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TRIVIAL IDENTITIES WITH SKEW DERIVATIONS

BY

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Abstract. Assume that Ω is an expansion closed word set. Let Ω' be a subset of Ω such that the expansion formula of each $\Delta \in \Omega'$ involoves only words in Ω'. So Ω' is also an expansion closed word set. We prove that $\Im(\Omega') = \wp(\Omega') \cap \Im(\Omega)$, where $\wp(\Omega')$ is the set of polynomials with words in Ω' and where $\Im(\Omega')$ and $\Im(\Omega)$ respectively are the ideals of trivial identities with words in Ω' and in Ω respectively. We also prove that any basis of Ω' modulo $\Im(\Omega')$ can be extended to a basis of Ω modulo $\Im(\Omega)$.

This paper is a continuation of [**3**]. Our aim here is to prove something we promised there. We will keep the notations of [**3**]: Throughout here, R is always a prime associative ring and U is its left Utumi quotient ring. The extended centroid C of R is defined to be the center of U. (See [**1**] or [**8**] for a definition.) We let U^{-1} denote the set of all invertible elements in U. Objects of our investigation are the following generalization of products of (higher) skew derivations:

Definition: ([**3**], pp. 294) By an *expansion closed* word set, we mean a set Ω of *symbols* satisfying the following three conditions:

1. There exist two *disjoint* subsets Ω^+, Ω^- of Ω such that $\Omega = \Omega^+ \cup \Omega^-$. There exist two *increasing* sequences of subsets of Ω^+ and Ω^- respectively

$$\Omega_0^+ \subseteq \cdots \Omega_n^+ \subseteq \Omega_{n+1}^+ \subseteq \cdots \subseteq \Omega^+, \qquad \Omega_0^- \subseteq \cdots \Omega_n^- \subseteq \Omega_{n+1}^- \subseteq \cdots \subseteq \Omega^-$$

such that $\Omega^+ = \bigcup_{n\geq 0} \Omega_n^+$ and $\Omega^- = \bigcup_{n\geq 0} \Omega_n^-$.

2. Each symbol $g \in \Omega_0^+$ is associated with an automorphism $x \in R \mapsto x^g \in R$ and the polynomial $\pi_g(x,y) = x^g y^g$. Each symbol $h \in \Omega_0^-$ is associated with an antiautomorphism $x \in R \mapsto x^h \in R$ and the polynomial $\pi_h(x,y) = y^h x^h$.

3. Let $n \geq 1$. Each symbol $\Delta \in \Omega_n^+ \setminus \Omega_0^+$ is associated with an additive map $\Delta : x \in R \mapsto x^\Delta \in U$ and a polynomial $\pi_\Delta(x,y)$ in the form

$$(\dagger^+) \qquad \pi_\Delta(x,y) = x^\Delta y^h + x^g y^\Delta + \sum_i a_i x^{\Delta_i} b_i y^{\Delta_i'} c_i,$$

with $g, h \in \Omega_0^+$, each $a_i, b_i, c_i \in U$ and each $\Delta_i, \Delta_i' \in \Omega_{n-1}^+$, such that for all $x, y \in R$, the identity $(xy)^\Delta = \pi_\Delta(x,y)$ holds. Each symbol $\Delta \in \Omega_n^- \setminus \Omega_0^-$ is associated with an additive map $\Delta : x \in R \mapsto x^\Delta \in U$ and a polynomial $\pi_\Delta(x,y)$ in the form

$$(\dagger^-) \qquad \pi_\Delta(x,y) = y^\Delta x^h + y^g x^\Delta + \sum_i a_i y^{\Delta_i} b_i x^{\Delta_i'} c_i,$$

with $g, h \in \Omega_0^-$, each $a_i, b_i, c_i \in U$ and each $\Delta_i, \Delta_i' \in \Omega_{n-1}^-$, such that for all $x, y \in R$, the identity $(xy)^\Delta = \pi_\Delta(x,y)$ holds.

