, otherwise. For convenience we give the statement of Kharchenko's theorem here. We remark that Kharchenko's theorem holds for nonzero ideals (see [11] and [12]).

Kharchenko's Theorem. *Let R be a prime ring. If $\phi(\Delta_i(X_j))$ is a differential identity for a nonzero ideal of R, where Δ_i are distinct regular words and X_j are distinct indeterminates, then $\phi(Z_{ij})$ is a* GPI *for R.*

Our first main result is the following

Theorem 1. *Let R be a prime ring with extended centroid C. Suppose that δ is a derivation of R algebraic over C such that $R^{(\delta)}$ is a* PI*-ring. Then R is a* PI*-ring.*

To prove Theorem 1 we need a result on the theory of prime rings with GPIs. We denote by $\mathrm{soc}(R)$ the socle of R. By a centrally closed prime C-algebra we mean a prime ring R with extended centroid C such that $R = RC$.

Theorem 2. *Let R be a centrally closed prime C-algebra with $\mathrm{soc}(R) \neq 0$. Suppose that $f(X_1, \ldots, X_t)$ is a multilinear* GPI *for R with coefficients in Q. If $f(X_1, \ldots, X_t)$ assumes the following form $\sum_{i=1}^m a_i X_1 g_i(X_2, \cdots, X_t) + f_1(X_1, \cdots, X_t)$, where $a_i \in Q$ are C-independent modulo $\mathrm{soc}(R)$ and where each monomial $b_1 X_{i_1} b_2 X_{i_2} \cdots b_t X_{i_t} b_{t+1}$ occurring in $f_1(X_1, \cdots, X_t)$ is either $b_1 \in \mathrm{soc}(R)$ or $i_1 \neq 1$, then each $g_i(X_2, \cdots, X_t)$ is a* GPI *for R for $1 \leq i \leq m$.*

Since Theorem 2 plays an important role in the proof of Theorem 1, we first give its proof. To prove Theorem 2 we need two preliminary lemmas. The first is a useful lemma due to Martindale [14]. Applying [4, Theorem 2] we give its statement in the following form.

Lemma 1. *Let R be a prime ring. Suppose that $\sum_{i=1}^n a_i x b_i = 0$ for all $x \in R$, where $a_i, b_i \in Q$. If the elements $a_1, a_2, \cdots, a_n$ are C-independent, then $b_i = 0$ for each $i = 1, \cdots, n$.*

Lemma 2. *Let R be a prime* GPI*-ring. Then $\mathrm{soc}(RC) = \mathrm{soc}(Q) \neq 0$.*

Proof. Since R is a prime GPI–ring, it follows from Martindale's theorem [14] that RC is a primitive ring with nonzero socle. It is easy to check that Q is contained in the two–sided Martindale quotient ring of RC. Thus, by [3, Theorem 4.3.6], $\mathrm{soc}(RC) = \mathrm{soc}(Q)$.

This proves the lemma.

We are now ready to give the

Proof of Theorem 2. We remark that R and Q satisfy the same GPIs with coefficients in Q [4, Theorem 2]. For $t = 1$, we can write

$$f(X_1) = \sum_{i=1}^{m} a_i X_1 b_i + \sum_{j=1}^{n} c_j X_1 d_j,$$

where $a_i, b_i, c_j, d_j \in Q$, $c_j \in \mathrm{soc}(R)$ and the a_i are C–independent modulo $\mathrm{soc}(R)$. Choose a basis $\{u_1, \cdots, u_s\}$ for the C–subspace of Q generated by $c_1, \cdots, c_n$. Since $a_1, \cdots, a_m$ are C–independent modulo $\mathrm{soc}(R)$, the elements $a_1, \cdots, a_m, u_1, \cdots, u_s$ are C–independent. Rewrite $f(X_1) = \sum_{i=1}^{m} a_i X_1 b_i + \sum_{j=1}^{s} u_j X_1 v_j$ for some $v_j \in Q$. By assumption, $f(X_1)$ is a GPI for R. In view of Lemma 1, $b_i = 0$ for each i, as desired.

Suppose next that $t > 1$. Let $x_2, \cdots, x_t \in \mathrm{soc}(R)$. Then

$$f(X_1, x_2, \cdots, x_t) = \sum_{i=1}^{m} a_i X_1 g_i(x_2, \cdots, x_t) + f_1(X_1, x_2, \cdots, x_t)$$

is a GPI for R. Write $f_1(X_1, x_2, \cdots, x_t) = \sum_{j=1}^{n} c_j X_1 d_j$ for some $c_j, d_j \in Q$. Let $b_1 X_{i_1} b_2 X_{i_2} \cdots b_t X_{i_t} b_{t+1}$ be a monomial occurring in $f_1(X_1, \cdots, X_t)$. By assumption, either $b_1 \in \mathrm{soc}(R)$ or $i_1 \neq 1$. Since, by Lemma 2, $Q\,\mathrm{soc}(R) \subseteq \mathrm{soc}(R)$, we have each $c_j \in \mathrm{soc}(R)$. In view of the case that $t = 1$, we see that $g_i(x_2, \cdots, x_t) = 0$ for $1 \leq i \leq m$. Thus $g_i(X_2, \cdots, X_t)$ is a GPI for $\mathrm{soc}(R) \neq 0$ and hence for R [4, Theorem 2] for $1 \leq i \leq m$. This proves the theorem.

