行政院國家科學委員會補助專題研究計畫成果報告

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※ 三維流形的坐標架 ※
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摘　　要

當一三維球閉流形其同調類於三維球同構時，我們發現它有一個自然的坐標架。

關鍵詞：三維流形，坐標架

A Natural Framing for Asymptotically Flat Integral Homology 3-Sphere

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Communicated with W. H. Lin

Abstract

For an integral homology 3-sphere embedded asymptotically flatly in an Euclidean space, we find a natural framing extending the standard trivialization on the asymptotically flat part.

Suppose $\overline{M}$ is a 3-dimensional closed smooth manifold which has the same integral homology groups as the 3-sphere S^3. x_0 is a fixed point in $\overline{M}$. Embed $\overline{M}$ in a Euclidean space $\mathbb{R}^n$ such that x_0 is the infinite point of the 3-dimensional flat space $\mathbb{R}^3 \times \{0\}$ of $\mathbb{R}^n$ and a neighborhood of x_0 contains the whole flat space $\mathbb{R}^3 \times \{0\}$ except a compact set. Precisely, for any positive number r, let B_r denote the closed ball of radius r in $\mathbb{R}^3$ and $N_r = (\mathbb{R}^3 - B_r) \times \{0\}$; there exists r_0, a positive number, such that N_{r_0} is contained in $\overline{M}$ and $N_{r_0} \cup \{x_0\}$ is an open neighborhood of x_0 in $\overline{M}$.

Let $M = \overline{M} - \{x_0\}$, it is an asymptotically flat 3-dimensional manifold with acyclic homology. The main purpose of this article is to define a natural framing for M. If we identify the tangent spaces of points in the flat part N_{r_0} with $\mathbb{R}^3 \times \{0\}$, then the tangent bundle of M can be thought as a 3-dimensional vector bundle over the closed manifold $M_0 = M/\overline{N}_s$, where s is a number greater than r_0 and $\overline{N}_s$ is the closure of N_s; we shall call this vector bundle the tangent bundle $T(M_0)$ of M_0. And our natural framing is just a trivialization of $T(M_0)$, which corresponds to a trivialization of the tangent bundle $T(M)$ whose restriction to the flat part is the standard trivialization on $\mathbb{R}^3$. Because M_0 is a closed 3-manifold, there are countably infinite many choices of framings associated with the infinite elements in $[M_0, SO(3)]$. (When $H_*(M_0) \approx H_*(S^3)$, $[M_0, SO(3)] \approx [S^3, SO(3)] \approx \mathbf{Z}$.) Therefore, our natural framing is a special choice from the infinite many.

On the other hand, this natural framing for $T(M_0)$ can also provide a special one-to-one correspondence between the infinite framings of S^3 and that of $\overline{M}$. (Note: Here, we do not think that $\overline{M}$ and M_0 have the same tangent bundle. Conversely, we may think that the tangent bundle of $\overline{M}$ is equal to the connected sum of the tangent bundles of M_0 and S^3.)

There are two main steps to the natural framing on $T(M_0)$.

Step 1 A special map from $C_2(M)$ to S^2

We define $C_2(M)$ at first.

For any set X, $\Delta(X)$ denote the diagonal subset $\{(x,x) \in X \times X, x \in X\}$ of $X \times X$ and $C_2(X) = X \times X - \Delta(X)$. Thus $C_2(M)$ is the configuration space of all pairs of distinct two points in M.

Fix some large number s such that $M \subset (B_s \times \mathbb{R}^{n-3}) \cup N_s$.

For any $r \geq s$, let $B_r = \{x \in \mathbb{R}^3 : |x| \leq r\}$, $N_r = (\mathbb{R}^3 - B_r) \times \{0\}$ and $M_r = M - N_r$.

Let Y denote the union of the following three subsets of $C_2(M)$:

$$\text{(i)} \quad Y_0 = C_2(N_s)$$

$$\text{(ii)} \quad Y_1 = \cup_{r \geq s}(N_{r+s} \times M_r)$$

$$\text{(iii)} \quad Y_2 = \cup_{r \geq s}(M_r \times N_{r+s})$$

Let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^3$ denote the projection

$$\pi(t_1, t_2, \cdots, t_n) = (t_1, t_2, t_3)$$

and $f : Y \longrightarrow S^2$ denote the map

$$f(x, y) = \frac{\pi(y - x)}{|\pi(y - x)|}$$

for $(x, y) \in Y$, $x, y \in M$.