Symbols of Ω are called *words* and the polynomial $\pi_\Delta(x,y)$ is called the *expansion formula* of the word $\Delta \in \Omega$. Set $\Omega_n \stackrel{\text{def.}}{=} \Omega_n^+ \cup \Omega_n^-$. We define $\bar{\Omega}_0 \stackrel{\text{def.}}{=} \Omega_0$ and $\bar{\Omega}_n \stackrel{\text{def.}}{=} \Omega_n \backslash \Omega_{n-1}$ for $n \geq 1$. Words in $\bar{\Omega}_n$ are said to be of *order* n. For $g, h \in \Omega_0^+$ (or $g, h \in \Omega_0^-$ respectively), let $\Omega_n(g,h)$, $n \geq 1$, be the set consisting of all $\Delta \in \Omega_n$ with $\pi_\Delta(x,y)$ described in the form $(\dagger^+)$ (or $(\dagger^-)$ respectively). We set $\Omega(g,h) \stackrel{\text{def.}}{=} \bigcup_{n=1}^\infty \Omega_n(g,h)$ and $\bar{\Omega}_n(g,h) \stackrel{\text{def.}}{=} \Omega(g,h) \cap \bar{\Omega}_n$ for $n \geq 1$.

We will investigate two expansion closed word sets related in the following way:

Definition: Let Ω be an expansion closed word set. A subset Ω' of Ω such that $\pi_\delta(x,y) \in \wp(\Omega')$ for any $\delta \in \Omega'$ is also an expansion closed word set. We indicate this by simply saying that Ω/Ω' *are expansion closed word sets.*

We fix an infinite set $X = \{x_1, x_2, \dots\}$ of *noncommuting* indeterminates $x_1, x_2, \dots$. For any subset Σ of Ω, let $\wp(\Sigma)$ be the set of all generalized polynomials with coefficients in U and in the indeterminates x^Δ, where $x \in X$ and $\Delta \in \Sigma$. We will regard $\wp(\Sigma)$ as a subset of $\wp(\Omega)$ in a natural way. Elements of $\wp(\Omega)$ are simply called polynomials for brevity. To avoid confusion, the zero element of $\wp(\Omega)$ is called the *zero* polynomial instead of the trivial polynomial, which has a technical meaning below:

Definition: ([**3**], pp. 298-299) Let Ω be an expansion closed word set.

1. Let $\Im$ be an ideal of $\wp(\Omega)$. For $g, h \in \Omega_0^+$ or $g, h \in \Omega_0^-$, a polynomial $\varphi(x) \in \wp(\Omega)$, in the indeterminate $x \in X$ only, is called a *basic* (g,h)-polynomial modulo $\Im$, if

$$\varphi(x+y) - \varphi(x) - \varphi(y) \in \Im \qquad \text{and} \qquad \varphi(xy) - \tilde{x}^g\varphi(\tilde{y}) - \varphi(\tilde{x})\tilde{y}^h \in \Im,$$

where $\tilde{x} \stackrel{\text{def.}}{=} x, \tilde{y} \stackrel{\text{def.}}{=} y$ for $g, h \in \Omega_0^+$ and $\tilde{x} \stackrel{\text{def.}}{=} y, \tilde{y} \stackrel{\text{def.}}{=} x$ for $g, h \in \Omega_0^-$.

2. The ideal of trivial identities of R, denote by $\Im(\Omega)$, is defined to be the *minimal* ideal $\Im$ of $\wp(\Omega)$ such that $\Im$ contains all the basic identities modulo $\Im$. An identity of R is said to be *trivial* or *nontrivial* according it is in $\Im(\Omega)$ or not.

Our main aim here is the following:

Theorem 1. *If Ω/Ω' are expansion closed word sets, then $\Im(\Omega') = \wp(\Omega') \cap \Im(\Omega)$.*

If each Ω_i are expansion closed word sets, then so is their union $\Omega \stackrel{\text{def.}}{=} \bigcup_i \Omega_i$ in a natural way. Theorem 1 above say that $\Im(\Omega_i) = \wp(\Omega_i) \cap \Im(\Omega)$ for each i. This coherence of trivial identities justifies their appropriateness. This problem is *not* encountered in [**6**] or its extensions [**1**], [**2**], and [**7**], because there is essentially only one word set involoved. As the word set Ω of our theory can be various in applications and also as our notion of trivial identities is rather complicate, we need Theorem 1 to assure that our theory does nicely generalize all old ones, such as [1], [2], [4], [5], [6] and [7].