For any integer $n \geq 1$, we define $\phi_n(b, X) = b^{n-1}X + b^{n-2}Xb + \cdots + bXb^{n-2} + Xb^{n-1}$ for $b \in Q$, where X is a noncommuting variable. Note that $\phi_1(b, X) = X$. A direct computation proves that $[b, \phi_n(b, X)] = [b^n, X]$. More generally, if $g(X) = X^n + \beta_1 X^{n-1} + \cdots + \beta_{n-1} X + \beta_n$ where $\beta_i \in C$, we define $\hat{g}_b(X) = \phi_n(b, X) +$

$\beta_1\phi_{n-1}(b,X)+\cdots+\beta_{n-1}\phi_1(b,X)$. We have also $[b,\hat{g}_b(y)]=[g(b),y]$ for $y\in Q$. In particular, if $g(b)=0$, then $\hat{g}_b(y)\in C_Q(b)$ for $y\in Q$.

Lemma 3. *Let R be a prime ring and let δ be a derivation of R algebraic over C. If $R^{(\delta)}$ is a PI–ring, then R itself is a GPI–ring.*

Proof. Note that $R^{(\delta)}=R$ if $\delta=0$. Thus we may assume that $\delta\neq 0$. Since $R^{(\delta)}$ is a PI–ring, $R^{(\delta)}$ satisfies a nonzero multilinear PI $f(X_1,\ldots,X_t)$ with coefficients ± 1 (see [1]). Suppose that $b\in Q$ is an associated element of δ as explained in (1).

Suppose first that δ is X–inner. Then $\delta=\mathrm{ad}(b)$ with $b\notin C$. Thus, by assumption, $\mathrm{ad}(b)$ is an algebraic derivation of Q over C. So b is algebraic over C. Let $h(X)=X^m+\alpha_1X^{m-1}+\cdots+\alpha_n$ where $\alpha_i\in C$ be the minimal polynomial of b over C. Note that $m\geq 2$. Choose an nonzero ideal I of R such that $\hat{h}_b(y)\in R$ for $y\in I$. Thus we see that $\hat{h}_b(y)\in C_R(b)(=R^{(\delta)})$ for $y\in I$. In particular, $f(\hat{h}_b(X_1),\cdots,\hat{h}_b(X_t))$ is a GPI for I and hence for R [4, Theorem 2]. Since $1,b,\cdots,b^{m-1}$ are C–independent over C, $f(\hat{h}_b(X_1),\cdots,\hat{h}_b(X_t))$ is nontrivial. This proves R to be a GPI–ring.

Suppose next that δ is X–outer. In this case, we have $\operatorname{char} R=p>0$. We keep the notations explained in (1) for δ. Let $h(X)=X^s+\mu_1X^{s-1}+\cdots+\mu_n$, where $\mu_i\in C$, be the minimal polynomial of b over C. Choose an nonzero ideal J of R such that $\hat{h}_b(z)\in R$ and $H(\hat{h}_b(z))\in R$ for $z\in J$. Let $z\in J$. Then $\hat{h}_b(z)\in C_R(b)$ and hence $\ell(\delta)(\hat{h}_b(z))=\mathrm{ad}(b)(\hat{h}_b(z))=[b,\hat{h}_b(z)]=[h(b),z]=0$. But $\ell(\delta)(x)=\delta(H(x))$ for all $x\in Q$ and this implies that $H(\hat{h}_b(z))\in R^{(\delta)}$. Thus

$$f\big(H(\hat{h}_b(X_1)),\cdots,H(\hat{h}_b(X_t))\big) \tag{2}$$

is a differential identity for J. Since $\hat{h}_b(X_i)=\phi_s(b,X_i)+\mu_1\phi_{s-1}(b,X_i)+\mu_{s-1}\phi_1(b,X_i)$, we see that

$$H(\hat{h}_b(X_i)) = \hat{h}_b(\delta^{p^m-1}(X_i)) + \cdots, \tag{3}$$

where the dots denote a sum of terms $a\delta^j(X_i)c$ in which $j < p^m - 1$ and $a, c \in Q$. Recall that $d_i = \delta^{p^i}$, $0 \le i \le m-1$, are C–independent modulo Q–inner derivations. These $\delta, \delta^2, \cdots, \delta^{p^m-1}$ are distinct regular words in $d_0, d_1, \cdots, d_{m-1}$. Use (3) to transform the differential identity (2) into its reduced form. Applying Kharchenko's theorem to its reduced form by substituting $\delta^{p^m-1}(X_i)$ by X_i and $\delta^j(X_i)$ by 0 for $j < p^m - 1$, we see that $f(\hat{h}_b(X_1), \cdots, \hat{h}_b(X_t))$ is a GPI for R. It is nontrivial, since $1, b, \cdots, b^{s-1}$ are C–independent. Thus R is a GPI–ring, proving the lemma.

Lemma 4. *Let R be a centrally closed prime C'-algebra, δ a derivation of R and I an ideal of R invariant under δ. Suppose that $b \in Q$ is algebraic over C modulo I such that $\delta(b) \in C$. If $g(X)$ is the minimal polynomial of b over C modulo I, then $\delta(\hat{g}_b(z)) = \hat{g}_b(\delta(z))$ for all $z \in R$.*

Proof. Write $g(X) = X^n + \beta_1 X^{n-1} + \cdots + \beta_n$ for some $\beta_i \in C$ with $n \ge 1$. Note that δ can be uniquely extended to a derivation of Q. Since $\delta(b) \in C$, we see that $\delta(g(b)) = \delta(b)g'(b) + g^\delta(b) \in I$, where $g'(X) = nX^{n-1} + (n-1)\beta_1 X^{n-2} + \cdots + \beta_{n-1}$ and $g^\delta(X) = \delta(\beta_1)X^{n-1} + \delta(\beta_2)X^{n-2} \cdots + \delta(\beta_n)$. By the minimal choice of n, this implies that

$$(n-k)\beta_k\delta(b) + \delta(\beta_{k+1}) = 0 \tag{4}$$

for $k = 0, 1, \cdots, n-1$, where $\beta_0 = 1$. Let $z \in R$. Then, using (4), we have

$$\delta(\hat{g}_b(z)) = \delta(\phi_n(b,z) + \beta_1\phi_{n-1}(b,z) + \cdots + \beta_{n-1}\phi_1(b,z))$$

$$
\begin{aligned}
&= \phi_n(b, \delta(z)) + \beta_1 \phi_{n-1}(b, \delta(z)) + \cdots + \beta_{n-1}\phi_1(b, \delta(z)) \\
&\quad + \sum_{j=0}^{n-2} \sum_{k=0}^{j} \left((n-k)\beta_k \delta(b) + \delta(\beta_{k+1})\right) b^{j-k} z b^{n-j-2} \\
&= \hat{g}_b(\delta(z)),
\end{aligned}
$$

as desired.

