行政院國家科學委員會專題研究計畫成果報告

計畫名稱: Sharp Estimate for the Green Function of the Boltzmann Equation

計畫編號: NSC 89-2115-M-002-011

執行期限： 88 年 8 月 1 日至 89 年 7 月 31 日

主持人: 陳俊全

執行機構及單位名稱:國立台灣大學 數學系

ABSTRACT. This project continues the study of the Boltzmann equation last year. We verified the decay rate of the error of an approximate Green function which is conjectured last year. More clear steps to obtain shock wave from the evolution of the equation have been figured out.

1. INTRODUCTION

This project continues the study of the Boltzmann equation last year. We verified the decay rate of the error of an approximate Green function which is constructed last year. It seems along a constant Maxwellian, this is quite good for later steps. However, when the asymptotic Maxwellians when $x \to -\infty$ and $x \to \infty$ are different, the errors for the approximate Green function are still too large and some new idea is needed.

Consider the Boltzmann equation

$$\frac{\partial}{\partial t} f + \xi \cdot \frac{\partial}{\partial x} f = Q(f,f) \tag{1.1}$$

in the rarefied gas dynamicd with a cut-off hard potential in the sense of Gradwhere $f = f(x,v,t)$ with $x \in \mathbb{R}^3$, $\xi \in \mathbb{R}^3$ and $t \geq 0$, and

$$Q(f,f) = \int_{\mathbb{R}^3} \int_{S_+} (f' f'_* - f f_*) |V \cdot n| \, d\xi_* \, dn$$

is the collision term. To understand microscopic dynamics nowadays, the Boltzmann equation is more and more important. However, many fundamental topics such as, for example,

(1) rigorus validity of the Boltzmann equation,
(2) existence and uniqueness of a global solution with a general initial value,
(3) existence for more general initial-boundary value problemsare,
(4) hydrodynamical limits,
(5) interaction of waves of the Boltzmann quation

are still not well understood. In this paper, we mainly concern with a very first step to understand the last topic, which are also close related to the others.

If we consider the full equation with a quadratic collision term, then we can not avoid nonlinear wave phenomenon. Since in the hydrodynamic regime, the Euler equations and the Navier-Stokes equations have shock and travelling wave solutions, it is nature to consider similar problem for the Boltzmann quation. The existence of a weak shock wave (travelling wave) for the Boltzmann quation was obtain by Caflisch and Nicolaenko [1]. They used a exact travelling wave of the Navier-Stokes equations as an approximate solution. Then the solution was found by a Lyapunov-Schmidt method as a bifurcation from a constant Maxwellian state. Unfortunately, they can not show this solution is nonnegative and it could be of no physical meaning. Caflisch and Nicolaenko in the same paper also proved a uniqueness result for the shock profile solution near a Maxwellian. Hence if we believe there is a weak shock profile solution with physical meaning, it must be the one constructed in [1].

Inspired by the works on shock profile solutions of conservation laws in Liu and Zeng [9], Liu [6], Liu and Wang [7] and Liu and Yu [8], it seems we can understand more about shock profile solutions and wave interaction from a better estimate of the Green function. The ideas are: (1) obtain pointwise estimates for the Green function of the linearized equation near a constant state; (2) obtain pointwise estimates for the Green function of the linearized equation near a approxmate shock profile solution; (3) use these estimates to trace the interaction of waves and show the convergence to a shock profile solution. The main difficulty to apply these ideas to the Boltzmann equation is that there is one more variable ξ in the equation. When linearized around a constant Maxwellian, the known results by Ukai [12] and Nishida and Imai [11] are L^2 type estimates for the semigroup. These estimates are not sharp enough to trace movement of waves.

In this paper, we linearize the equation around a constant Maxwellian. We use the semigroup to represent a solution and drop terms which decay very fast in time. Then we transfer the dominant terms into a convolution of the initial value and the sorce term with the Green function. From this, an approximate Green function is obtained. To avoid the difficult of estimating the terms appear in the semigroup representation, we go back to equation to verify that the approximatete Green function is good enough.

2. Linearized Theory

Let $M = (2\pi)^{-\frac{3}{2}} exp(-\frac{|\xi|^2}{2})$. We linearize the equation around M and write $f = M + M^{\frac{1}{2}} h$ and the collision term

$$Q(f,f) = Lh + \nu\Gamma(h,h),$$

where $L = 2M^{-\frac{1}{2}}Q(M^{\frac{1}{2}}h, M)$ is the linear part. The operator L is nonpositive, i.e.,

$$(Lh,h) \leq 0 \text{ for } h \in D(L)$$

and satisfies

$$Lh = 0 \text{ iff } h \in span\{M^{\frac{1}{2}}, \xi_j M^{\frac{1}{2}}, |\xi|^2 M^{\frac{1}{2}}\}.$$

Moreover, it can be decomposed as

$$Lh = -\nu(|\xi|)h + Kh'$$

where $\nu(|\xi|)$ satisfies

$$0 < \nu_o \leq \nu(|\xi|) \leq \nu_1(1 + |\xi|)$$

and K is a compact operator in L^2. Now we consider the linearized equation

$$\frac{\partial}{\partial t}h + \xi \cdot \frac{\partial}{\partial x}h = Lh$$

with

$$h(x,\xi,0) = h_o(x,\xi).$$

Let $\hat{h}$ denote the Fourier transform of h in x. Then $\hat{h}(k,\xi,t)$ satisfies

$$\frac{\partial}{\partial t}\hat{h} + i\xi \cdot k\hat{h} = L\hat{h}.$$

Let

$$B(k) = L - i\xi \cdot kI.$$

We can represent $\hat{h}$ in the form of semigroup. See [12] and [11].

