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Partial pole assignment for the vibrating system with aerodynamic effect

Wen-Wei Lin[1,†] and Jenn-Nan Wang[2,*,‡]

[1] *Department of Mathematics, National Tsing-Hua University, Hsinchu 300, Taiwan*
[2] *Department of Mathematics, National Taiwan University, Taipei 106, Taiwan*

SUMMARY

The partial pole assignment (PPA) problem is the one of reassigning a few unwanted eigenvalues of a control system by feedback to suitably chosen ones, while keeping the remaining large number of eigenvalues unchanged. The problem naturally arises in modifying dynamical behaviour of the system. The PPA has been considered by several authors in the past for standard state–space systems and for quadratic matrix polynomials associated with second-order systems. In this paper, we consider the PPA for a cubic matrix polynomial arising from modelling of a vibrating system with aerodynamics effects and derive explicit formulas for feedback matrices in terms of the coefficient matrices of the polynomial. Our results generalize those of a quadratic matrix polynomial by Datta *et al.* (*Linear Algebra Appl.* 1997;**257**:29) and is based on some new orthogonality relations for eigenvectors of the cubic matrix polynomial, which also generalize the similar ones reported in Datta *et al.* (*Linear Algebra Appl.* 1997;**257**:29) for the symmetric definite quadratic pencil. Besides playing an important role in our solution for the PPA, these orthogonality relations are of independent interests, and believed to be an important contribution to linear algebra in its own right. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: aerodynamic effect; cubic pencils; partial pole assignment; orthogonality relations

1. INTRODUCTION

In this paper, we will study the vibrating system with aerodynamic effect originated from a dynamic loads analysis system (DYLOFLEX) [1]. The model is described in the form

$$M\ddot{q} + (C_1 + \zeta(s)C_2)\dot{q} + (K_1 + \zeta(s)K_2)q = H(s,t) \tag{1}$$

where M is the inertia matrix, C_1 and K_1 are the structural damping and stiffness matrices, respectively, C_2 and K_2 are aerodynamic damping and stiffness matrices, respectively. The

*Correspondence to: Jenn-Nan Wang, Department of Mathematics, National Taiwan University, Taipei 106, Taiwan
†E-mail: wwlin@am.nthu.edu.tw
‡E-mail: jnwang@math.ntu.edu.tw

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non-homogeneous term $H(s,t)$ represents the forcing function which is the combination of the generalized forces and gust inputs. In practice, the matrices M, K_1, K_2 are real positive definite and C_1, C_2 are real symmetric. Theoretically though, we assume throughout the paper that M, C_1, C_2, K_1, K_2 are real symmetric and M is non-singular. The factor $\zeta(s)$ in (1) is called the Wagner lift-growth buildup function which is due to an instantaneous change in angle of attack [2]. For our study, we take $\zeta(s)$ in the form

$$\zeta(s) = \rho + \frac{\rho}{s - \omega}$$

with constants $\rho \neq 0$ and $\omega \neq 0$. In this expression, the parameter s is interpreted as the Laplace transform parameter. Reinterpreting s as the t derivative and setting $q = x\mathrm{e}^{\lambda t}$, system (1) will lead to an open-loop cubic pencil

$$\begin{aligned} P(\lambda) = {} & M\lambda^3 + (C_1 + \rho C_2 - \omega M)\lambda^2 + [(K_1 + \rho K_2) - \omega(C_1 + \rho C_2) + \rho C_2]\lambda \\ & + [\rho K_2 - \omega(K_1 + \rho K_2)] \\ = {} & M\lambda^3 + C\lambda^2 + K\lambda + L \end{aligned} \tag{2}$$

where

$$\begin{aligned} C &= C_1 + \rho C_2 - \omega M \\ K &= (K_1 + \rho K_2) - \omega(C_1 + \rho C_2) + \rho C_2 \\ L &= \rho K_2 - \omega(K_1 + \rho K_2) \end{aligned} \tag{3}$$

On the other hand, by choosing the control force $H(s,t) = BF^{\mathrm{T}}\dot{q} + B(G_1^{\mathrm{T}} + \zeta(s)G_2^{\mathrm{T}})q$ in (1) we obtain a controlled system, which gives rise to a closed-loop cubic pencil

$$\begin{aligned} P_{\mathrm{c}}(\lambda) = {} & M\lambda^3 + [(C_1 - BF^{\mathrm{T}}) + \rho C_2 - \omega M]\lambda^2 + [(K_1 - BG_1^{\mathrm{T}}) + \rho(K_2 - BG_2^{\mathrm{T}}) \\ & - \omega(C_1 - BF^{\mathrm{T}}) - \omega\rho C_2 + \rho C_2]\lambda + [\rho(K_2 - BG_2^{\mathrm{T}}) - \omega(K_1 - BG_1^{\mathrm{T}}) \\ & - \omega\rho(K_2 - BG_2^{\mathrm{T}})] \\ = {} & M\lambda^3 + (C - BF^{\mathrm{T}})\lambda^2 + (K - BG_1^{\mathrm{T}} - \rho BG_2^{\mathrm{T}} + \omega BF^{\mathrm{T}})\lambda \\ & + (L - \rho BG_2^{\mathrm{T}} + \omega BG_1^{\mathrm{T}} + \omega\rho BG_2^{\mathrm{T}}) \end{aligned} \tag{4}$$

Here $B \in \mathbb{R}^{n\times p}$ is the control matrix and $F, G_1, G_2 \in \mathbb{R}^{n\times p}$ are the gain matrices, where $1 \leqslant p \leqslant n$. Without loss of generality, we assume throughout that B has full column rank. Let $\{\lambda_j\}_{j=1}^{3n}$ be the spectrum of $P(\lambda)$. Clearly, this is a self-conjugate set. Now let $\{\mu_j\}_{j=1}^{k}$ be another

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self-conjugate set with $1 \leqslant k < 3n$. Then the *partial pole assignment problem by state feedback control* is to find real gain matrices F, G_1, G_2 such that $\{\{\mu_j\}_{j=1}^k, \{\lambda_j\}_{j=k+1}^{3n}\} = \sigma(P_c)$. Hereafter, we denote $\sigma(Q)$ the spectrum of the pencil $Q(\lambda)$ or the spectrum of the matrix Q. In other words, one would like to use the low rank perturbations BF^T, BG_1^T and BG_2^T to assign a self-conjugate set $\{\lambda_j\}_{j=1}^k \subset \sigma(P)$ into $\{\mu_j\}_{j=1}^k$, while keeping the rest of $\sigma(P)$ unchanged. The main result of this paper is that a self-conjugate set $\{\lambda_j\}_{j=1}^k$ of $\sigma(P)$ can be assigned to prescribed self-conjugate set $\{\mu_j\}_{j=1}^k$ by appropriate choices of real matrices F, G_1 and G_2 whenever a special generalized Cauchy matrix is non-singular (see Theorem 3.2). It turns out when $p = 1$ (single-input), this generalized Cauchy matrix is invertible if and only if $x_j^T b \neq 0$ for all $1 \leqslant j \leqslant k$, where $B = b \in \mathbb{R}^{n\times 1}$, and $\{\lambda_j\}_{j=1}^k$ and $\{\mu_j\}_{j=1}^k$ are two sets with distinct elements. Further investigation shows that $x_j^T b \neq 0$ for all $1 \leqslant j \leqslant k$ and the distinctness of $\{\lambda_j\}_{j=1}^k$ are sufficient to construct a solution to the PPA problem with single-input. These conditions are exactly the conditions used in Reference [3] for the quadratic pencil.

The PPA problem by single-input state feedback control for the first- and second-order systems are studied in References [3, 4], respectively. Although it is not explicitly proved, the gain vectors found in Reference [3] are real as long as the assignable eigenvalues and the target values are self-conjugate sets. As for the PPA problem by multiple-input control for the second-order system, several results are obtained in References [5–8]. In Reference [5], the authors also present an algorithm for the PPA problem. Nevertheless, their method does not preserve the eigenvalues that we do not intend to relocate. The partial eigenstructure assignment problem is the main focus of Reference [6] in which both the eigenvalues and the eigenvectors are relocated. To solve this eigenstructure assignment problem, the control matrix B needs to be chosen as well. In Reference [8], the authors use the multiple-input control to increase the robustness of the PPA problem. Numerical evidences show that the multiple-input control out-performs the single-input control for the PPA problem. Comparing our results with those in the aforementioned papers, an obvious distinct feature is that we are dealing with a meaningful third-order system. Moreover, the number of eigenvalues we want to relocate are arbitrary and the gain matrices we choose can be shown to be real whenever both the assignable eigenvalues and the target values are self-conjugate sets. Similar results for the second-order system can be found in Reference [8, Section 3.2]. However, the fact that the gain matrices are real, which is not trivial in the multiple-input control, is not proven there.