We are now ready to give the proof of Theorem 1.

Proof of Theorem 1. In view of Lemma 3, R is a prime GPI–ring. It follows from Martindale's theorem [9] that RC is a centrally closed prime C–algebra with $\mathrm{soc}(RC) \neq 0$. We keep the notations explained in (1) for δ. Let $g(X)$ be the minimal polynomial over C of b modulo $\mathrm{soc}(RC)$. That is, $g(X) = X^n + \beta_1 X^{n-1} + \cdots + \beta_n$, where each $\beta_i \in C$, is the polynomial of minimal degree n such that $g(b) = b^n + \beta_1 b^{n-1} + \cdots + \beta_n \in \mathrm{soc}(RC)$. Such a polynomial does exist because b is algebraic over C. In view of Litoff's theorem [6], there exists an idempotent $e \in \mathrm{soc}(RC)$ such that $g(b) \in eRCe$. If $e = 1$, then RC is a finite–dimensional central simple C–algebra. We are done in this case. Thus we may suppose $e \neq 1$. We claim that

(5) $$f\left(\hat{g}_b((1-e)X_1(1-e)), \cdots, \hat{g}_b((1-e)X_t(1-e))\right)$$

is a GPI for RC.

Suppose first that δ is X–inner. In this case $\delta = \mathrm{ad}(b)$, where $b \in Q$. Choose a nonzero ideal I of R such that $\hat{g}_b((1-e)y(1-e)) \in R$ for $y \in I$. Let $y \in I$. Then $[b, \hat{g}_b((1-e)y(1-e))] = [g(b), (1-e)y(1-e)] = 0$, implying that $\hat{g}_b((1-e)y(1-e)) \in C_R(b)$. Note that $R^{(\delta)} = C_R(b)$. Since $f(X_1, \ldots, X_t)$ is a PI for $C_R(b)$, we see that $f\left(\hat{g}_b((1-e)X_1(1-e)), \cdots, \hat{g}_b((1-e)X_t(1-e))\right)$ is a GPI for I and hence for RC [4, Theorem 2] as desired.

Suppose next that δ is X–outer. Since $\mathrm{soc}(RC)$ is invariant under δ and $\delta(b) \in C$, it follows from Lemma 4 that $\delta(\hat{g}_b(z)) = \hat{g}_b(\delta(z))$ for all $z \in R$. Since $g(b) \in \mathrm{soc}(RC)$, there exists an idempotent $e \in \mathrm{soc}(RC)$ such that $g(b) \in eRCe$ [6]. Choose a nonzero ideal I of R such that

$$(1-e)y(1-e), H((1-e)y(1-e)), \hat{g}_b(H((1-e)y(1-e))) \in R$$

for $y \in I$. We see that

$$\begin{aligned} 0 &= [g(b), (1-e)y(1-e)] = [b, \hat{g}_b((1-e)y(1-e))] \\ &= \hat{g}_b([b, (1-e)y(1-e)]) = \hat{g}_b(\ell(b)((1-e)y(1-e))) \\ &= \hat{g}_b(\delta(H((1-e)y(1-e)))) = \delta(\hat{g}_b(H((1-e)y(1-e)))), \end{aligned}$$

where the last equality follows from Lemma 4. This implies that $\hat{g}_b(H((1-e)y(1-e))) \in R^{(\delta)}$ for all $y \in I$. Thus we have that

$$f\Big(\hat{g}_b(H((1-e)X_1(1-e))), \cdots, \hat{g}_b(H((1-e)X_t(1-e)))\Big) \tag{6}$$

is a differential identity for I. Recall that these $\delta, \delta^2, \cdots, \delta^{p^m-1}$ are distinct regular words in $d_0, d_1, \cdots, d_{m-1}$, where $d_i = \delta^{p^i}$ for $0 \le i \le m-1$. In (6) we write

$$H((1-e)X_i(1-e)) = (1-e)\delta^{p^m-1}(X_i)(1-e) + \cdots, \tag{7}$$

where the dots denote a sum of terms $a\delta^j(X_i)c$ in which $j < p^m - 1$ and $a, c \in Q$. Using (7) to transform (6) into its reduced form and then applying Kharchenko's theorem to the reduced form by substituting $\delta^{p^m-1}(X_i)$ by X_i and $\delta^j(X_i)$ by 0 for $j < p^m - 1$, we see that $f\Big(\hat{g}_b((1-e)X_1(1-e)), \cdots, \hat{g}_b((1-e)X_t(1-e))\Big)$ is a GPI for R and hence for RC, as desired. This proves our claim.