Since our understanding of the ideal $\Im(\Omega)$ of trivial identities mainly comes from bases of Ω modulo $\Im(\Omega)$, we will deduce this from the following result, which is both important and interesting in itself:

Theorem 2. *If Ω/Ω' are expansion closed word sets, then any basis of Ω' modulo $\Im(\Omega')$ can be extended to a basis of Ω modulo $\Im(\Omega)$.*

In our further researches, we often have to extend the given expansion closed word set Ω' to a bigger one Ω and most results of our investigation can be stated in terms of bases or reduced forms. This theorem will enable us to push such results of Ω back to the given Ω'. Granted Theorem 2, we now give

Proof of Theorem 1: Let Ω/Ω' be expansion closed word sets. Obviously, $\wp(\Omega')\cap\Im(\Omega)$ is an ideal of $\wp(\Omega')$. Firstly, if $\varphi \in \wp(\Omega')$ is basic modulo $\wp(\Omega')\cap\Im(\Omega)$, then, as a polynomial in $\wp(\Omega)$, it is also basic modulo $\Im(\Omega)$, since $\wp(\Omega')\cap\Im(\Omega) \subseteq \Im(\Omega)$, and hence $\varphi \in \wp(\Omega')\cap\Im(\Omega)$ by the defining closure property of $\Im(\Omega)$. So the ideal $\wp(\Omega')\cap\Im(\Omega)$ of $\wp(\Omega')$ also enjoys the defining closure property of $\Im(\Omega')$. We hence have $\wp(\Omega') \cap \Im(\Omega) \supseteq \Im(\Omega')$ by the defining minimality of $\Im(\Omega')$. For the other inclusion, we assume the validity of Theorem 2. Fix a basis Ξ of Ω' modulo $\Im(\Omega')$ and extend it to a basis $\Xi \cup \Sigma$ of Ω modulo $\Im(\Omega)$, where Σ is a subset of $\Omega\backslash\Omega'$. Given $\varphi \in \wp(\Omega')\cap\Im(\Omega)$, let $\psi \in \wp(\Xi)$ be its Ξ-reduced form modulo $\Im(\Omega')$. So $\varphi - \psi \in \Im(\Omega')$. Since we have shown $\Im(\Omega') \subseteq \Im(\Omega)$, we also have $\varphi \equiv \psi$ modulo $\Im(\Omega)$. So ψ is also the $\Xi \cup \Sigma$-reduced form of φ modulo $\Im(\Omega)$. Since $\varphi \in \wp(\Omega') \cap \Im(\Omega) \subseteq \Im(\Omega)$, the $\Xi \cup \Sigma$-reduced form ψ of φ must be the zero polynomial. Hence $\varphi = \varphi - \psi \in \Im(\Omega')$, as asserted.

We introduce the following generalized notion of bases modulo ideals:

Definition: Let Ω/Ω' be expansion closed word sets. Let $\Im$, $\Im'$ be ideals of $\wp(\Omega)$, $\wp(\Omega')$ respectively such that $\Im \supseteq \Im'$. (We indicate this by simply saying that $\Im/\Im'$ *are ideals of* Ω/Ω'.) By a basis of Ω/Ω' modulo $\Im/\Im'$, we mean a subset Σ of $\Omega \setminus \Omega'$ satisfying the property: For any given $\varphi \in \wp(\Omega)$, there exists $\psi \in \wp(\Omega' \cup \Sigma)$, which is *unique* up to within equivalence modulo the ideal generated by $\Im'$ in $\wp(\Omega)$, such that $\varphi \equiv \psi$ modulo $\Im$. That is, any given $\varphi \in \wp(\Omega)$ is equivalent to some $\psi \in \wp(\Omega' \cup \Sigma)$ modulo $\Im$ and any two such ψ's are equivalent modulo the ideal generated by $\Im'$ in $\wp(\Omega)$. The polynomial ψ is called the Σ-*reduced* form of φ modulo $\Im/\Im'$. We also observe the equivalence: A subset Σ of $\Omega \setminus \Omega'$ forms a basis of Ω/Ω' modulo $\Im/\Im'$ if and only if $\wp(\Omega' \cup \Sigma) + \Im = \wp(\Omega)$ and $\Im \cap \wp(\Omega' \cup \Sigma)$ is included in the ideal generated by $\Im'$. The first condition is equivalent to the existence of $\psi \in \wp(\Omega' \cup \Sigma)$ which is equivalent to a given $\varphi \in \wp(\Omega)$ modulo $\Im$. The second condition is equivalent to the uniqueness (modulo the ideal generated by $\Im'$) of such ψ, if there exists any.