It should be pointed out that when the input is multiple, i.e. $p > 1$, we have certain degrees of freedom in choosing the eigenvector associated with the assigned pole μ_j, $1 \leqslant j \leqslant k$. This fact paves the way for the discussion of robustness issue. The degrees of freedom in the choice of eigenvectors will obviously give rise to the degrees of freedom in the choice of gain matrices. The *robust* pole assignment problem is to choose appropriate gain matrices so that the assigned eigenvalues are as insensitive as possible to perturbations in the coefficient matrices of the closed-loop system. We will report this matter elsewhere.

This paper is organized as follows. In Section 2, we derive some orthogonality relations for the cubic pencil $P(\lambda)$. One of the orthogonality relations will play a key role in the PPA problem. In Section 3, the solutions to the PPA problem with multiple-input state feedback control for (1) are explicitly constructed. Some numerical results are provided in Section 4.

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2. ORTHOGONALITY RELATIONS FOR $P(\lambda)$

In this section, we will derive several orthogonality relations for the cubic pencil $P(\lambda)$ as in (2). Similar orthogonality relations were derived for the symmetric definite quadratic pencil in Reference [3]. If we assume that all eigenvalues of $P(\lambda)$ are distinct, then the argument in Reference [3] can be directly applied to obtain the orthogonality relations for $P(\lambda)$. However, we take a slightly different approach here. Most importantly, our method is applicable for more general eigenvalues

Let (X, Λ) and $(\tilde{X}, \tilde{\Lambda})$ be the eigenmatrix pairs of the cubic pencil $P(\lambda)$, where the matrices X, Λ, $\tilde{X}$ and $\tilde{\Lambda}$ are of sizes $n \times \ell$, $\ell \times \ell$, $n \times \tilde{\ell}$, and $\tilde{\ell} \times \tilde{\ell}$, respectively, with $2 \leqslant \ell + \tilde{\ell} \leqslant 3n$. In view of the definition of the eigenmatrix pair (see Reference [9, Chapter 6]), we have that

$$P(\Lambda)X \equiv MX\Lambda^3 + CX\Lambda^2 + KX\Lambda + LX = 0 \tag{5}$$

and

$$P(\tilde{\Lambda})\tilde{X} \equiv M\tilde{X}\tilde{\Lambda}^3 + C\tilde{X}\tilde{\Lambda}^2 + K\tilde{X}\tilde{\Lambda} + L\tilde{X} = 0 \tag{6}$$

Transposing (5) and multiplying it on the right by $\tilde{X}$ yields

$$\Lambda^3 X^{\mathrm{T}} M\tilde{X} + \Lambda^2 X^{\mathrm{T}} C\tilde{X} + \Lambda X^{\mathrm{T}} K\tilde{X} + X^{\mathrm{T}} L\tilde{X} = 0 \tag{7}$$

Next, multiplying (6) on the left by X^{T} gives

$$X^{\mathrm{T}} M\tilde{X}\tilde{\Lambda}^3 + X^{\mathrm{T}} C\tilde{X}\tilde{\Lambda}^2 + X^{\mathrm{T}} K\tilde{X}\tilde{\Lambda} + X^{\mathrm{T}} L\tilde{X} = 0 \tag{8}$$

Eliminating the term $X^{\mathrm{T}} L\tilde{X}$ in (7) and (8), we get that

$$\Lambda^3 X^{\mathrm{T}} M\tilde{X} - X^{\mathrm{T}} M\tilde{X}\tilde{\Lambda}^3 + \Lambda^2 X^{\mathrm{T}} C\tilde{X} - X^{\mathrm{T}} C\tilde{X}\tilde{\Lambda}^2 + \Lambda X^{\mathrm{T}} K\tilde{X} - X^{\mathrm{T}} K\tilde{X}\tilde{\Lambda} = 0 \tag{9}$$

Rearranging Equation (9) yields

$$\begin{aligned}
&\Lambda(\Lambda^2 X^{\mathrm{T}} M\tilde{X} + X^{\mathrm{T}} M\tilde{X}\tilde{\Lambda}^2 + \Lambda X^{\mathrm{T}} M\tilde{X}\tilde{\Lambda} + \Lambda X^{\mathrm{T}} C\tilde{X} + X^{\mathrm{T}} C\tilde{X}\tilde{\Lambda} + X^{\mathrm{T}} K\tilde{X}) \\
&\quad - (X^{\mathrm{T}} M\tilde{X}\tilde{\Lambda}^2 + \Lambda^2 X^{\mathrm{T}} M\tilde{X} + \Lambda X^{\mathrm{T}} M\tilde{X}\tilde{\Lambda} + X^{\mathrm{T}} C\tilde{X}\tilde{\Lambda} + \Lambda X^{\mathrm{T}} C\tilde{X} + X^{\mathrm{T}} K\tilde{X})\tilde{\Lambda} \\
&\quad = 0
\end{aligned} \tag{10}$$

Now if we assume that

$$\sigma(\Lambda) \cap \sigma(\tilde{\Lambda}) = \emptyset \tag{11}$$

then in view of (10) we have the first orthogonality relation

$$\Lambda^2 X^{\mathrm{T}} M\tilde{X} + X^{\mathrm{T}} M\tilde{X}\tilde{\Lambda}^2 + \Lambda X^{\mathrm{T}} M\tilde{X}\tilde{\Lambda} + \Lambda X^{\mathrm{T}} C\tilde{X} + X^{\mathrm{T}} C\tilde{X}\tilde{\Lambda} + X^{\mathrm{T}} K\tilde{X} = 0 \tag{12}$$

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Next, eliminating terms containing $X^{\mathrm{T}}K\tilde{X}$ in (7) and (8) (by multiplying (7) on the right by $\tilde{\Lambda}$ and multiplying (8) on the left by Λ) leads to

$$\Lambda(\Lambda^2 X^{\mathrm{T}}M\tilde{X}\tilde{\Lambda} + \Lambda X^{\mathrm{T}}M\tilde{X}\tilde{\Lambda}^2 + \Lambda X^{\mathrm{T}}C\tilde{X}\tilde{\Lambda} - X^{\mathrm{T}}L\tilde{X}) \\ -(\Lambda X^{\mathrm{T}}M\tilde{X}\tilde{\Lambda}^2 + \Lambda^2 X^{\mathrm{T}}M\tilde{X}\tilde{\Lambda} + \Lambda X^{\mathrm{T}}C\tilde{X}\tilde{\Lambda} - X^{\mathrm{T}}L\tilde{X})\tilde{\Lambda} = 0 \tag{13}$$

Likewise, from (13) we can obtain the second orthogonality relation

$$\Lambda X^{\mathrm{T}}M\tilde{X}\tilde{\Lambda}^2 + \Lambda^2 X^{\mathrm{T}}M\tilde{X}\tilde{\Lambda} + \Lambda X^{\mathrm{T}}C\tilde{X}\tilde{\Lambda} - X^{\mathrm{T}}L\tilde{X} = 0 \tag{14}$$

provided that (11) is satisfied. Subsequently, by eliminating terms containing $X^{\mathrm{T}}C\tilde{X}$ and $X^{\mathrm{T}}M\tilde{X}$ in (7) and (8), respectively, and using the same argument as above, we are able to derive the third orthogonality relation

$$\Lambda^2 X^{\mathrm{T}}M\tilde{X}\tilde{\Lambda}^2 - \Lambda X^{\mathrm{T}}K\tilde{X}\tilde{\Lambda} - X^{\mathrm{T}}L\tilde{X}\tilde{\Lambda} - \Lambda X^{\mathrm{T}}L\tilde{X} = 0 \tag{15}$$

and the fourth orthogonality relation

$$\Lambda^2 X^{\mathrm{T}}C\tilde{X}\tilde{\Lambda}^2 + \Lambda X^{\mathrm{T}}K\tilde{X}\tilde{\Lambda}^2 + \Lambda^2 X^{\mathrm{T}}K\tilde{X}\tilde{\Lambda} + \Lambda^2 X^{\mathrm{T}}L\tilde{X} + \Lambda X^{\mathrm{T}}L\tilde{X}\tilde{\Lambda} + X^{\mathrm{T}}L\tilde{X}\tilde{\Lambda}^2 = 0 \tag{16}$$

respectively, provided that (11) is satisfied.