We may assume, without loss of generality, that the monomial $X_1X_2\cdots X_t$ occurs in $f(X_1, \ldots, X_t)$. Write

$$f(X_1, \ldots, X_t) = X_1 f_1(X_2, \cdots, X_t) + h_1(X_1, \cdots, X_t), \tag{8}$$

where $h_1(X_1, \cdots, X_t)$ consists of the monomials starting with $\pm X_j$ for some $j > 1$. By (5) and (8) we see that

$$(9) \qquad \hat{g}_b((1-e)X_1(1-e))f_1(\hat{g}_b((1-e)X_2(1-e)), \cdots, \hat{g}_b((1-e)X_t(1-e)))$$
$$+h_1(\hat{g}_b((1-e)X_1(1-e)), \cdots, \hat{g}_b((1-e)X_t(1-e)))$$

is a GPI for R. We claim that the elements $b^{n-1}(1-e), \cdots, b(1-e), (1-e)$ are C-independent modulo $\mathrm{soc}(RC)$. Indeed, suppose that $\sum_{i=1}^n \mu_i b^{n-i}(1-e) \in \mathrm{soc}(RC)$ for some $\mu_i \in C$. Now, by Lemma 2, we have $\mathrm{soc}(RC) = \mathrm{soc}(Q)$. Thus $\sum_{i=1}^n \mu_i b^{n-i} = \sum_{i=1}^n \mu_i b^{n-i}(1-e) + \sum_{i=1}^n \mu_i b^{n-i}e \in \mathrm{soc}(RC)$. By the minimal choice of n, we see that $\mu_i = 0$ for each i. This proves our claim. Applying Theorem 2 to the monomials in (4) starting with $b^{n-1}(1-e)X_1$, we conclude that

$$(10) \qquad (1-e)f_1\Big(\hat{g}_b((1-e)X_2(1-e)), \cdots, \hat{g}_b((1-e)X_t(1-e))\Big)$$

is a GPI for R. Next, an analogous argument proves that the elements $(1-e)b^{n-1}(1-e), \cdots, (1-e)b(1-e), (1-e)$ are C-independent modulo $\mathrm{soc}(RC)$. Note that the degree of $f_1(X_2, \cdots, X_t)$ is $t-1$ and contains the monomial $X_2X_3\cdots X_t$. Write

$$(11) \qquad f_1(X_2, \cdots, X_t) = X_2 f_2(X_3, \cdots, X_t) + h_2(X_2, \cdots, X_t),$$

where $h_2(X_2, \cdots, X_t)$ consists of the monomials starting with $\pm X_j$ for some $j > 2$. Applying (10) and (11), we can repeat using Theorem 2 to handle the present case and conclude that

$$(1-e)f_2\Big(\hat{g}_b((1-e)X_3(1-e)), \cdots, \hat{g}_b((1-e)X_t(1-e))\Big)$$

is a GPI for R. Hence, by induction, we obtain the conclusion that $\hat{g}_b((1-e)X(1-e))$ is a GPI for R. By Theorem 2 again, we see that $e = 1$, a contradiction. This proves the theorem.

The next main result is the following result.

Theorem 3. *Let R be a prime ring and let δ be a derivation of R algebraic over C. Suppose that $b \in Q$ is an associated element of δ. Then $R^{(\delta)}$ and $C_R(b)$ satisfy the same* PI*s over C.*

The following lemma will be used in the proof of Theorem 3.

Lemma 5. *Let R be a prime* PI*–ring with center Z and extended centroid C. Suppose that $\sum_{i=1}^{n} \phi_i(\Delta_j(x_k))a_i = 0$ for all $x_k \in Z$, where each $\phi_i(Z_{jk})$ is a polynomial over C in commuting variables Z_{jk}, Δ_j are distinct regular words in derivations of Z and $a_i \in Q$. Then $\sum_{i=1}^{n} \phi_i(z_{jk})a_i = 0$ for all $z_{jk} \in C$.*

Proof. Choose a basis $\{b_1, \cdots, b_t\}$ of $\sum_{i=1}^{n} C\, a_i$. Write each $a_i = \sum_{m=1}^{t} \beta_{im} b_m$, where $\beta_{im} \in C$. The assumption that $\sum_{i=1}^{n} \sum_{m=1}^{t} \beta_{im} b_m \phi_i(\Delta_j(x_k)) = 0$ for all $x_k \in Z$ implies that each $\sum_{i=1}^{n} \beta_{im} \phi_i(\Delta_j(x_k)) = 0$ for all $x_x \in Z$. Note that C is the quotient field of Z. Applying Kharchenko's theorem, we obtain each $\sum_{i=1}^{n} \beta_{im} \phi_i(Z_{jk})$ is a GPI for C. Thus $\sum_{i=1}^{n} \phi_i(z_{jk})a_i = 0$ for all $z_{jk} \in C$. This proves the lemma.

Our Lemma 5 will be used in the following manner: Let R be a prime PI–ring with center Z and extended centroid C. A derivation of R vanishing on C must be X–inner (see [12, p.68]). Hence, the restriction to Z of a basis for X–outer derivations of R gives a C–independent set of derivations of Z and can be linearly ordered in the same way as the original basis of R. In this way, regular words for R give rise to regular words for Z. We are now ready to give the

Proof of Theorem 3. If δ is X–inner, then $\delta = \mathrm{ad}(b)$ and hence $R^{(\delta)} = C_R(b)$. The conclusion is trivially true in this case. Thus we may assume that δ is X–outer and keep the notations explained in (1) and (1)'. It is clear that $R^{(\delta)} \subseteq C_R(b)$, implying that each PI for $C_R(b)$ is also satisfied by $R^{(\delta)}$. For the reverse direction, let $f(X_1, \ldots, X_t)$ be a nonzero polynomial over C which is a PI for $R^{(\delta)}$. Then $R^{(\delta)}C$ satisfies a nonzero multilinear polynomial over C, implying that $R^{(\delta)}C$ satisfies a nonzero polynomial with coefficients ± 1 (see [1]). Thus $R^{(\delta)}C$ is a PI–ring and so is $R^{(\delta)}$.