For the well-defining of the map f, we should check that $|\pi(y - x)|$ is a non-zero value. When (x, y) is in Y_0, $|\pi(y - x)| = |y - x|$, it is non-zero. When (x, y) is in Y_1, (x, y) is in $N_{r+s} \times M_r$ for some $r \geq s$; thus $\pi(x)$ is outside of B_{r+s} and $\pi(y)$ is in B_r, and hence $\pi(y - x) = \pi(y) - \pi(x)$, it has also a non-zero norm. It is similar for the case that (x, y) is in Y_2.

The following proposition describes some homology properties for the space Y and the map f.

Proposition 1

(i) $H_*(Y) \approx H_*(S^2)$

(ii) $f_*; H_2(Y) \longrightarrow H_2(S^2)$ is an isomorphism.

(iii) Let $j : Y \longrightarrow C_2(M)$ denote the inclusion map.

$$j_* : H_i(Y) \longrightarrow H_i(C_2(M))$$

is isomorphic, for all integer $i \geq 0$. ∎

In the proof of the proposition, we strongly use the assumption that $H_*(M)$ is acyclic.

Remark: All the homologies in this article are with integral coefficients.

By Proposition 1, the continuous map $f : Y \longrightarrow S^2$ uniquely extends to a continuous map $\overline{f} : C_2(M) \longrightarrow S^2$ up to homotopy relative to the subspace Y. (That is, if both $\overline{f}_1$ and $\overline{f}_2$ are the extensions of f to the whole space $C_2(M)$, then there is a homotopy $F : C_2(M) \times [0, 1] \longrightarrow S^2$ such that

$F(\xi, 0) = \overline{f}_1(\xi)$, $F(\xi, 1) = \overline{f}_2(\xi)$, for all $\xi \in C_2(M)$, and $F(\xi'. t) = f(\xi')$ for all $\xi' \in Y$ and $t \in [0, 1]$.)

Usually, the homotopy class of a map from $C_2(M)$ to S^2 can not give any framing on $T(M_0)$. But the extension of f does give a framing on $T(M_0)$ as shown in Step 2.

Step 2 The framing determined by the map $\overline{f}$ on $C_2(M)$

The normal bundle of $\Delta(M)$ in $M \times M$ can be identified as the tangent bundle $T(M)$ of M. Consider a suitable compactification of $C_2(M)$, the spherical bundle $S(TM)$ become a part of boundary of $C_2(M)$. Let $h : S(TM) \longrightarrow S^2$ denote the restriction of $\overline{f}$ to $S(TM)$. On the flat part N_s of M, the spherical bundle $S(TN_s) = N_s \times S^2$ and h on $S(TN_s)$ is equal to the map restricted from f which is exactly the projection from $N_s \times S^2$ to S^2. Thus h induces a map $h_0 : S(TM_0) \longrightarrow S^2$.

$S(TM_0)$ is a $SO(3)$-bundle over M_0.

Can $h_0 : S(TM_0) \longrightarrow S^2$ determine uniquely an orthogonal map, that is, a fibrewise orthogonal map? (An orthogonal map is exactly a framing for the vector bundle.) There is also an interesting question that can h_0 be homotopic to an orthogonal map; if such an orthogonal map exists, is it unique up to homotopy? We shall answer the questions partially.

Choose a framing for $S(TM_0)$ and we may think h_0 as a map from $M_0 \times S^2$ to S^2. Let y_0 denote the point in M_0 representing the set N_s. Then the restriction of h_0 to $y_0 \times S^2$ is the identity map of S^2. Thus the restriction of h_0 to each fibre $x \times S^2$, $x \in M_0$, is also a homotopy equivalence; and hence, h_0 induces a map $\widehat{h}_0$ from M_0 to $G(3)$, the space of all homotopy equivalences of S^2 to itself. Choose a base point z_0 in S^2, and consider the subspace $F(3)$ of $G(3)$ consisting of all the homotopy equivalences which fix the base point z_0. Then $F(3)$ is the fibre of the fibration $G(3)$ over S^2.

it is the key fact for the homotopic computations.

For any two spaces X_1 and X_2 with base points x_1 and x_2, respectively, $[X_1, X_2]$ denotes the set of homotopy classes of continuous maps from X_1 to X_2 and sending x_1 to x_2. In the following, $M_0$is with base point y_0 representing the set $\overline{N}_s$; $SO(3)$, $G(3)$ and $F(3)$ are with the base point the identity of S^2. We shall consider only the maps sending the base point to base point and consider only the homotopies which keep the base point fixed.