To compare with the results in Reference [3] where the second-order system is treated, we disregard the aerodynamic effect, i.e. taking $\rho = 0$ in $\zeta(s)$. In this situation, Equations (3) become

$$C = C_1 - \omega M, \quad K = K_1 - \omega C_1, \quad L = -\omega K_1 \tag{17}$$

First of all, we substitute (17) into the first orthogonality relation (12) and get that

$$\Lambda^2 X^{\mathrm{T}}M\tilde{X} + X^{\mathrm{T}}M\tilde{X}\tilde{\Lambda}^2 + \Lambda X^{\mathrm{T}}M\tilde{X}\tilde{\Lambda} + \Lambda X^{\mathrm{T}}C_1\tilde{X} + X^{\mathrm{T}}C_1\tilde{X}\tilde{\Lambda} \\ + X^{\mathrm{T}}K_1\tilde{X} - \omega(\Lambda X^{\mathrm{T}}M\tilde{X} + X^{\mathrm{T}}M\tilde{X}\tilde{\Lambda} + X^{\mathrm{T}}C_1\tilde{X}) = 0 \tag{18}$$

Next, we have to ignore the effect comes from the Laplace transform parameter s which results in an extra t-differentiation. Equivalently, we would like relation (18) to hold for all $\omega \in \mathbb{R}$. This immediately implies that

$$\Lambda X^{\mathrm{T}}M\tilde{X} + X^{\mathrm{T}}M\tilde{X}\tilde{\Lambda} + X^{\mathrm{T}}C_1\tilde{X} = 0 \tag{19}$$

which is the third orthogonality relation shown in Reference [3]. Likewise, substituting (17) into (14), (15), and (16), respectively, and considering those relations to be satisfied for all $\omega \in \mathbb{R}$, we can obtain that

$$\Lambda X^{\mathrm{T}}M\tilde{X}\tilde{\Lambda} - X^{\mathrm{T}}K_1\tilde{X} = 0 \tag{20}$$

$$\Lambda X^{\mathrm{T}}C_1\tilde{X}\tilde{\Lambda} + X^{\mathrm{T}}K_1\tilde{X}\tilde{\Lambda} + \Lambda X^{\mathrm{T}}K_1\tilde{X} = 0 \tag{21}$$

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and

$$\Lambda^2 X^{\mathrm{T}} M \tilde{X} \tilde{\Lambda}^2 + \Lambda X^{\mathrm{T}} C_1 \tilde{X} \tilde{\Lambda}^2 + \Lambda^2 X^{\mathrm{T}} C_1 \tilde{X} \tilde{\Lambda} + \Lambda^2 X^{\mathrm{T}} K_1 \tilde{X} + \Lambda X^{\mathrm{T}} K_1 \tilde{X} \tilde{\Lambda} + X^{\mathrm{T}} K_1 \tilde{X} \tilde{\Lambda}^2 = 0 \tag{22}$$

respectively. Formulas (20) and (21) are essentially the first and second orthogonality relations obtained in Reference [3]. Notice that (22) is redundant and it can be derived by (20) and (21) immediately. It should be pointed out that the orthogonality relations derived in Reference [3] are based on the assumption that all eigenvalues are distinct. This condition is, apparently, more restrictive than (11).

3. SOLUTIONS TO THE PPA PROBLEM

In this section, we will show how to solve the PPA problem by multiple-input feedback control for the cubic pencil $P(\lambda)$. For this case, we aim to relocate k eigenvalues of $P(\lambda)$. More precisely, let $\{\lambda_j\}_{j=1}^k \subset \sigma(P)$ and $\{\mu_j\}_{j=1}^k$ $(1 \leqslant k \leqslant 3n)$ be two self-conjugate sets of complex numbers. Then we want to find appropriate $F, G_1, G_2 \in \mathbb{R}^{n\times p}$ such that $\{\mu_1, \ldots, \mu_k, \lambda_{k+1}, \ldots, \lambda_{3n}\} = \sigma(P_c)$. Notice that the eigenvalues $\{\lambda_j\}_{j=k+1}^{3n}$ of $P(\lambda)$ remain unchanged in the feedback control. Following the ideas in Reference [3], we shall use one of the orthogonality relations derived in the previous section to find the forms of F, G_1, G_2 such that $\{\lambda_j\}_{j=k+1}^{3n} \in \sigma(P_c)$. With abuse of notations, let us define

$$\begin{aligned} \Lambda &= \mathrm{diag}(\lambda_1, \ldots, \lambda_k) \\ \sigma(\tilde{\Lambda}) &= \{\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_{3n}\} \\ X &= [x_1, \ldots, x_k] \\ \tilde{X} &= [x_{k+1}, x_{k+2}, \ldots, x_{3n}] \end{aligned}$$

where $\{(\lambda_j, x_j)\}_{j=1}^k$ and $(\tilde{\Lambda}, \tilde{X})$ are eigenpairs and an eigenmatrix pair of $P(\lambda)$, respectively. Now we can prove that

Theorem 3.1
Assume that (11) is satisfied, i.e.

$$\sigma(\Lambda) \cap \sigma(\tilde{\Lambda}) = \emptyset$$

Let $\rho \neq 0$, $\xi \in \mathbb{C}^{k\times p}$, and

$$\begin{aligned} F &= MX\Lambda\xi \\ G_1 &= [-\omega(K_1 + \rho K_2) + \rho K_2]X\xi + (1-\omega)[MX\Lambda^2 + (C_1 + \rho C_2)X\Lambda]\xi \\ G_2 &= \frac{1}{\rho}\{[\omega(K_1 + \rho K_2) - \rho K_2]X\xi + \omega[MX\Lambda^2 + (C_1 + \rho C_2)X\Lambda]\xi\} \end{aligned} \tag{23}$$

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then

$$\begin{aligned}P_c(\tilde{\Lambda})\tilde{X} &= M\tilde{X}\tilde{\Lambda}^3 + [(C_1 - BF^T) + \rho C_2 - \omega M]\tilde{X}\tilde{\Lambda}^2 + [(K_1 - BG_1^T) + \rho(K_2 - BG_2^T)\\ &\quad - \omega(C_1 - BF^T) - \omega\rho C_2 + \rho C_2]\tilde{X}\tilde{\Lambda} + [\rho(K_2 - BG_2^T) - \omega(K_1 - BG_1^T)\\ &\quad - \omega\rho(K_2 - BG_2^T)]\tilde{X}\\ &= 0\end{aligned}$$

That is, $(\tilde{\Lambda}, \tilde{X})$ is an eigenmatrix pair of the cubic pencil $P_c(\lambda)$.

Proof
This is the place where we use one of the orthogonality relations. Since

$$M\tilde{X}\tilde{\Lambda}^3 + C\tilde{X}\tilde{\Lambda}^2 + K\tilde{X}\tilde{\Lambda} + L\tilde{X} = 0$$

where M, C, K, L are given in (3), by using (23), we obtain that

$$\begin{aligned}P_c(\tilde{\Lambda})\tilde{X} &= M\tilde{X}\tilde{\Lambda}^3 + C\tilde{X}\tilde{\Lambda}^2 + K\tilde{X}\tilde{\Lambda} + L\tilde{X}\\ &\quad - \{BF^T\tilde{X}\tilde{\Lambda} + [(BG_1^T + \rho BG_2^T) - \omega BF^T]\tilde{X}\tilde{\Lambda}\\ &\quad + [\rho BG_2^T - \omega(BG_1^T + \rho BG_2^T)]\tilde{X}\}\\ &= -B\xi^T\{\Lambda X^T M\tilde{X}\tilde{\Lambda}^2 + [\Lambda^2 X^T M + \Lambda X^T(C_1 + \rho C_2 - \omega M)]\tilde{X}\tilde{\Lambda}\\ &\quad + X^T[\omega(K_1 + \rho K_2) - \rho K_2]\tilde{X}\}\\ &= 0 \end{aligned} \tag{24}$$

Note that the last equality in (24) comes from the second orthogonality relation (14), i.e.

$$\begin{aligned}&\Lambda X^T M\tilde{X}\tilde{\Lambda}^2 + \Lambda^2 X^T M\tilde{X}\tilde{\Lambda} + \Lambda X^T C\tilde{X}\tilde{\Lambda} - X^T L\tilde{X}\\ &\quad = \Lambda X^T M\tilde{X}\tilde{\Lambda} + [\Lambda^2 X^T M + \Lambda X^T(C_1 + \rho C_2 - \omega M)]\tilde{X}\tilde{\Lambda}\\ &\qquad + X^T[\omega(K_1 + \rho K_2) - \rho K_2]\tilde{X}\\ &\quad = 0\end{aligned}$$

provided that (11) is satisfied. □

Theorem 3.1 implies that if the modes $\{\lambda_j\}_{j=1}^k$ that we want to relocate are entirely different from other eigenvalues of $P(\lambda)$, then the choices F, G_1, G_2 in (23) will keep the rest of eigenvalues (also eigenvectors) of $P(\lambda)$ unchanged. Next, it remains to prove that the modes

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$\{\lambda_j\}_{i=1}^k$ can be assigned to the appropriate prescribed values $\{\mu_j\}_{i=1}^k$ with the real matrices F, G_1 and G_2 defined by (23). Similar to the technique used in Reference [3], we will show that this can be done by choosing appropriate ξ.