It follows from Theorem 2 that R is a prime PI–ring and so Z, the center of R, is nonzero [16]. Since C is the quotient field of Z, we can choose $\mu \in Z$ such that $\mu\alpha_i \in Z$ for each $1 \leq i \leq m$. Note that $\mu^p\alpha_i \in Z$ and $\delta(\mu^p\alpha_i) = 0$ for each i. Let $y_1, \cdots, y_t \in C_R(b)$ and $\beta_1, \cdots, \beta_t \in Z$. Then $\mu^p H(\beta_i y_i) \in R$ for each i. Since $\delta(\mu^p H(\beta_i y_i)) = \mu^p[b, \beta_i y_i] = 0$, implying that $\mu^p H(\beta_i y_i) \in R^{(\delta)}$. Thus we see that $f(\mu^p H(\beta_1 y_1), \cdots, \mu^p H(\beta_t y_t)) = 0$. Using the fact that $\delta, \delta^2, \cdots, \delta^{p^m-1}$ are distinct regular words in $d_0, d_1, \cdots, d_{m-1}$, we transform this identity into a reduced identity for Z as the type given in Lemma 5. Applying Lemma 5 by replacing $\delta^{p^m-1}(\beta_i)$ with β_i and $\delta^j(\beta_i)$ with 0 for $j \neq p^m - 1$, we obtain that $f(\mu^p\beta_1 y_1, \cdots, \mu^p\beta_t y_t) = 0$ for all $\beta_i \in Z$ and hence for all $\beta_i \in C$. In particular, $f(y_1, \cdots, y_t) = 0$ as asserted. This proves the theorem.

§3. Estimation of PI–degrees

In this section we will estimate the PI–degree of a prime ring R in terms of the PI–degree of $R^{(\delta)}$ and the algebraic degree of the associated elements of δ, where δ is a derivation of R algebraic over C. We need this estimation to extend Theorem 1 to the semiprime case in the next section. The following theorem expresses the minimal polynomial of δ in terms of $\ell(X)$ and the minimal polynomial of $\mathrm{ad}(b)$.

Theorem 4. *Let R be a prime ring and δ a derivation algebraic over C and let*

$\ell(X)$ and b be as described in (1) and (1)'. If $g(X)$ is the minimal polynomial of $\mathrm{ad}(b)$ *over C, then $g(\ell(X))$ is the minimal polynomial of δ over C. In particular,* $\deg \ell(X) \deg_C b \le \deg_C \delta$.

Proof. Denote by $p(X)$ the minimal polynomial of δ over C. It is clear that $p(X)$ and $g(X)$ lie in $C^{(\delta)}[X]$, where $C^{(\delta)}$ is the subfield of constants of δ in C. Set $A(X) = g(\ell(X)) \in C^{(\delta)}[X]$. Since $\delta(\alpha_i) = 0$ for each i, we see that $A(\delta) = g(\ell(\delta)) = g(\mathrm{ad}(b)) = 0$. Thus $p(X)$ divides $A(X)$ in $C^{(\delta)}[X]$. We can write $p(X) = \sum_{i=0}^{t} \ell(X)^i q_i(X)$, where $q_i(X) \in C^{(\delta)}[X]$ and $\deg q_i(X) < \deg \ell(X)$ for each i: $q_0(X)$ is simply the remainder of $p(X)$ divided by $\ell(X)$ and $q_1(X)$ is the remainder of the quotient obtained above divided by $\ell(X)$ again and so on. Since $p(X)$ and $\ell(X)$ have their coefficients in $C^{(\delta)}$, so has each $q_i(X)$. Let $x \in R$. Then

$$0 = p(\delta)(x) = \sum_{i=0}^{t} \ell(\delta)^i q_i(\delta)(x) = \sum_{i=0}^{t} \mathrm{ad}(b)^i q_i(\delta)(x)$$

That is,

$$\sum_{i=0}^{t} \mathrm{ad}(b)^i (q_i(\delta)(X)) \tag{12}$$

is a differential identity for R. Note that $q_t(X) \neq 0$. Write $q_t(X) = \beta_s X^s + \cdots + \beta_0 \in C^{(\delta)}[X]$, where $\beta_s \neq 0$ and $0 \le s \le p^m - 1$. For $j \neq t$, write $q_j(X) = \cdots + \mu_j X^s + \cdots$. Since these $\delta, \delta^2, \cdots, \delta^{p^m-1}$ are distinct regular words in $d_0, d_1, \cdots, d_{m-1}$, applying Kharchenko's theorem to (12) by replacing $\delta^s(X)$ with the variable X and $\delta^j(X)$ with 0 for $j \neq s$ we see that $\sum_{j=0}^{t-1} \mu_j \mathrm{ad}(b)^j(X) + \beta_s \mathrm{ad}(b)^t(X)$ is a GPI for R. This implies that $g(X)$ is a divisor of $\beta_s X^t + \sum_{j=0}^{t-1} \mu_j X^j$ and, hence, $\deg g(X) \le t$. Thus

$$\deg p(X) \ge t(\deg \ell(X)) \ge \deg \ell(X) \deg g(X) = \deg A(X),$$

implying that $p(X) = A(X) = g(\ell(X))$.

To prove the last statement, it suffices to show that $\deg_C b \le \deg_C \mathrm{ad}(b)$. Set $m = \deg_C \mathrm{ad}(b)$. Then there exist $\beta_1, \cdots, \beta_{m-1} \in C$ such that $\mathrm{ad}(b)^m(x) + \beta_1 \mathrm{ad}(b)^{m-1}(x) + \cdots + \beta_{m-1}\mathrm{ad}(b)(x) = 0$ for all $x \in R$. Expanding this into a GPI and collecting terms according to their left coefficients $1, b, \cdots, b^m$, we have $b^m x + \sum_{i=1}^m b^{m-i} x v_i = 0$ for all $x \in R$, where $v_i \in Q$. In view of Lemma 1, the elements $1, b, \cdots, b^m$ are C–dependent, implying that $\deg_C b \le \deg_C \mathrm{ad}(b)$, as desired.