If $\Omega' = \emptyset$, then $\wp(\Omega')$ is just the ring U and we can merely let $\Im'$ be the zero ideal. By taking $\Omega' = \emptyset$ and $\Im' = 0$, bases of Ω/Ω' modulo $\Im/\Im'$ are merely bases of Ω modulo $\Im$ as defined in [**3**]. In this sense, our notion of bases here generalize the old one in [**3**]. This generalized notion of bases are related to our problem via the following:

Theorem 3. *Let $\Im/\Im'$ be ideals of expansion closed word sets Ω/Ω'. If Ω' possesses a basis modulo $\Im'$, then the following are equivalent for a subset Σ of $\Omega \setminus \Omega'$:*
(1) The set Σ forms a basis of Ω/Ω' modulo $\Im/\Im'$.
(2) For any basis Ξ of Ω' modulo $\Im'$, $\Xi \cup \Sigma$ forms a basis of Ω modulo $\Im$.
(3) There exists a basis Ξ of Ω' modulo $\Im'$ such that $\Xi \cup \Sigma$ forms a basis of Ω modulo $\Im$.

In view of this, Theorem 2 is equivalent to the existence of bases of Ω/Ω' modulo $\Im(\Omega)/\Im(\Omega')$, since we have already shown $\Im(\Omega') \subseteq \Im(\Omega)$ in the proof of Theorem 1. Our construction of bases thus generalized essentially follows the line of [**3**]. But, because of the complexity of the argument, it seems rather difficult to indicate merely where and how such modifications should be made. For the sake of clarity, we will supply all the details. For convenience of reference, we will also recall all important notions of [**3**]. Actually, we intend to make this paper self-contained so that corresponding results of [**3**] can be read off here as special instances by letting $\Omega' = \emptyset$ and $\Im' = 0$. We start with the following generalization of Fact 1 [**3**]:

Fact 1. *Let $\Im/\Im'$ be ideals of expansion closed word sets Ω/Ω' and let Σ be a subset of $\Omega^c \stackrel{\text{def.}}{=} \Omega \setminus \Omega'$. Assume that for each $\Delta \in \Omega^c \setminus \Sigma$, there is assigned a polynomial $\lambda_\Delta(x) \in \wp(\Omega' \cup \Sigma)$ in the indeterminate x only. Then the set Σ forms a basis of Ω/Ω' modulo $\Im/\Im'$ with $\lambda_\Delta(x)$ being the Σ-reduced form of x^Δ for each $\Delta \in \Omega^c \setminus \Sigma$ if and only if $\Im$ is the ideal of $\wp(\Omega)$ generated by $\Im'$ together with all $x^\Delta - \lambda_\Delta(x)$ for $\Delta \in \Omega^c \setminus \Sigma$.*

Proof: The sufficiency ($\Rightarrow$) is simple: Assume that Σ forms a basis of Ω/Ω' modulo $\Im/\Im'$ and that $\lambda_\Delta(x)$ is the Σ-reduced expression of x^Δ for $\Delta \in \Omega^c \setminus \Sigma$. Then each $x^\Delta - \lambda_\Delta(x) \in \Im$. Set $\Im^\dagger$ to be the ideal of $\wp(\Omega)$ generated by $\Im'$ and all such $x^\Delta - \lambda_\Delta(x)$. Then $\Im^\dagger \subseteq \Im$. For $\varphi \in \wp(\Omega)$, let $\varphi' \in \wp(\Omega' \cup \Sigma)$ be the expression obtained from φ by replacing every occurrence of x^Δ, where $\Delta \in \Omega^c \setminus \Sigma$, by $\lambda_\Delta(x)$. We have $\varphi' \in \wp(\Omega' \cup \Sigma)$ and $\varphi - \varphi' \in \Im^\dagger \subseteq \Im$. Hence φ' is the Σ-reduced form of φ modulo $\Im$. Suppose $\varphi \in \Im$. By the uniqueness of reduced forms, the Σ-reduced expression φ' of φ is equivalent to the zero polynomial modulo the ideal generated by $\Im'$. Since $\Im^\dagger \supseteq \Im'$, we also have $\varphi' \equiv 0$ modulo $\Im^\dagger$, that is, $\varphi' \in \Im^\dagger$. So $\varphi = (\varphi - \varphi') + \varphi' \in \Im^\dagger$. Since $\varphi \in \Im$ is arbitrary, $\Im \subseteq \Im^\dagger$ follows. So $\Im = \Im^\dagger$, as asserted.