To begin with, let the self-conjugate set $\{\lambda_j\}_{j=1}^k$ be arranged in the following way:

$$\{\lambda_j\}_{j=1}^k = \{\{\lambda_{2\ell-1}, \lambda_{2\ell}\}_{\ell=1}^{m_1}, \{\lambda_j\}_{j=2m_1+1}^k\}$$

where $0 \leqslant m_1 \leqslant k/2$, $\{\lambda_{2\ell-1}, \lambda_\ell\}_{\ell=1}^{m_1}$ are pairs of conjugate complex numbers with non-zero imaginary parts, and $\{\lambda_j\}_{j=2m_1+1}^k$ are all real numbers. We aim to assign $\{\lambda_j\}_{j=1}^k$ into a self-conjugate set of complex numbers $\{\mu_j\}_{j=1}^k$. Of course, we assume that

$$\{\mu_j\}_{j=1}^k \cap \sigma(P) = \emptyset \tag{25}$$

Likewise, we put

$$\{\mu_j\}_{j=1}^k = \{\{\mu_{2r-1}, \mu_{2r}\}_{r=1}^{m_2}, \{\mu_j\}_{j=2m_2+1}^k\}$$

where $0 \leqslant m_2 \leqslant k/2$, $\{\mu_{2r-1}, \mu_{2r}\}_{r=1}^{m_2}$ are pairs of conjugate complex numbers with non-zero imaginary parts, and $\{\mu_j\}_{j=2m_2+1}^k$ are all real numbers. Correspondingly, the eigenvectors associated with $\{\lambda_j\}_{j=1}^k$ are grouped into

$$\{\{x_{2\ell-1}, x_{2\ell}\}_{\ell=1}^{m_1}, \{x_j\}_{j=2m_1+1}^k\}$$

where $x_{2\ell-1} = \bar{x}_{2\ell}$ for all $1 \leqslant \ell \leqslant m_1$ and $\{x_j\}_{j=2m_1+1}^k$ are real vectors. Notice that m_1 is not necessarily equal to m_2. Now suppose that $\mathscr{U} = [u_1, \ldots, u_k]$ is a $p \times k$ complex matrix with column vectors $u_j \neq 0$ satisfying

$$\begin{cases} u_{2r-1} = \bar{u}_{2r} & \text{for } 1 \leqslant r \leqslant m_2 \\ u_j \in \mathbb{R}^{p\times 1} & \text{for } 2m_2 + 1 \leqslant j \leqslant k \end{cases} \tag{26}$$

Subsequently, let z_j be the jth column of $B\mathscr{U}$, i.e. $z_j = Bu_j \neq 0$ for $1 \leqslant j \leqslant k$. Notice that $\{z_j\}_{j=1}^k$ satisfy the same relations as in (26). In view of (25), we define

$$y_j = P(\mu_j)^{-1} z_j, \quad 1 \leqslant j \leqslant k$$

That is, y_j satisfies

$$P(\mu_j) y_j = M y_j \mu_j^3 + C y_j \mu_j^2 + K y_j \mu_j + L y_j = z_j, \quad 1 \leqslant j \leqslant k$$

Notice that $y_j \neq 0$ for all $1 \leqslant j \leqslant k$. In the following, we shall show that y_j is an eigenvector of the closed-loop cubic pencil $P_c(\lambda)$ related to the pole μ_j. It should be noted that the degrees of freedom in the choice of y_j is reflected by the degrees of freedom in choosing z_j for any given μ_j. Finally, we denote

$$\mathscr{C} = \begin{bmatrix} \dfrac{x_1^{\mathrm{T}} z_1}{\mu_1 - \lambda_1} & \cdots & \dfrac{x_1^{\mathrm{T}} z_k}{\mu_k - \lambda_1} \\ \vdots & \ddots & \vdots \\ \dfrac{x_k^{\mathrm{T}} z_1}{\mu_1 - \lambda_k} & \cdots & \dfrac{x_k^{\mathrm{T}} z_k}{\mu_k - \lambda_k} \end{bmatrix} = \begin{bmatrix} \dfrac{x_1^{\mathrm{T}} Bu_1}{\mu_1 - \lambda_1} & \cdots & \dfrac{x_1^{\mathrm{T}} Bu_k}{\mu_k - \lambda_1} \\ \vdots & \ddots & \vdots \\ \dfrac{x_k^{\mathrm{T}} Bu_1}{\mu_1 - \lambda_k} & \cdots & \dfrac{x_k^{\mathrm{T}} Bu_k}{\mu_k - \lambda_k} \end{bmatrix} \in \mathbb{C}^{k\times k}$$

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which is a generalized Cauchy matrix [10]. Now we are in a position to show that eigenpairs $\{(\lambda_j, x_j)\}_{j=1}^k$ of $P(\lambda)$ can be assigned to $\{(\mu_j, y_j)\}_{j=1}^k$ by the choices of real gain matrices F, G_1, G_2 in (23) with an appropriate ξ provided that $\mathscr{C}$ is invertible.

Theorem 3.2
Assume that the sets $\{(\lambda_j, x_j), \mu_j\}_{j=1}^k$ and the matrix $\mathscr{U} = [u_1, \ldots, u_k]$ defined by (26) satisfy

$$\det \mathscr{C} \neq 0 \tag{27}$$

and $\{\lambda_j\}_{j=1}^k$ are all non-zero. Let F, G_1, G_2 be chosen as in (23) with

$$\xi^{\mathrm{T}} = \beta \Lambda^{-1} \tag{28}$$

where $\beta = \mathscr{U}\mathscr{C}^{-1} \in \mathbb{C}^{p \times k}$, then $\{(\mu_j, y_j)\}_{j=1}^k$ are eigenpairs of $P_{\mathrm{c}}(\lambda)$. Moreover, the gain matrices F, G_1 and G_2 are real.

Proof
First of all, we want to check that $\{(\mu_j, y_j)\}_{j=1}^k$ are eigenpairs of $P_{\mathrm{c}}(\lambda)$. To this end, we compute

$$\begin{aligned}
P_{\mathrm{c}}(\mu_j) y_j = {} & M y_j \mu_j^3 + [(C_1 - BF^{\mathrm{T}}) + \rho C_2 - \omega M] y_j \mu_j^2 + [(K_1 - BG_1^{\mathrm{T}}) + \rho(K_2 - BG_2^{\mathrm{T}}) \\
& - \omega(C_1 - BF^{\mathrm{T}}) - \omega\rho C_2 + \rho C_2] y_j \mu_j + [\rho(K_2 - BG_2^{\mathrm{T}}) - \omega(K_1 - BG_1^{\mathrm{T}}) \\
& - \omega\rho(K_2 - BG_2^{\mathrm{T}})] y_j \\
= {} & z_j - B\{F^{\mathrm{T}} y_j \mu_j^2 + (G_1^{\mathrm{T}} + \rho G_2^{\mathrm{T}} - \omega F^{\mathrm{T}}) y_j \mu_j + [\rho G_2^{\mathrm{T}} - \omega(G_1^{\mathrm{T}} + \rho G_2^{\mathrm{T}})] y_j\} \\
= {} & z_j - B\xi^{\mathrm{T}}\{\Lambda X^{\mathrm{T}} M y_j \mu_j^2 + [\Lambda^2 X^{\mathrm{T}} M + \Lambda X^{\mathrm{T}}(C_1 + \rho C_2 - \omega M)] y_j \mu_j \\
& + X^{\mathrm{T}}[\omega(K_1 + \rho K_2) - \rho K_2] y_j\} \\
= {} & z_j - B\xi^{\mathrm{T}}\{\Lambda X^{\mathrm{T}} M y_j \mu_j^2 + (\Lambda^2 X^{\mathrm{T}} M + \Lambda X^{\mathrm{T}} C) y_j \mu_j - X^{\mathrm{T}} L y_j\}
\end{aligned} \tag{29}$$