We are now ready to state the main theorem in this section.

Theorem 5. *Let R be a prime ring and let δ be a derivation of R algebraic over C. Suppose further that $R^{(\delta)}$ satisfies a nonzero PI over C of degree t. Then R satisfies $S_{mt}(X_1, \cdots, X_{mt})$, where m is the reduced degree of δ. In particular, R satisfies $S_{st}(X_1, \cdots, X_{st})$, where $s = \deg_C \delta$.*

For simplicity, we adopt the following terminology from Herstein's "Topics in Algebra" [7]: A $m \times n$ matrix is said to be of size m by n. An n by n square matrix is said to be of size n only. By the principal diagonal of a $m \times n$ matrix, we mean the entries in the positions (i, i), and by the super diagonal of a $m \times n$ matrix, we mean the entries in the positions $(i, i+1)$, whenever they are defined. Consider matrices over a field F. By a Jordan block belonging to $\lambda \in F$, we mean a square matrix with λ on the principal diagonal, 1's on the super diagonal and 0's elsewhere:

$$J = \begin{pmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & \lambda \end{pmatrix}. \tag{13}$$

A square matrix is said to be in the Jordan normal form, if it is of the form

$$\begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_k \end{pmatrix}, \tag{14}$$

where each J_i is a Jordan block belonging to some $\lambda_i \in F$.

Lemma 6. *Assume that R is the ring of square matrices of size n over a field F and $b \in R$. Suppose that $C_R(b)$ satisfies a nonzero* PI *over F of degree d. Then $dm \geq 2n$, where $m = \deg_F b$. In particular, R satisfies $S_{dm}(X_1, \cdots, X_{dm})$.*

Proof. Without loss of generality, we may assume that F is algebraically closed and that b is in its Jordan normal form (14).

Case 1. The minimum polynomial of b is of the form $(x-\lambda)^m$.

So each J_s in (9) is a Jordan block of size m_s belonging to λ. We may further assume that $m = m_1 \geq m_2 \geq \cdots \geq m_k$. Any element $a \in R$ may be written in the form:

$$a = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{pmatrix}, \tag{15}$$

where A_{ij} is the matrix of size m_i by m_j. For each $1 \leq i \leq j \leq k$, let $e_{ij} \in R$ be the matrix (15), written as above, such that $A_{st} = 0$ for $(s,t) \neq (i,j)$ and such that A_{ij} is the m_i by m_j matrix with 1 on its principal diagonal and 0 elsewhere. We verify easily that each $e_{ij} \in C_R(b)$ for each $1 \leq i \leq j \leq k$. A direct computation shows that

$$e_{ij}e_{st} = \begin{cases} 0, & \text{if } j \neq s; \\ e_{it}, & \text{if } j = s, \end{cases}$$

Assume towards a contradiction that the PI-degree d of $C_R(b)$ is $< 2k$. Without loss of generality, we may assume that $C_R(b)$ satisfies a multilinear PI of the form:

$$\mu(X_1, X_2, \ldots, X_{2k-1}) = X_1X_2\cdots X_{2k-1} + \cdots,$$

where the dots denote the sum of terms different from $X_1X_2\cdots X_{2k-1}$. Set $X_1 = e_{11}$, $X_2 = e_{12}$, $X_3 = e_{22}$, ..., $X_{2k-1} = e_{k,k}$. We have $\mu(e_{11}, e_{12}, e_{22}, \ldots, e_{k,k}) = e_{1k} \neq 0$, a contradiction. So the PI-degree d of $C_R(b)$ is $\geq 2k$. With this, we have $dm \geq 2km \geq 2(m_1 + m_2 + \cdots + m_k) = 2n$, as asserted.

Case 2. The minimum polynomial of b is of the form $(x-\lambda_1)^{m_1}\cdots(x-\lambda_s)^{m_s}$, where $\lambda_1, \ldots, \lambda_s$ are distinct.

We may assume that, for each $i = 1, \ldots, s$, $J_{k_{i-1}+1}, \ldots, J_{k_{i-1}+k_i}$ consist of all Jordan blocks belonging to λ_i and n_i is the sum of their sizes (set $k_0 = 0$ in the above). We may thus write b in the form:

$$b = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_s \end{pmatrix},$$

where each B_i is the n_i by n_i matrix:

$$B_i = \begin{pmatrix} J_{k_{i-1}+1} & 0 & \cdots & 0 \\ 0 & J_{k_{i-1}+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{k_{i-1}+k_i} \end{pmatrix}.$$

Let R_i be the ring of n_i by n_i matrices over F and let d_i be the PI-degree of $C_{R_i}(B_i)$. The minimal polynomial of B_i is obviously $(X-\lambda_i)^{m_i}$. We have $d_im_i \geq 2n_i$ by Case 1. Any $a \in R$ may be written in the form:

$$a = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{ss} \end{pmatrix},$$

where A_{ij} is the matrix of size n_i by n_j. A direct computation shows that $a \in C_R(b)$ if and only if $A_{ij} = 0$ for $i \neq j$ and each $A_{ii} \in C_{R_i}(B_i)$. So the PI-degree d of $C_R(b)$ is $\geq$ the PI-degree d_i of each $C_{R_i}(B_i)$. Thus $dm = d(m_1 + \cdots + m_s) \geq d_1 m_1 + \cdots d_s m_s \geq 2(n_1 + \cdots n_s) = 2n$, as asserted. By Amitsur–Levitzki's theorem [10, p.21], R satisfies $S_{2n}(X_1, \cdots, X_{2n})$ and hence it satisfies $S_{dm}(X_1 \cdots, X_{dm})$. This proves the lemma.