Theorem. *There exist $\delta > 0$, $b_1 > 0$ and $b_2 > 0$ such that for $\hat{h}_o = \hat{h}(k, \xi, 0) \in D(B(k))$, (a) For any k with $|k|\delta$,*

$$\hat{h} = e^{tB(k)}\hat{h}_o = \lim_{r\to\infty} \frac{1}{2\pi i}\int_{-b_1-ir}^{-b_1+ir} e^{t\lambda}(\lambda - B(k))^{-1}\hat{h}_o \, d\lambda$$
$$+ \sum_{i=1}^{5} e^{td_j(k)}(\psi_j(-k), \hat{h}_o)\psi_j(k),$$

where $d_j(k)$ and $\psi_j(k)$ are the eigenvalues and the corresponding eigenfunctions of $B(k)$.
(b) For $|k| \geq \delta$,

$$\hat{h} = e^{tB(k)}\hat{h}_o = \lim_{r\to\infty} \frac{1}{2\pi i}\int_{-b_2-ir}^{-b_2+ir} e^{t\lambda}(\lambda - B(k))^{-1}\hat{h}_o \, d\lambda$$

3. Decay Rate for Approximate Green Function

Taking inverse Fourier transform in k, we have

$$h(x, \xi, t) = \int \hat{h}\, dk = \int_{|k|\geq\delta} \hat{h}\, dk + \int_{|k|\leq\delta} \hat{h}\, dk.$$

By the spectrum property of $B(k)$, we have

$$d_j(k) = i\alpha_j\kappa - \beta_j\kappa^2 + O(|k|^3),$$

where $\kappa = \equiv|k|$, $\alpha_j \in \mathbb{R}$ and $\beta_j > 0$ for $j = 1, 2, .., 5$ and the limit terms in Theorem 1 satisfy with some $b > 0$

$$\lim_{r\to\infty} \frac{1}{2\pi i}\int_{-b_1-ir}^{-b_1+ir} e^{t\lambda}(\lambda - B(k))^{-1}\hat{h}_o \, d\lambda = O(e^{-bt})$$

for $|k| \leq \delta$ and

$$\lim_{r\to\infty} \frac{1}{2\pi i}\int_{-b_2-ir}^{-b_2+ir} e^{t\lambda}(\lambda - B(k))^{-1}\hat{h}_o \, d\lambda = O(e^{-bt})$$

for $|k| \geq \delta$. Hence

$$\int_{|k|\geq\delta} \hat{h}\, dk = O(e^{-bt})$$

and

$$\begin{aligned}
\int_{|k|\leq\delta} \hat{h}\, dk =& O(e^{-bt}) + \sum_{j=1}^{5}\int_{|k|\leq\delta} e^{-t[i\alpha_j\kappa - \beta_j\kappa^2 + O(|k|^3)]}(\psi_j(-k), \hat{h}_o)\psi_j(k)\, dk \\
=& O(e^{-bt}) + \sum_{j=1}^{5}\int_{|k|\leq\delta} e^{-t[i\alpha_j\kappa - \beta_j\kappa^2]}(\psi_j(-k), \hat{h}_o)\psi_j(k)\, dk \\
&+ \sum_{j=1}^{5}\int_{|k|\leq\delta} e^{-t[i\alpha_j\kappa - \beta_j\kappa^2]}[e^{O(|k|^3)} - 1](\psi_j(-k), \hat{h}_o)\psi_j(k)\, dk \\
=& \sum_{j=1}^{5}\int_{\mathbb{R}^3} e^{-t[i\alpha_j\kappa - \beta_j\kappa^2]}(\psi_j(-k), \hat{h}_o)\psi_j(k)\, dk \\
&+ O(t^{-2}\log^3 t) + O(e^{-b_3 t})
\end{aligned}$$

for some $b_3 > 0$. The final form we obtained is as follows. Let $\psi_j(\xi)$ denote the eigenfunction above when $k = 0$.

Therorem A. *Let $K_j(x,y,t)$ be the heat kernel*

$$K_j(x,y,t) = (4\beta_j\pi)^{-\frac{3}{2}}e^{-\frac{(x-y-\alpha_j t)^2}{4\beta_j t}}.$$

Define the approximate Green function

$$G(x,y,t,\xi,\bar{\xi}) = \sum_{j=1}^{5} G_j(x,y,t,\xi,\bar{\xi}),$$

where G_j has the form

$$\begin{aligned}
&G_j(x,y,t,\xi,\bar{\xi}) = \\
&K_j(x,y,t)\psi_j(\xi)\psi_j(\bar{\xi}) \\
&- i\partial_x K_j(x,y,t)[\psi_j(\xi)\frac{\partial}{\partial k}\psi_j(\bar{\xi}) + \frac{\partial}{\partial k}\psi_j(\xi)\psi_j(\bar{\xi})] \\
&+ \frac{1}{2}\partial_x^2[2(\frac{\partial}{\partial k}\psi_j(\xi), \frac{\partial}{\partial k}\psi_j(\bar{\xi}) + \psi_j(\xi)\frac{\partial^2}{\partial k^2}\psi_j(\bar{\xi}) + \frac{\partial^2}{\partial k^2}\psi_j(\xi)\psi_j(\bar{\xi})]
\end{aligned}$$

Then

$$\frac{\partial}{\partial t}G + i\xi \cdot kG = LG + O(t^{-3})$$

for large t.

The decay rate of the error in Theorem A is good enough to trace the evolution for small data in many cases. We hope in a coming work the full estimate for the evolution of the nonlinear equation with small data can be derived from the approximate Green function.

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DEPARTMENT OF MATHEMATICS, NATIONAL TAIWAN UNIVERSITY, TAIPEI, TAIWAN
E-mail address: chchchen @math.ntu.edu.tw