附件：封面格式

行政院國家科學委員會補助專題研究計畫成果報告

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※ 利用無限次接合法造極小曲面 ※

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計畫類別：☑個別型計畫　　☐整合型計畫

計畫編號：NSC 89－2115－M－002－005－

執行期間：88年8月1日至89年7月31日

計畫主持人：王藹農

共同主持人：

計畫參與人員：

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中　華　民　國　89年　10月　31日

行政院國家科學委員會專題研究計畫成果報告

計畫編號：NSC 89-2115-M-002-005
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主持人：王藹農 執行機構及單位名稱 台大數學系
共同主持人： 執行機構及單位名稱
計畫參與人員： 執行機構及單位名稱

一、中文摘要 本篇報告描述 Collin 與 Rosenberg 建構 Nadirashvili 有界極小曲面之細節

關鍵詞： 有界極小曲面

Abstract This report describes the details of Collin and Rosenberg construction of Nadirashvili's bounded minimal surface

Keywords: bounded minimal surfaces

NOTES ON NADIRASHVILI'S PROOF OF CALABI-YAU CONJECTURE

AI-NUNG WANG

This note is written upon the request of Professor Yi Fang and we thank him for notifying us the publication of [1]. To fix the notation for Nadirashvili's labyrinth, let $N \geq 3$ be an integer, $r_i = 1 - \frac{i}{N^3}$, $i = 0, 1, ..., 2N^2$, $S_r = \partial D_r$, $\mathcal{A} = D_1 - D_{1-\frac{2}{N}}$ and

$$A = \bigcup_{i=0}^{N^2-1} D_{r_{2i}} - D_{r_{2i+1}} \qquad \tilde{A} = \bigcup_{i=0}^{N^2-1} D_{r_{2i+1}} - D_{r_{2i+2}};$$

$$L = \bigcup_{i=0}^{N-1} l_{\frac{2i\pi}{N}} \cap A \qquad \tilde{L} = \bigcup_{i=0}^{N-1} l_{\frac{(2i+1)\pi}{N}} \cap \tilde{A} \qquad S = \bigcup_{i=0}^{2N^2} S_{r_i}.$$

Now we define the labyrinth $H = L \cup \tilde{L} \cap S$ and $\Omega = \mathcal{A} - U[\frac{1}{4N^3}](H)$ with $2N^3$ connected components, here $U[r](H)$ denotes r-neighborhood of H.

In the following we assume $N \geq 10$.

Assertion: Let $ds = \lambda|dz|$ be a metric such that

$$\begin{cases} \lambda \geq 1 & on\ D_1; \\ \lambda \geq N^4 & on\ \Omega. \end{cases}$$

then for all paths σ from 0 to ∂D_1, $\int_\sigma ds \geq N$.

In fact one does not need to raise N to such a high power, Collin uses N^4 simply to avoid changing too much from Nadirashvili's notations.

Now we define recursively a sequence of minimal immersions $F_0 = X, F_1, ..., F_{2N}$ from $\bar{D}_1$ to $\mathbb{R}^3$ satisfying ($\phi^i = 2\frac{dF_i}{dz}$):

$$((\mathcal{H}_i)_{1\leq i\leq N}) \qquad \begin{cases} ||\phi^i - \phi^{i-1}|| \leq \frac{\epsilon}{2N^2} & on\ D_1 - \omega_i' \\ ||\phi^i|| \geq \frac{\nu}{2}N^{3.5} & on\ \omega_i \\ ||\phi^i|| \geq \frac{\nu}{2\sqrt{N}} & on\ \omega_i' \end{cases}$$

where ω_i is the union of the line segment $l_{\frac{i\pi}{N}} \cap \mathcal{A}$ and the N^2 components of Ω intersecting it, and $\omega_i' = U[\frac{1}{8N^3}](\omega_i)$ thus D_1 is divided into $4N+1$ disjoint sets :

$$\begin{cases} D_1 - \cup_{i=1}^{2N}\omega_i' \\ \omega_i' - \omega_i,\ \ i = 1, ..., 2N; \\ \omega_i',\ \ i = 1, ..., 2N. \end{cases}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Collin adds two more controling parameters: $\mu = ||X||_{C^2} + 1$ and $\nu = inf_{D_1}||\phi||$. The idea behind Nadirashvili's construction is actually quite simple: using López-Ros transform, we can deform a minimal surface in the direction perpendicular to the radial direction. Therefore if we increase the metric by $\frac{1}{n}$ each time, the embedding increases at most $\frac{1}{n^2}$ hence remains bounded.

We further assume that $N \geq 10, \frac{2}{\epsilon}, (\frac{3(\rho+s)}{\nu})^2, (7\mu)^2, \frac{2\epsilon}{\nu}, \ldots$

Construction of F_i. –Suppose that $F_1, \ldots, F_{i-1}$ have been constructed satisfying $\mathcal{H}_1, \ldots, \mathcal{H}_{i-1}$; and $G_k : D_1 \to S^2$ denotes the Gauss map of F_k. On ω_i', $||\phi^{i-1}|| \leq \mu$(since $||\phi^0|| \leq \mu - 1$ and $||\phi^{i-1} - \phi^0|| \leq i\frac{\epsilon}{2N^2} \leq \frac{\epsilon}{N}$). Because the diameter $\delta(\omega_i') \leq \frac{7}{N}$, we have $\delta(F_{i-1}(\omega_i')) \leq \frac{7\mu}{N}$. On the other hand, by definition of μ, $\delta(G_0(\omega_i')) \leq \frac{7\mu}{N}$. Since on ω_i', $||G_{i-1} - G_0|| \leq \frac{2\epsilon}{\nu N}$(since $||\phi^{i-1} - \phi^0|| \leq \frac{\epsilon}{N}$), we also have $\delta(G_{i-1}(\omega_i')) \leq \frac{7\mu}{N} + \frac{2\epsilon}{\nu N}$.