Proof of Theorem 5. In view of Theorem 1, R is a PI–ring. Suppose first that δ is X–inner. Thus $\delta = \mathrm{ad}(b)$ for some $b \in Q$. By Posner's theorem [10, p.57], $Q = RC$ is a finite–dimensional central simple C–algebra. Moreover, $Z(R)$, the center of R, is nonzero and C is the quotient field of $Z(R)$. A direct computation shows that $C_{RC}(b) = C_R(b)C$. Set $F = C$ if C is finite and let F denote the algebraic closure of C if C is infinite. Then $RC \otimes_C F \cong \mathrm{M}_n(F)$ for some $n \geq 1$ and $C_{RC\otimes_C F}(b \otimes 1) = C_R(b)C \otimes_C F$. Moreover, $C_R(b)C \otimes_C F$, $C_R(b)C$ and $C_R(b)$ satisfy the same PIs over C. Hence, the case is reduced to Lemma 6 and is proved.

Thus we may assume that δ is X–outer. We keep the notations explained in (1) and (1)'. In view of Theorem 3, $R^{(\delta)}$ and $C_R(b)$ satisfy the same PIs over C. So $C_R(b)$ satisfies a nonzero PI over C of degree t since $R^{(\delta)}$ does. It follows from Lemma 6 that R satisfies $S_{mt}(X_1, \cdots, X_{mt})$, where $m = \deg_C b$. But, by Theorem 4, $\deg \ell(X) \deg_C b \leq \deg_C \delta$. So $m = \deg_C(b) \leq \deg_C(\delta)$ and hence R satisfies $S_{ts}(X_1, \cdots, X_{ts})$, where $s = \deg_C \delta$, proving the theorem.

§4. The Semiprime Case

We will use Beidar and Mikhalëv's theory of orthogonal completeness [2] to push our result from the prime case to the semiprime case. We recall precisely the result we need from this theory: We will assume some familiarity with the basic notions of the first order logic with equality such as formulae, sentences (that is, formulae without free variables), conjunctions, disjunctions and quantifications. Our logical symbols are: $\vee$ (*or*), $\wedge$ (*and*), $\neg$ (*not*), $\rightarrow$ (*if ..., then ...*), $\forall$ (*for all ...*), $\exists$ (*there exists ...*) and $=$ (*equals*). By a language, we mean a set of nonlogical symbols (or, equivalently, proper symbols). The language of ring theory consists of two binary function symbols $+$ (*plus*), $\cdot$ (*times*) and a constant symbol 0. The language $\mathcal{L}$ we need here is the language of ring theory expanded by adjoining a function symbol δ intended to denote the derivation δ under consideration. For clarity, we also adopt the convention of omitting the multiplication function symbol $\cdot$ in writing formulae. The concept of *Horn* formula is defined inductively as follows:

(1) an atomic formula is a Horn formula;

(2) a disjunction of negated atomic formulae is a Horn formula;

(3) if $\Theta_1, \ldots, \Theta_n, \Theta$ are atomic formulae, then the formula $(\Theta_1 \wedge \cdots \wedge \Theta_n) \rightarrow \Theta$ is a Horn formula;

(4) if v is a variable and Θ is a Horn formula, then $\forall v\Theta$ and $\exists v\Theta$ are also Horn formulae;

(5) if Θ_1 and Θ_2 are Horn formulae, then so is $\Theta_1 \wedge \Theta_2$;

(6) all Horn formulae are obtained in this way.

A sentence of $\mathcal{L}$ is said to be *hereditary*, if its truth on any given ring implies its truth on any direct summand of this given ring. The main result of the theory of orthogonal completeness for semiprime rings is the following:

Theorem [2]. *Let R be a semiprime ring which is orthogonally complete with respect* *to the Boolean ring B consisting of all idempotents in its extended centroid C. Let $\mathcal{L}$ be the language as described above. (1) Let Θ be a sentence of $\mathcal{L}$ which is hereditary and whose negation $\neg\Theta$ is logically equivalent to a Horn sentence. If Θ holds on R, then Θ also holds on any R/P, where P is any arbitrary minimal prime ideal of R. (2) Let Θ be logically equivalent to a Horn sentence of $\mathcal{L}$. If Θ holds on all R/P, where P are minimal prime ideals of R, then Θ also holds on R.*

Assume that R is a semiprime ring and Q is its two-sided Martindale quotient ring. Let $Q^{(\delta)} = \{x \in Q \mid x^\delta = 0\}$. It is not obvious that our assumption that constants of δ satisfy a polynomial identity carries over to the orthogonal completion of R. Instead, we observe the following:

Lemma 7. *Suppose that $f(X_1, \dots, X_t)$ is a PI for $R^{(\delta)}$ with coefficients ± 1. For any $c_i \in Q^{(\delta)}$, $i = 0, \dots, n-1$, for any $b \in Q$ and for any linear polynomial $l(x) = \sum_{j=1}^m a_j x b_j$, if $\sum_{i=1}^n c_i \delta^i = \mathrm{ad}(b)$ and if b commutes with $l(x)$, then the differential identity*

$$f\Big(\sum_{i=1}^n c_i \delta^{i-1}(l(y_1)), \dots, \sum_{i=1}^n c_i \delta^{i-1}(l(y_t))\Big) \tag{16}$$

holds on Q.