By virtue of the relation

$$\Lambda X^{\mathrm{T}} K = -(\Lambda^3 X^{\mathrm{T}} M + \Lambda^2 X^{\mathrm{T}} C + X^{\mathrm{T}} L)$$

we can get

$$
\begin{aligned}
\Lambda X^{\mathrm{T}} z_j &= \Lambda X^{\mathrm{T}}(M y_j \mu_j^3 + C y_j \mu_j^2 + K y_j \mu_j + L y_j) \\
&= \Lambda X^{\mathrm{T}} M y_j \mu_j^3 + \Lambda X^{\mathrm{T}} C y_j \mu_j^2 - (\Lambda^3 X^{\mathrm{T}} M + \Lambda^2 X^{\mathrm{T}} C + X^{\mathrm{T}} L) y_j \mu_j + \Lambda X^{\mathrm{T}} L y_j \\
&= \Lambda X^{\mathrm{T}} M y_j \mu_j^3 + \Lambda^2 X^{\mathrm{T}} M y_j \mu_j^2 + \Lambda X^{\mathrm{T}} C y_j \mu_j^2 - X^{\mathrm{T}} L y_j \mu_j \\
&\quad - (\Lambda^3 X^{\mathrm{T}} M y_j \mu_j + \Lambda^2 X^{\mathrm{T}} M y_j \mu_j^2 + \Lambda^2 X^{\mathrm{T}} C y_j \mu_j - \Lambda X^{\mathrm{T}} L y_j) \\
&= \mu_j (\Lambda X^{\mathrm{T}} M y_j \mu_j^2 + \Lambda^2 X^{\mathrm{T}} M y_j \mu_j + \Lambda X^{\mathrm{T}} C y_j \mu_j - X^{\mathrm{T}} L y_j) \\
&\quad - \Lambda(\Lambda^2 X^{\mathrm{T}} M y_j \mu_j + \Lambda X^{\mathrm{T}} M y_j \mu_j^2 + \Lambda X^{\mathrm{T}} C y_j \mu_j - X^{\mathrm{T}} L y_j), \quad 1 \leqslant j \leqslant k
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
\Lambda X^{\mathrm{T}} M y_j \mu_j^2 + \Lambda^2 X^{\mathrm{T}} M y_j \mu_j + \Lambda X^{\mathrm{T}} C y_j \mu_j - X^{\mathrm{T}} L y_j &= (\mu_j I - \Lambda)^{-1} \Lambda X^{\mathrm{T}} z_j \\
&= \Lambda(\mu_j I - \Lambda)^{-1} X^{\mathrm{T}} z_j
\end{aligned} \tag{30}
$$

Therefore, if $\xi^{\mathrm{T}} = \beta \Lambda^{-1}$ with $\beta = \mathscr{U}\mathscr{C}^{-1}$, then we obtain from (29) and (30) that

$$
P_{\mathrm{c}}(\mu_j) y_j = z_j - B\beta\Lambda^{-1}\Lambda(\mu_j I - \Lambda)^{-1} X^{\mathrm{T}} z_j = z_j - B\beta(\mu_j I - \Lambda)^{-1} X^{\mathrm{T}} z_j = z_j - B u_j = 0
$$

for all $1 \leqslant j \leqslant k$. In other words, we have that $\{\mu_j\}_{j=1}^{k} \subset \sigma(P_{\mathrm{c}})$.

Now we want to show that F, G_1 and G_2 are real matrices. In view of their structures, it suffices to show that matrices $X\Lambda\xi$, $X\xi$ and $X\Lambda^2\xi$ are real. It is useful to look at the form of $\bar{\mathscr{C}}$. We can see that

$$
\bar{\mathscr{C}} = \begin{bmatrix}
\dfrac{\bar{x}_1^{\mathrm{T}} \bar{z}_1}{\bar{\mu}_1 - \bar{\lambda}_1} & \dfrac{\bar{x}_1^{\mathrm{T}} \bar{z}_2}{\bar{\mu}_2 - \bar{\lambda}_1} & \cdots & \dfrac{\bar{x}_1^{\mathrm{T}} z_{2m_2+1}}{\mu_{2m_2+1} - \bar{\lambda}_1} & \cdots & \dfrac{\bar{x}_1^{\mathrm{T}} z_k}{\mu_k - \bar{\lambda}_1} \\
\dfrac{\bar{x}_2^{\mathrm{T}} \bar{z}_1}{\bar{\mu}_1 - \bar{\lambda}_2} & \dfrac{\bar{x}_2^{\mathrm{T}} \bar{z}_2}{\bar{\mu}_2 - \bar{\lambda}_2} & \cdots & \dfrac{\bar{x}_2^{\mathrm{T}} z_{2m_2+1}}{\mu_{2m_2+1} - \bar{\lambda}_2} & \cdots & \dfrac{\bar{x}_2^{\mathrm{T}} z_k}{\mu_k - \bar{\lambda}_2} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\dfrac{x_{2m_1+1}^{\mathrm{T}} \bar{z}_1}{\bar{\mu}_1 - \lambda_{2m_1+1}} & \dfrac{x_{2m_1+1}^{\mathrm{T}} \bar{z}_2}{\bar{\mu}_2 - \lambda_{2m_1+1}} & \cdots & \dfrac{x_{2m_1+1}^{\mathrm{T}} z_{2m_2+1}}{\mu_{2m_2+1} - \lambda_{2m_1+1}} & \cdots & \dfrac{x_{2m_1+1}^{\mathrm{T}} z_k}{\mu_k - \lambda_{2m_1+1}} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\dfrac{x_k^{\mathrm{T}} \bar{z}_1}{\bar{\mu}_1 - \lambda_k} & \dfrac{x_k^{\mathrm{T}} \bar{z}_2}{\bar{\mu}_2 - \lambda_k} & \cdots & \dfrac{x_k^{\mathrm{T}} z_{2m_2+1}}{\mu_{2m_2+1} - \lambda_k} & \cdots & \dfrac{x_k^{\mathrm{T}} z_k}{\mu_k - \lambda_k}
\end{bmatrix}
$$

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On account of the complex conjugation, we can see that

$$\bar{\mathscr{C}} = \begin{bmatrix} \dfrac{x_2^{\mathrm{T}} z_2}{\mu_2 - \lambda_2} & \dfrac{x_2^{\mathrm{T}} z_1}{\mu_1 - \lambda_2} & \cdots & \dfrac{x_2^{\mathrm{T}} z_{2m_2+1}}{\mu_{2m_2+1} - \lambda_2} & \cdots & \dfrac{x_2^{\mathrm{T}} z_k}{\mu_k - \lambda_2} \\ \dfrac{x_1^{\mathrm{T}} z_2}{\mu_2 - \lambda_1} & \dfrac{x_1^{\mathrm{T}} z_1}{\mu_1 - \lambda_1} & \cdots & \dfrac{x_1^{\mathrm{T}} z_{2m_2+1}}{\mu_{2m_2+1} - \lambda_1} & \cdots & \dfrac{x_1^{\mathrm{T}} z_k}{\mu_k - \lambda_1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \dfrac{x_{2m_1+1}^{\mathrm{T}} z_2}{\mu_2 - \lambda_{2m_1+1}} & \dfrac{x_{2m_1+1}^{\mathrm{T}} z_1}{\mu_1 - \lambda_{2m_1+1}} & \cdots & \dfrac{x_{2m_1+1}^{\mathrm{T}} z_{2m_2+1}}{\mu_{2m_2+1} - \lambda_{2m_1+1}} & \cdots & \dfrac{x_{2m_1+1}^{\mathrm{T}} z_k}{\mu_k - \lambda_{2m_1+1}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \dfrac{x_k^{\mathrm{T}} z_2}{\mu_2 - \lambda_k} & \dfrac{x_k^{\mathrm{T}} z_1}{\mu_1 - \lambda_k} & \cdots & \dfrac{x_k^{\mathrm{T}} z_{2m_2+1}}{\mu_{2m_2+1} - \lambda_k} & \cdots & \dfrac{x_k^{\mathrm{T}} z_k}{\mu_k - \lambda_k} \end{bmatrix} \tag{31}$$

Let $S_{\mathrm{col}} \in \mathbb{R}^{k\times k}$ be the matrix such that for any $A \in \mathbb{C}^{k\times k}$ AS_{col} is equal to the matrix derived by swapping the $(2r-1)$th and $(2r)$th columns of A for all r with $1 \leqslant r \leqslant m_2$. Likewise, we define $S_{\mathrm{row}} \in \mathbb{R}^{k\times k}$ to be the matrix such that $S_{\mathrm{row}}A$ is obtained by exchanging the $(2\ell-1)$th and (2ℓ)th rows of A for all ℓ with $1 \leqslant \ell \leqslant m_1$. Using S_{col} and S_{row}, we can get

$$\bar{\beta}\bar{\mathscr{C}}S_{\mathrm{col}} = \bar{\beta}S_{\mathrm{row}}^{-1}S_{\mathrm{row}}\bar{\mathscr{C}}S_{\mathrm{col}} = \bar{\mathscr{U}}S_{\mathrm{col}}$$

From (31) it follows that $S_{\mathrm{row}}\bar{\mathscr{C}}S_{\mathrm{col}} = \mathscr{C}$. Similarly, the definition of $\mathscr{U}$ implies $\bar{\mathscr{U}}S_{\mathrm{col}} = \mathscr{U}$. Consequently, we obtain that

$$\bar{\beta}S_{\mathrm{row}}^{-1} = \mathscr{U}\mathscr{C}^{-1} = \beta \tag{32}$$