There is a $q_i \in S^2$ such that:
(a) If

$$dist_{\mathbb{R}^3}(0, F_{i-1}(\omega_i')) \geq \frac{1}{\sqrt{N}},$$

then the angle

$$\angle(q_i, F_{i-1}(\omega_i')) \leq \frac{7\mu}{\sqrt{N}};$$

(b)

$$dist_{S^2}(\pm q_i, G_{i-1}(\omega_i')) \geq \frac{1}{\sqrt{N}}.$$

In fact, if $dist_{\mathbb{R}^3}(0, F_{i-1}(\omega_i')) \leq \frac{1}{\sqrt{N}}$, then the condition (a) is empty and the condition (b) can be achieved by at least one q_i since

$$\delta(G_{i-1}(\omega_i')) \leq \frac{7\mu}{N} + \frac{2\epsilon}{\nu N} \leq \frac{1}{\sqrt{N}}$$

On the other hand, if $dist_{\mathbb{R}^3}(0, F_{i-1}(\omega_i')) \geq \frac{1}{\sqrt{N}}$, then $F_{i-1}(\omega_i')$ is in a cone of vertex angle $\frac{7\mu}{\sqrt{N}}$ whose axis q_i satisfies (a). Since $\delta(G_{i-1}(\omega_i')) \leq \frac{1}{\sqrt{N}}$, one can modify q_i by an angle $\frac{2}{\sqrt{N}}$ to make it satisfy (b) too.

Now we fix the coordinate of $\mathbb{R}^3$ so that $e_3 = q^i$. The Weierstrass representation of $F_i - 1$ determines through $\phi^{i-1} = (\phi_1^{i-1}, \phi_2^{i-1}, \phi_3^{i-1})$ two holomorphic functions $f = \phi_1^{i-1} - i\phi_2^{i-1}$ and $fg = -\phi_1^{i-1} - i\phi_2^{i-1}$. The metric is given by

$$\lambda_{F_{i-1}} = \frac{1}{\sqrt{2}}||\phi^{i-1}|| = \frac{1}{2}(|f| + |fg^2|).$$

Let $T_i > 1$ and $h = h[T_i, D_1 - \omega_i', \omega_i, g]$ be defined by Proposition (4.3) [2]. The functions $\tilde{f} = fh$ and $\widetilde{fg^2} = \frac{fg^2}{h}$ determine a new minimal immersion F_i such that

$$\phi_3^i = \phi_3^{i-1}.$$

It remains to verify that $\mathcal{H}_i$ hold for sufficiently large T_i.

• On $D_i - \omega_i'$:

$$|\tilde{f} - f| = |f(h-1)| \leq \frac{sup_{D_i}|f|}{T_i} \leq \frac{\epsilon}{4N^2}$$

and

$$|\widetilde{fg^2} - fg^2| = |fg^2(\frac{1-h}{h})| \leq \frac{sup_{D_i}|fg^2|}{T_i - 1} \leq \frac{\epsilon}{4N^2}$$

thus $||\phi^i - \phi^{i-1}|| \leq \frac{\epsilon}{2N^2}$.

• On ω_i':

$$\lambda_{F_i} = \frac{1}{2}(|f||h| + \frac{|fg^2|}{|h|}) \geq |fg| = \lambda_{F_{i-1}} \frac{2|g|}{(1+|g|^2)} \geq \frac{\lambda_{F_{i-1}}}{\sqrt{N}}$$

for by (b), $\frac{2}{\sqrt{N}} \leq |g| \leq \frac{\sqrt{N}}{2}$. Therefore, since $||\phi^{i-1} - \phi^0|| \leq i\frac{\epsilon}{2N^2} \leq \frac{\nu}{2}$ on ω_i', we have

$$||\phi^{i-1}|| \geq \frac{\nu}{2}$$

and

$$||\phi^i|| = \sqrt{2}\lambda_{F_i} \geq \frac{\nu}{2\sqrt{N}}$$

• On ω_i:

$$\lambda_{F_i} \geq \frac{1}{2}|f|T_i = \frac{\lambda_{F_{i-1}} T_i}{(1+|g|^2)} \geq \frac{\lambda_{F_{i-1}} T_i}{N}$$

Therefore, as above, $||\phi^i|| \geq \frac{\nu T^i}{2N} \geq \frac{\nu}{2}N^{3.5}$ for sufficiently large T_i.

The formula (16) in [2] was a mis-print, Nadirashvili actually meant our formula (a) above.

References

[1] Pascal Collin and Harold Rosenberg, *Notes sur la Démonstration de N. Nadirashvili des Conjectures de Hadamard et Calabi-Yau*, Bull. Sci. math. **123** (1999), p.563-575.

[2] Nikolai Nadirashvili, *Hadamard's and Calabi-Yau's conjectures on negatively curved and minimal surfaces.*, Invent. math. **126** (1996), 457-465.