Proof. Let $c_i \in Q^{(\delta)}$, $b \in Q$ and $l(x) = \sum_{j=1}^m a_j x b_j$ be as said above. For any $x \in Q$,

$$0 = [b, l(x)] = \sum_{i=1}^n c_i \delta^i(l(x)) = \delta\Big(\sum_{i=1}^n c_i \delta^{i-1}(l(x))\Big),$$

where the last equality holds since $c_i \in Q^{(\delta)}$. Pick a dense two-sided ideal I of R such that for any $x \in I$, $c_i \delta^{i-1}(\ell(x)) \in R$ for each i. For $x \in I$, $\sum_{i=1}^n c_i \delta^{i-1}(l(x)) \in R^{(\delta)}$. The identity (16) holds for $y_1, \dots, y_t \in I$ and thus also holds for $y_1, \dots, y_t \in Q$ [13, Theorem 3].

We now can prove the main theorem of this paper.

Theorem 6. *Let R be a semiprime ring and let δ be a derivation of R integral over C. Suppose further that $R^{(\delta)}$ is a* PI*-ring. Then R is a* PI*-ring. In addition, if $R^{(\delta)}$ satisfies a nonzero* PI *with coefficients ± 1 of degree t, then R satisfies $S_{st}(X_1, \cdots, X_{st})$, where s is the integral degree of δ over C.*

Proof. We consider the following formulae in our language:

$$\Theta_m: \forall \mathrm{x}\Big(\sum_{j=1}^{m} \mathrm{b}\mathrm{a}_j\mathrm{x}\mathrm{b}_j = \sum_{j=1}^{m} \mathrm{a}_j\mathrm{x}\mathrm{b}_j\mathrm{b}\Big)$$

$$\Psi_n: \delta(\mathrm{c}_1) = 0 \wedge \cdots \wedge \delta(\mathrm{c}_n) = 0 \wedge \forall \mathrm{x}\Big(\sum_{i=1}^{n} \mathrm{c}_i\delta^i(\mathrm{x}) = [\mathrm{b}, \mathrm{x}]\Big)$$

$$\Phi_{m,n}: \forall \mathrm{y}_1 \cdots \forall \mathrm{y}_t \Big(\mathrm{f}\Big(\sum_{i=1}^{n} \mathrm{c}_i\delta^{i-1}(\sum_{j=1}^{m} \mathrm{a}_j\mathrm{y}_1\mathrm{b}_j) \ \ldots, \sum_{i=1}^{n} \mathrm{c}_i\delta^{i-1}(\sum_{j=1}^{m} \mathrm{a}_j\mathrm{y}_t\mathrm{b}_j)\Big) = 0\Big).$$

Let $\mathrm{b}, \mathrm{c}_i, \mathrm{a}_j, \mathrm{b}_j$ denote $b, c_i, a_j, b_j \in Q$. Then Θ_m asserts that b commutes with the linear polynomial $\ell(x) = \sum_{j=1}^{m} a_jxb_j$, Ψ_n asserts that $c_1, \ldots, c_n \in Q^{(\delta)}$ and $\sum_{i=1}^{n} c_i\delta^i = \mathrm{ad}(b)$, and finally $\Phi_{m,n}$ asserts that the differential identity (16) holds on Q. The above lemma can thus be expressed as

$$\forall \mathrm{b}\forall \mathrm{a}_1 \cdots \mathrm{a}_m \forall \mathrm{b}_1 \cdots \mathrm{b}_m \forall \mathrm{c}_1 \ldots \mathrm{c}_n (\Theta_m \wedge \Psi_n \to \Phi_{m,n}).$$

This sentence is obvious hereditary. Its negation is equivalent to:

$$\exists \mathrm{b}\exists \mathrm{a}_1 \cdots \mathrm{a}_m \exists \mathrm{b}_1 \cdots \mathrm{b}_m \exists \mathrm{c}_1 \ldots \mathrm{c}_n (\Theta_m \wedge \Psi_n \wedge \neg\Phi_{m,n}). \tag{17}$$

Since Θ_m, Ψ_n and $\neg\Phi_{m,n}$ are obviously Horn, this negation is Horn.

By Lemma 7, the sentence (17) holds on Q, which is surely orthogonally complete. By the first part of Beidar and Mikhalëv's theorem cited above, the sentence (17) also holds on any Q/P, where P is any arbitrary minimal prime ideal of R. In other words, the assertion made in Lemma 7 is true in Q/P for any minimal prime ideal P of Q. A close examination of our argument in the prime case shows that the assertion made in Lemma 7 is what we need to prove that R satisfies the identity $S_{kt}(X_1, \ldots, X_{kt})$ there. Therefore, for any minimal prime ideal P of Q, the quotient ring Q/P also satisfies the identity $S_{kt}(X_1, \ldots, X_{kt})$, where k is the reduced degree of δ on Q/P. But the reduced degree k of δ on Q/P is obviously $\leq$ the integral degree s of δ over C by Theorem 4. So for any minimal prime ideal P of Q, the quotient ring Q/P satisfies the identity $S_{st}(X_1, \ldots, X_{st})$. Let $S_{st}(X_1, \ldots, X_{st})$ be expressed in our language by the expression

$$\mathbf{S}_{st}(\mathbf{x}_1, \ldots, \mathbf{x}_{st}).$$

The sentence

$$\forall \mathbf{x}_1 \cdots \mathbf{x}_{st} (\mathbf{S}_{st}(\mathbf{x}_1, \ldots, \mathbf{x}_{st}) = 0).$$

is obviously Horn and holds on Q/P for any minimal prime ideal P of Q. By the second part of Beidar and Mikhalëv's theorem, this sentence also holds on Q. Thus, Q also satisfies $S_{st}(X_1, \ldots, X_{st})$ as asserted.

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