Now let $\beta = [\beta_1, \ldots, \beta_k]$ with $\beta_j \in \mathbb{C}^{p\times 1}$ for $1 \leqslant j \leqslant k$, then relation (32) is equivalent to

$$\bar{\beta}_{2\ell-1} = \beta_{2\ell} \text{ for } 1 \leqslant \ell \leqslant m_1 \quad \text{and} \quad \beta_j \in \mathbb{R}^{p\times 1} \text{ for } 2m_1+1 \leqslant j \leqslant k \tag{33}$$

Calculating $X\Lambda\xi$, $X\xi$ and $X\Lambda^2\xi$ reveal that

$$\begin{aligned} X\Lambda\xi &= X\Lambda\Lambda^{-1}\beta^{\mathrm{T}} = \sum_{\ell=1}^{m_1}(x_{2\ell-1}\beta_{2\ell-1}^{\mathrm{T}} + x_{2\ell}\beta_{2\ell}^{\mathrm{T}}) + \sum_{j=2m_1+1}^{k} x_j\beta_j^{\mathrm{T}} \\ X\xi &= X\Lambda^{-1}\beta^{\mathrm{T}} = \sum_{\ell=1}^{m_1}(x_{2\ell-1}\lambda_{2\ell-1}^{-1}\beta_{2\ell-1}^{\mathrm{T}} + x_{2\ell}\lambda_{2\ell}^{-1}\beta_{2\ell}^{\mathrm{T}}) + \sum_{j=2m_1+1}^{k} x_j\lambda_j^{-1}\beta_j^{\mathrm{T}} \\ X\Lambda^2\xi &= X\Lambda^2\Lambda^{-1}\beta^{\mathrm{T}} = \sum_{\ell=1}^{m_1}(x_{2\ell-1}\lambda_{2\ell-1}\beta_{2\ell-1}^{\mathrm{T}} + x_{2\ell}\lambda_{2\ell}\beta_{2\ell}^{\mathrm{T}}) + \sum_{j=2m_1+1}^{k} x_j\lambda_j\beta_j^{\mathrm{T}} \end{aligned} \tag{34}$$

Combining (33) and (34), we find that $X\Lambda\xi$, $X\xi$ and $X\Lambda^2\xi$ are real matrices. The proof of theorem is now complete. □

Now we would like to make some remarks on condition (27) in Theorem 3.2. It is clear to see that if $x_j^{\mathrm{T}}B = 0$ for any j with $1 \leqslant j \leqslant k$, then the matrix $\mathscr{C}$ is never invertible regardless

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how we choose $\mathscr{U}$. In control theory, $x_j^{\mathrm{T}}B=0$ corresponds to the fact that the mode λ_j is *uncontrollable*. In other words, this eigenmode cannot be replaced. On the contrary, if $x_j^{\mathrm{T}}B\neq 0$ for all $1\leqslant j\leqslant k$ and the elements of the set $\{\lambda_j,\mu_j\}_{j=1}^k$ are distinct, then $\mathscr{C}$ is invertible for some choices of $\mathscr{U}$. To see this, we first recall that the determinant of the usual Cauchy matrix

$$\mathscr{C}_0=\begin{bmatrix}\dfrac{1}{\mu_1-\lambda_1} & \cdots & \dfrac{1}{\mu_k-\lambda_1}\\ \vdots & \ddots & \vdots\\ \dfrac{1}{\mu_1-\lambda_k} & \cdots & \dfrac{1}{\mu_k-\lambda_k}\end{bmatrix}$$

is given by

$$\det\mathscr{C}_0=\frac{\prod_{1\leqslant i<j\leqslant k}(\mu_i-\mu_j)\prod_{1\leqslant i<j\leqslant k}(\lambda_i-\lambda_j)}{\prod_{i,j=1}^k(\mu_i-\lambda_j)} \tag{35}$$

(see e.g. Reference [11]). Hence, $\mathscr{C}_0$ is non-singular when elements of $\{\lambda_j,\mu_j\}_{j=1}^k$ are distinct. Since $x_j^{\mathrm{T}}B\neq 0$ for all $1\leqslant j\leqslant k$, we can find a non-singular real $p\times p$ matrix R such that the first entries of $x_j^{\mathrm{T}}BR$ are not zero for all $1\leqslant j\leqslant k$. Now we choose $u_1=u_2=\cdots=u_k=R[1,0,\ldots,0]^{\mathrm{T}}\in\mathbb{R}^{p\times 1}$. Then we get that

$$x_j^{\mathrm{T}}Bu_i=x_j^{\mathrm{T}}BR[1,0,\ldots,0]^{\mathrm{T}}=a_j\neq 0,\quad 1\leqslant i,j\leqslant k$$

which implies

$$\det\mathscr{C}=\left(\prod_{j=1}^k a_j\right)\det\mathscr{C}_0\neq 0$$

In addition, it is well known that the Cauchy matrix is ill-conditioned. To compute the solution β for linear system $\beta\mathscr{C}=\mathscr{U}$ more accurately and stably, we use the following strategies. Let $\mathscr{C}=\mathfrak{X}\Sigma\mathfrak{Y}^*$ be the SVD of $\mathscr{C}$, where $\mathfrak{X}$, $\mathfrak{Y}$ are unitary and $\Sigma=\mathrm{diag}(\sigma_1,\ldots,\sigma_k)$ with $\sigma_1\geqslant\cdots\geqslant\sigma_k\geqslant 0$. If $\mathscr{C}$ is nearly singular, i.e. $\sigma_k\approx 0$, then we adopt Chan's idea [12] in which a deflated decomposition of the solution to a nearly singular system were introduced. On the other hand, if $\sigma_1\gg 1$ and $\sigma_k\approx O(1)$, i.e. $\mathscr{C}$ has a bad condition number, we compute β by

$$\beta=\mathscr{U}\mathfrak{Y}\Sigma^{-1}\mathfrak{X}^* \tag{36}$$

Notice that all unitary operations are numerically stable and Σ^{-1} in (36) can be explicitly calculated.

As indicated in Reference [3], the assumption that all $\{\lambda_j\}_{j=1}^k$ are all non-zero in Theorem 3.2 can be removed because we always can construct a new cubic pencil $\tilde{P}(\lambda)$ out of the original one $P(\lambda)$ so that all eigenvalues of $\tilde{P}(\lambda)$ do not vanish. To see this, let the new cubic pencil $\tilde{P}(\lambda)$ be defined by

$$\tilde{P}(\lambda)=M\lambda^3+\tilde{C}\lambda^2+\tilde{K}\lambda+\tilde{L}$$

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where

$$\tilde{C} = C + 3\eta M$$

$$\tilde{K} = K + 2\eta C + 3\eta^2 M$$

$$\tilde{L} = L + \eta K + \eta^2 C + \eta^3 M$$

then it can be easily verified that $\{\lambda_j - \eta\}_{j=1}^{3n} = \sigma(\tilde{P})$.

According to the last remark, if we want to move $\{\lambda_i\}_{j=1}^{k}$ to $\{\mu_j\}_{j=1}^{k}$ and some of $\{\lambda_j\}_{j=1}^{k}$ vanish, then we can first construct a shifted pencil

$$\tilde{P}(\lambda) = M\lambda^3 + \tilde{C}\lambda^2 + \tilde{K}\lambda + \tilde{L}$$

with a real shift $\eta \neq 0$ such that all $\{\lambda_j - \eta\}_{j=1}^{k}$ are not zero. It should be mentioned that if $P(\lambda_j)x_j = 0$, then $\tilde{P}(\lambda_j - \eta)x_j = 0$ as well. Therefore, the generalized Cauchy matrix $\mathscr{C}$ does not change when $\{\lambda_j, \mu_j\}_{j=1}^{k}$ are shifted to $\{\lambda_j - \eta, \mu_j - \eta\}_{j=1}^{k}$. Moreover, if the original pencil $P(\lambda)$ is symmetric, then so is the shifted pencil $\tilde{P}(\lambda)$. Thus, we can perform the pole assignment for $\tilde{P}(\lambda)$. To restore the shift, we simply add η to each eigenvalues of the shifted closed pencil $\tilde{P}_c(\lambda)$. More precisely, let $\tilde{F}, \tilde{G}_1, \tilde{G}_2$ be the feedback matrices for $\tilde{P}(\lambda)$, then the shifted closed pencil becomes

$$\begin{aligned}\tilde{P}_c(\lambda) &= M\lambda^3 + (\tilde{C} - B\tilde{F}^{\mathrm{T}})\lambda^2 + (\tilde{K} - B\tilde{G}_1^{\mathrm{T}} - \rho B\tilde{G}_2^{\mathrm{T}} + \omega B\tilde{F}^{\mathrm{T}})\lambda \\ &\quad + (\tilde{L} - \rho B\tilde{G}_2^{\mathrm{T}} + \omega B\tilde{G}_1^{\mathrm{T}} + \omega\rho B\tilde{G}_2^{\mathrm{T}}) \\ &= \mathrm{M}\lambda^3 + \hat{C}\lambda^2 + \hat{K}\lambda + \hat{L}\end{aligned}$$

where

$$\hat{C} = \tilde{C} - B\tilde{F}^{\mathrm{T}}$$

$$\hat{K} = \tilde{K} - B\tilde{G}_1^{\mathrm{T}} - \rho B\tilde{G}_2^{\mathrm{T}} + \omega B\tilde{F}^{\mathrm{T}}$$

$$\hat{L} = \tilde{L} - \rho(1 - \omega)B\tilde{G}_2^{\mathrm{T}} + \omega B\tilde{G}_1^{\mathrm{T}}$$