M_0 has the same homology as S^3. Usually, we can not expect they also have the same homotopy behavior. But we still have the following proposition.

Proposition 2 Suppose $\phi : M_0 \longrightarrow S^3$ is a degree 1 map. Then the homotopy classes $[M_0, SO(3)]$, $[M_0, G(3)]$, $[M_0, F(3)]$ are all groups, and the group homomorphisms induced by ϕ,

$$[S^3, SO(3)] \xrightarrow{\phi^\sharp} [M_0, SO(3)]$$

$$[S^3, G(3)] \xrightarrow{\phi^\sharp} [M_0, G(3)]$$

$$[S^3, F(3)] \xrightarrow{\phi^\sharp} [M_0, F(3)]$$

$$[S^3, S^2] \xrightarrow{\phi^\sharp} [M_0, S^2]$$

are all isomorphisms of groups. ∎

There are further relations between these homotopy classes.

Proposition 3 Let $p : SO(3) \longrightarrow G(3)$ and $q : F(3) \longrightarrow G(3)$ denote the inclusions. Then, for any integral homology 3-sphere M_0, the homomorphism

$$p_* \oplus q_* : [M_0, SO(3)] \oplus [M_0, F(3)] \longrightarrow [M_0, G(3)]$$

is an isomorphism.
Especially, when $M_0 = S^3$, we have

$$\pi_3(G(3)) \approx \pi_3(SO(3)) \oplus \pi_3(F(3)) \ .$$

∎

Furthermore, the group isomorphism

$$q_*^{-1} : [M_0, G(3)]/p_*([M_0, SO(3)] \longrightarrow [M_0, F(3)]$$

induces a group homomorphism

$$Q : [M_0, G(3)] \longrightarrow [M_0, F(3)] \approx \mathbf{Z}_2 \ .$$

For a continuous map $g : M_0 \times S^2 \longrightarrow S^2$, let $\widehat{g}$ denote the map from M_0 to $G(3)$ defined by $\widehat{g}(x)(y) = g(x, y)$, for $x \in M_0$ and $y \in S^2$ and let $Q(g) = Q([\widehat{g}])$.

Theorem 4 A continuous map $g : M_0 \times S^2 \longrightarrow S^2$ is homotopic to an orthogonal map, if and only if, $Q(g) = 0$ in $[M_0, F(3)]$. ∎

Now, h_0 still denotes the map from $S(TM_0)$ to S^2 given by the map $\overline{f} : C_2(M) \longrightarrow S^2$. Choose a framing for TM_0, $\psi : S(TM_0) \longrightarrow M_0 \times S^2$, it is a fibre map and fibrewise orthogonal. Then $h_0 \circ \psi^{-1}$ is a map from $M_0 \times S^2$ to S^2 and the value $Q(h_0 \circ \psi^{-1})$ is independent of the choice of the framing ψ. Therefore, $Q(h_0 \circ \psi^{-1})$ is an invariant of the integral homology 3-sphere $\overline{M}$, it is the obstruction for h_0 to be homotopic to an orthogonal map. We hope that this is not really an obstruction.

Conjecture 5 $Q(h_0 \circ \psi^{-1}) = 0$, for any integral homology 3-sphere $\overline{M}$. ∎

On the other hand, the group isomorphism

$$p_*^{-1} : [M_0, G(3)]/q_*([M_0, F(3)] \longrightarrow [M_0, SO(3)]$$

induces a group homomorphism

$$P : [M_0, G(3)] \longrightarrow [M_0, SO(3)] \quad .$$

For a continuous map $g : M_0 \times S^2 \longrightarrow S^2$, let $P(g) = P([\widehat{g}])$.

For the map h_0 and the corresponding element $P(h_0 \circ \psi^{-1})$ in $[M_0, SO(3)]$, choose an orthogonal map $g_0 : M_0 \times S^2 \longrightarrow S^2$ such that the associated map $\widehat{g}_0$ is in the homotopy class $P(h_0 \circ \psi^{-1})$. Then we get an orthogonal map $g_0 \circ \psi : S(TM_0) \longrightarrow S^2$ which represents a homotopy class of framings determined by h_0, also by the map $\overline{f} : C_2(M) \longrightarrow S^2$. This framing can also be characterized by the following theorem.

Theorem 6 There exists a framing $\psi_0 : S(TM_0) \longrightarrow M_0 \times S^2$ unique up to homotopy such that $P(h_0 \circ \psi_0^{-1}) = 0$. ∎

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