To prove the necessity ($\Leftarrow$), we define $\lambda'_\Delta(x) \stackrel{\text{def.}}{=} x^\Delta - \lambda_\Delta(x)$ for $\Delta \in \Omega^c \setminus \Sigma$. Assume that $\Im$ is the ideal of $\wp(\Omega)$ generated by $\Im'$ and all such $\lambda'_\Delta(x)$. Then $x^\Delta \equiv \lambda_\Delta(x)$ modulo $\Im$ for $\Delta \in \Omega^c \setminus \Sigma$. Given $\varphi \in \wp(\Omega)$, let φ' be the expression obtained from φ by replacing every occurrence of x^Δ, where $\Delta \in \Omega^c \setminus \Sigma$, by $\lambda_\Delta(x)$. Then $\varphi' \in \wp(\Omega' \cup \Sigma)$ and $\varphi \equiv \varphi'$ modulo $\Im$. To show the uniqueness of such φ', we must show that $\Im \cap \wp(\Omega' \cup \Sigma)$ is a subset of the ideal generated by $\Im'$ in $\wp(\Omega)$. Let $\varphi \in \Im \cap \wp(\Omega' \cup \Sigma)$ be given arbitrarily. Since the ideal $\Im$ is generated by $\Im'$ and the set of $\lambda'_\Delta(x)$, where $\Delta \in \Omega^c \setminus \Sigma$, we may write

$$\varphi = \sum_{\theta \in \Im'} \zeta\theta\eta + \sum_{\Delta \in \Omega^c \setminus \Sigma} \phi\lambda'_\Delta(x)\psi, \tag{0}$$

where $\zeta, \eta, \phi, \psi \in \wp(\Omega)$, where the first summation ranges over $\theta \in \Im'$ and where the second summation ranges over all possible $x \in X$, $\Delta \in \Omega^c \setminus \Sigma$. In ζ, η, ϕ and ψ, we first replace every occurrence of x^Δ, where $\Delta \in \Omega^c \setminus \Sigma$, by $\lambda'_\Delta(x) + \lambda_\Delta(x)$. We then substitute expressions of $\lambda_\Delta(x)$ as polynomials in $\wp(\Omega' \cup \Sigma)$ into the resulting expression but we leave all the occurrences of $\lambda'_\Delta(x)$ unaltered like *indeterminates* for a moment. That is, in ζ, η, ϕ and ψ, we replace every occurrence of x^Δ, where $\Delta \in \Omega^c \setminus \Sigma$, by

$$\lambda'_\Delta(x) + \cdots ,$$

where the dots denote the expression of $\lambda_\Delta(x)$ as a polynomial in $\wp(\Omega' \cup \Sigma)$ and where $\lambda'_\Delta(x)$ is treated like indeterminates for a moment. The final expression thus obtained from (0) can be expanded as a generalized polynomial in the "indeterminates" $\lambda'_\Delta(x)$, where $\Delta \in \Omega^c \setminus \Sigma$, in the "indeterminates" x^Δ, where $\Delta \in \Omega' \cup \Sigma$ and also in the "indeterminates" $\theta \in \Im'$. Observe that terms resulting from the first summation of (0) always contain one factor $\theta \in \Im'$ and also that terms resulting from the second summation of (0) must involve at least one factor $\lambda'_\Delta(x)$. We may incorporate all terms containing $\lambda'_\Delta(x)$, $\Delta \in \Omega^c \setminus \Sigma$, into the the second summation of (0) and express all occurrences of $\theta \in \Im'$ in such terms as polynomials in $\wp(\Omega')$. By doing so, we may hence rewrite (0) as

$$\varphi = \sum_{\theta \in \Im'} \zeta\theta\eta + \sum_{k \geq 1} \mu_0 \lambda'_{\Delta_1}(x_1)\mu_1 \cdots \lambda'_{\Delta_k}(x_k)\mu_k, \tag{1}$$