Restoring the shift η gives

$$\begin{aligned}P_c(\lambda) &= M\lambda^3 + (\hat{C} - 3\eta M)\lambda^2 + (\hat{K} - 2\eta\hat{C} + 3\eta^2 M)\lambda + (\hat{L} - \eta\hat{K} + \eta^2\hat{C} - \eta^3 M) \\ &= M\lambda^3 + (\tilde{C} - B\tilde{F}^{\mathrm{T}} - 3\eta M)\lambda^2 \\ &\quad + [\tilde{K} - B\tilde{G}_1^{\mathrm{T}} - \rho B\tilde{G}_2^{\mathrm{T}} + \omega B\tilde{F}^{\mathrm{T}} - 2\eta(\tilde{C} - B\tilde{F}^{\mathrm{T}}) + 3\eta^2 M]\lambda\end{aligned}$$

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$$
\begin{aligned}
&+[\tilde{L}-\rho B\tilde{G}_2^{\mathrm{T}}+\omega B\tilde{G}_1^{\mathrm{T}}+\omega\rho B\tilde{G}_2^{\mathrm{T}}-\eta(\tilde{K}-B\tilde{G}_1^{\mathrm{T}}-\rho B\tilde{G}_2^{\mathrm{T}}+\omega B\tilde{F}^{\mathrm{T}})\\
&+\eta^2(\tilde{C}-B\tilde{F}^{\mathrm{T}})-\eta^3 M]\\
=&\,M\lambda^3+(C+3\eta M-B\tilde{F}^{\mathrm{T}}-3\eta M)\lambda^2\\
&+[K+2\eta C+3\eta^2 M-B\tilde{G}_1^{\mathrm{T}}-\rho B\tilde{G}_2^{\mathrm{T}}+\omega B\tilde{F}^{\mathrm{T}}\\
&-2\eta(C+3\eta M-B\tilde{F}^{\mathrm{T}})+3\eta^2 M]\lambda\\
&+[L+\eta K+\eta^2 C+\eta^3 M-\rho B\tilde{G}_2^{\mathrm{T}}+\omega B\tilde{G}_1^{\mathrm{T}}+\omega\rho B\tilde{G}_2^{\mathrm{T}}\\
&-\eta(K+2\eta C+3\eta^2 M-B\tilde{G}_1^{\mathrm{T}}-\rho B\tilde{G}_2^{\mathrm{T}}+\omega B\tilde{F}^{\mathrm{T}})\\
&+\eta^2(C+3\eta M-B\tilde{F}^{\mathrm{T}})-\eta^3 M]\\
=&\,M\lambda^3+(C-B\tilde{F}^{\mathrm{T}})\lambda^2+(K-B\tilde{G}_1^{\mathrm{T}}-\rho B\tilde{G}_2^{\mathrm{T}}+\omega B\tilde{F}^{\mathrm{T}}+2\eta B\tilde{F}^{\mathrm{T}})\lambda\\
&+[L-\rho B\tilde{G}_2^{\mathrm{T}}+\omega B\tilde{G}_1^{\mathrm{T}}+\omega\rho B\tilde{G}_2^{\mathrm{T}}+\eta(B\tilde{G}_1^{\mathrm{T}}+\rho B\tilde{G}_2^{\mathrm{T}}-\omega B\tilde{F}^{\mathrm{T}})-\eta^2 B\tilde{F}^{\mathrm{T}}]\\
=&\,M\lambda^3+(C-B\tilde{F}^{\mathrm{T}})\lambda^2+[K-B((1-\eta)\tilde{G}_1+(2\eta-\omega\eta+\eta^2)\tilde{F}-\rho\eta\tilde{G}_2)^{\mathrm{T}}\\
&-\rho B((1+\eta)\tilde{G}_2+(\omega\eta-\eta^2)/\rho\tilde{F}+\eta/\rho\tilde{G}_1)^{\mathrm{T}}+\omega B\tilde{F}^{\mathrm{T}}]\lambda\\
&+[L-\rho(1-\omega)B((1+\eta)\tilde{G}_2+(\omega\eta-\eta^2)/\rho\tilde{F}+\eta/\rho\tilde{G}_1)^{\mathrm{T}}\\
&+\omega B((1-\eta)\tilde{G}_1+(2\eta-\omega\eta+\eta^2)\tilde{F}-\rho\eta\tilde{G}_2)^{\mathrm{T}}]
\end{aligned}
$$

Therefore, the whole process is equivalent to applying feedback matrices

$$
\begin{aligned}
F&=\tilde{F}\\
G_1&=(1-\eta)\tilde{G}_1+(2\eta-\omega\eta+\eta^2)\tilde{F}-\rho\eta\tilde{G}_2\\
G_2&=(1+\eta)\tilde{G}_2+(\omega\eta-\eta^2)/\rho\tilde{F}+\eta/\rho\tilde{G}_1
\end{aligned}
$$

to the original pencil $P(\lambda)$.

To connect with the results in Reference [3], it is useful to explore the single-input case ($p=1$) a bit further. Let $\{\lambda_j\}_{j=1}^k$, $k\in\mathbb{N}$, be a self-conjugate set of non-zero eigenvalues of $P(\lambda)$ with associated eigenvectors $\{x_j\}_{j=1}^k$ and $\{\mu_j\}_{j=1}^k$ be any self-conjugate set of complex numbers satisfying

$$\{\lambda_j\}_{j=1}^k\cap\{\mu_j\}_{j=1}^k=\emptyset \tag{37}$$

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In this situation, the associated Cauchy matrix is

$$\mathscr{C} = \begin{bmatrix} \dfrac{x_1^{\mathrm{T}} b u_1}{\mu_1 - \lambda_1} & \cdots & \dfrac{x_1^{\mathrm{T}} b u_k}{\mu_k - \lambda_1} \\ \vdots & \ddots & \vdots \\ \dfrac{x_k^{\mathrm{T}} b u_1}{\mu_1 - \lambda_k} & \cdots & \dfrac{x_k^{\mathrm{T}} b u_k}{\mu_k - \lambda_k} \end{bmatrix} \tag{38}$$

where $b = B \in \mathbb{R}^{n\times 1}$ is the control vector and $u_1, \ldots, u_k$ are non-zero scalars. In order to find the gain vectors we are required to solve the system of equations

$$\beta\mathscr{C} = [u_1, \ldots, u_k]$$

which is equivalent to

$$[\beta_1, \ldots, \beta_k] \begin{bmatrix} \dfrac{x_1^{\mathrm{T}} b}{\mu_1 - \lambda_1} & \cdots & \dfrac{x_1^{\mathrm{T}} b}{\mu_k - \lambda_1} \\ \vdots & \ddots & \vdots \\ \dfrac{x_k^{\mathrm{T}} b}{\mu_1 - \lambda_k} & \cdots & \dfrac{x_k^{\mathrm{T}} b}{\mu_k - \lambda_k} \end{bmatrix} = [1, \ldots, 1] \tag{39}$$

Let

$$\tilde{\mathscr{C}} = \begin{bmatrix} \dfrac{x_1^{\mathrm{T}} b}{\mu_1 - \lambda_1} & \cdots & \dfrac{x_1^{\mathrm{T}} b}{\mu_k - \lambda_1} \\ \vdots & \ddots & \vdots \\ \dfrac{x_k^{\mathrm{T}} b}{\mu_1 - \lambda_k} & \cdots & \dfrac{x_k^{\mathrm{T}} b}{\mu_k - \lambda_k} \end{bmatrix}$$

then in view of (35) its determinant is explicitly written as

$$\det \tilde{\mathscr{C}} = \prod_{i=1}^{k} x_i^{\mathrm{T}} b \, \frac{\prod_{1\leqslant i<j\leqslant k}(\mu_i - \mu_j)\prod_{1\leqslant i<j\leqslant k}(\lambda_i - \lambda_j)}{\prod_{i,j=1}^{k}(\mu_i - \lambda_j)}$$

Therefore, in addition to condition (37), if $x_j^{\mathrm{T}} b \neq 0$ for all $1 \leqslant j \leqslant k$ and both $\{\lambda_j\}_{j=1}^{k}$ and $\{\mu_j\}_{j=1}^{k}$ have distinct elements, then $\tilde{\mathscr{C}}$ is invertible. The solution to (39) is given by

$$\beta_j = \frac{1}{x_j^{\mathrm{T}} b} \, \frac{\prod_{i=1}^{k} \mu_i - \lambda_j}{\prod_{i=1, i\neq j}^{k} \lambda_i - \lambda_j}, \quad 1 \leqslant j \leqslant k \tag{40}$$

which is due to an identity

$$\sum_{i=1}^{k} \frac{\prod_{j=1, j\neq \ell}^{k} \mu_j - \lambda_i}{\prod_{j=1, j\neq i}^{k} \lambda_j - \lambda_i} = 1$$

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for any $1 \leqslant \ell \leqslant k$ (see Reference [3]). Hence, under the conditions that $x_j^{\mathrm{T}} b \neq 0$ for $1 \leqslant j \leqslant k$, $\{\lambda_j\}_{j=1}^k$ are all non-zero and distinct, and $\{\mu_j\}_{j=1}^k$ have distinct elements, we can construct the gain vectors

$$
\begin{aligned}
F &= MX\Lambda\xi \\
G_1 &= [-\omega(K_1 + \rho K_2) + \rho K_2]X\xi + (1-\omega)[MX\Lambda^2 + (C_1 + \rho C_2)X\Lambda]\xi \\
G_2 &= \frac{1}{\rho}\{[\omega(K_1 + \rho K_2) - \rho K_2]X\xi + \omega[MX\Lambda^2 + (C_1 + \rho C_2)X\Lambda]\xi\}
\end{aligned}
$$

where $\xi = [\xi_1, \ldots, \xi_k]^{\mathrm{T}} \in \mathbb{C}^k$ with

$$
\xi_j = \frac{\beta_j}{\lambda_j} = \frac{1}{x_j^{\mathrm{T}} b} \frac{\mu_j - \lambda_j}{\lambda_j} \prod_{\substack{i=1 \\ i \neq j}}^{k} \frac{\mu_i - \lambda_j}{\lambda_i - \lambda_j}, \quad 1 \leqslant j \leqslant k
$$

will move $\{\lambda_j\}_{j=1}^k$ to $\{\mu_j\}_{j=1}^k$ and keep the rest of $\sigma(P)$ unchanged. The vector ξ we derive here has the same form as obtained in Reference [3]. By straightforward computations, we can check that the gain vectors F, G_1 and G_2 are real. It is worth mentioning that the system (39) is solvable with one solution given by (40) whenever $x_j^{\mathrm{T}} b \neq 0$ for all $1 \leqslant j \leqslant k$ and $\{\lambda_j\}_{j=1}^k$ are distinct. These two conditions are exactly the same conditions used in Reference [3]. Of course, if $\{\mu_j\}_{j=1}^k$ are not distinct, then the choice of ξ is not unique. Now, if we ignore the aerodynamic effect in (1), i.e. $\rho = 0$, then we can see that G_2 does not appear in the closed-loop cubic pencil $P_{\mathrm{c}}(\lambda)$ and

$$
\begin{aligned}
F &= MX\Lambda\xi \\
G_1 &= -\omega K_1 X\xi + (1-\omega)(MX\Lambda^2 + C_1 X\Lambda)\xi = -KX\xi
\end{aligned}
$$

which are identical to the gain vectors derived in Reference [3].

4. NUMERICAL EXAMPLE

A set of pseudosimulation data is provided by The Boeing Company for testing purposes. The sizes of matrices M, C_1, C_2, K_1 and K_2 are all 42×42. Therefore, the total number of eigenvalues (counting multiplicity) is 126. Now we let

$$
\begin{aligned}
\{\lambda_j\}_{j=1}^{12} &= \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots, \lambda_{11}, \lambda_{12}\} \\
&= \{\lambda_1, \bar{\lambda}_1, \lambda_3, \bar{\lambda}_3, \ldots, \lambda_{11}, \bar{\lambda}_{11}\}
\end{aligned}
$$

be a set of six pairs of complex conjugate unwanted eigenvalues and

$$
\begin{aligned}
\{\mu_j\}_{j=1}^{12} &= \{\mu_1, \mu_2, \mu_3, \mu_4, \ldots, \mu_{11}, \mu_{12}\} \\
&= \{\mu_1, \bar{\mu}_1, \mu_3, \bar{\mu}_3, \ldots, \mu_{11}, \bar{\mu}_{11}\}
\end{aligned}
$$

be a set of prescribed values. Their specific values are given in Tables I–IV.

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Table I. Unwanted and prescribed eigenvalues.

j	λ_j	μ_j
1	$-1.9416357e+00$ $+5.7145254e+01i$	$-1.8907325e+00$ $+6.1436553e+01i$
2	$-1.9416357e+00$ $-5.7145254e+01i$	$-1.8907325e+00$ $-6.1436553e+01i$
3	$-6.9183335e-01$ $+4.1683158e+01i$	$-2.2785004e+00$ $+3.9639351e+01i$
4	$-6.9183335e-01$ $-4.1683158e+01i$	$-2.2785004e+00$ $-3.9639351e+01i$
5	$-2.6340898e+00$ $+3.5063988e+01i$	$-6.0938997e-01$ $+3.7364828e+01i$
6	$-2.6340898e+00$ $-3.5063988e+01i$	$-6.0938997e-01$ $-3.7364828e+01i$
7	$-2.5838961e+00$ $-2.8441096e+01i$	$-4.5168038e-01$ $-2.8518590e+01i$
8	$-2.5838961e+00$ $+2.8441096e+01i$	$-4.5168038e-01$ $+2.8518590e+01i$
9	$-9.9717738e-01$ $-1.5327875e+01i$	$-1.4440260e+00$ $-2.1440443e+01i$
10	$-9.9717738e-01$ $+1.5327875e+01i$	$-1.4440260e+00$ $+2.1440443e+01i$
11	$-3.2740196e-01$ $+1.4651202e+01i$	$-1.1006744e+00$ $+1.9167033e+01i$
12	$-3.2740196e-01$ $-1.4651202e+01i$	$-1.1006744e+00$ $-1.9167033e+01i$

Table II. Relative errors of assigned eigenvalues.

j	$\frac{\lvert\mu_j-\hat{\mu}_j\rvert}{\lvert\mu_j\rvert}$
1	$9.584286188571896e-11$
3	$1.348282068322105e-11$
5	$8.323364112144900e-12$
7	$4.054783553919088e-11$
9	$5.260232134411892e-11$
11	$8.279416920535226e-11$

Table III. Norms of gain matrices.

$\lVert F\rVert_2$	$5.304129423507520e+06$
$\lVert G_1\rVert_2$	$6.333435534363977e+08$
$\lVert G_2\rVert_2$	$6.009725627198257e+07$

Table IV. Errors of imaginary parts of gain matrices.

$\lVert F-\bar{F}\rVert_2$	$4.404408704029483e-08$
$\lVert G_1-\bar{G}_1\rVert_2$	$3.463194515350621e-06$
$\lVert G_2-\bar{G}_2\rVert_2$	$3.624612282800583e-07$

Here we choose a double-input $B=[b_1,b_2]$ with

$$b_1^{\mathrm{T}}=\frac{1}{\sqrt{21}}[1,0,1,0,\ldots,1,0]^{\mathrm{T}}$$

$$b_2^{\mathrm{T}}=\frac{1}{\sqrt{21}}[0,1,0,1,\ldots,0,1]^{\mathrm{T}}$$

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For comparison purpose, we denote $\{\lambda_{13},\ldots,\lambda_{126}\}$ the remaining eigenvalues of the open-loop pencil $P(\lambda)$ and $\{\hat{\mu}_1,\ldots,\hat{\mu}_{12},\hat{\lambda}_{13},\ldots,\hat{\lambda}_{126}\}$ the eigenvalues of the closed-loop pencil $P_c(\lambda)$. All computations are performed in MatLab version 6.0 on a Linux machine. We find that

$$\max\left\{\frac{|\lambda_i-\hat{\lambda}_i|}{|\lambda_i|},\ i=13,\ldots,126\right\}=8.577661179394325e-010$$

5. CONCLUSION

We have solved the PPA problem by multiple-input state feedback control for the vibrating system with aerodynamic effect. The choices of gain matrices reply on the invertibility of a generalized Cauchy matrix which is formed by the control matrix, unwanted eigenpairs, and prescribed eigenvalues. Under generic conditions, we can see that a self-conjugate set of unwanted eigenvalues can be relocated to almost arbitrary prescribed values which are closed under complex conjugation, while keeping other eigenvalues unchanged. This pole assignment can be achieved by the real gain matrices. Therefore, this control is realizable by means of physical devices. If the aerodynamic effect is neglected and the feedback control is governed by single-input, then we recover a solution to the PPA problem for the quadratic pencil derived in Reference [3].

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