where all $\zeta, \eta, \mu_i \in \wp(\Omega' \cup \Sigma)$. The second summation ranges over all $k \geq 1$, over all $x_i \in X$ (not necessarily distinct) and over all $\Delta_i \in \Omega^c \setminus \Sigma$ (not necessarily distinct). Fix any arbitrarily given $\lambda'_{\Delta_1}(x_1), \ldots, \lambda'_{\Delta_{k'}}(x_{k'})$ (not necessarily distinct) in this order. Let

$$\sum{}' \mu'_0 \lambda'_{\Delta_1}(x_1)\mu'_1 \cdots \lambda'_{\Delta_{k'}}(x_{k'})\mu'_{k'} \tag{2}$$

denote the sum of all terms in the second summation of (1) with $\lambda'_{\Delta_1}(x_1), \ldots, \lambda'_{\Delta_{k'}}(x_{k'})$ occurring in this order. We want to show that this summation (2) is equal to the zero polynomial in the "indeterminates" $\lambda'_{\Delta_i}(x)$. That is, we want to show that

$$\sum{}' \mu'_0 \otimes \mu'_1 \cdots \otimes \mu'_{k'} = 0,$$

where the tensor product $\otimes$ is over the extended centroid C. Assume on the contrary that there exists such a nonzero summation. Without loss of generality, we may assume that our k' is chosen to be the largest possible among all such nonzero summations in (1). We hence assume that (2) is such a nonzero summation for the k' so chosen. In the two expressions (1) and (2), we first replace each "indeterminate" $\lambda'_\Delta(x)$, where $\Delta \in \Omega^c \setminus \Sigma$, by the *polynomial* expression $x^\Delta - \lambda_\Delta(x)$ and we then expand the expressions thus resulted. Observe that the expansion of (2) in $\wp(\Omega)$ gives rise to the following sum

$$\sum{}' \mu'_0 x_1^{\Delta_1} \mu'_1 \cdots x_{k'}^{\Delta_{k'}} \mu'_{k'}.$$

All ζ, η, θ in the first summation of (1) fall in $\wp(\Omega' \cup \Sigma)$ and hence can*not* contribute to give such terms. By the maximality of k' in the second summation of (1), this sum consists of all terms in the expansion of (1) with the factors $x_1^{\Delta_1}, \ldots, x_{k'}^{\Delta_{k'}}$ occurring in this order. But there are no such terms in φ since $\varphi \in \wp(\Omega' \cup \Sigma)$ and $\Delta_1, \ldots, \Delta_{k'} \in \Omega^c \setminus \Sigma = \Omega \setminus (\Omega' \cup \Sigma)$. By comparing the expressions on both sides of (1) as polynomials in $\wp(\Omega)$, we thus obtain

$$\sum{}' \mu'_0 \otimes \mu'_1 \cdots \otimes \mu'_{k'} = 0.$$

This says that the summation $\sum' \mu'_0 \lambda'_{\Delta_1}(x_1)\mu'_1 \cdots \lambda'_{\Delta_{k'}}(x_{k'})\mu'_{k'}$ given in (2) is the zero polynomial in "indeterminates" $\lambda'_{\Delta_i}(x)$, a contradiction to our assumption. Hence the second summation on the right hand side of (1) is really equal to the zero polynomial in the "indeterminates" $\lambda'_{\Delta_i}(x)$. It follows that $\varphi = \sum_{\theta \in \Im'} \zeta\theta\eta$. Thus φ falls in the ideal generated by $\Im'$, as asserted.

We are now ready for

Proof of Theorem 3: For brevity, we set $\Omega^c \stackrel{\text{def.}}{=} \Omega \setminus \Omega'$ and let $\Im^\dagger$ denote the ideal generated by $\Im'$ in $\wp(\Omega)$.

(1) $\Rightarrow$ (2): Assume that Σ is a basis of Ω/Ω' modulo $\Im/\Im'$. For $\Delta \in \Omega^c \setminus \Sigma$, let $\lambda_\Delta(x) \in \wp(\Omega' \cup \Sigma)$ be the Σ-reduced form of x^Δ. Given any basis Ξ of Ω' modulo $\Im'$, we also let $\lambda_\delta(x) \in \wp(\Xi)$ be the Ξ-reduced form of x^δ for $\delta \in \Omega' \setminus \Xi$. By replacing all occurrences of x^δ, $\delta \in \Omega'$, by their Ξ-reduced forms (modulo $\Im'$), we may assume $\lambda_\Delta(x) \in \wp(\Xi \cup \Sigma)$ for $\Delta \in \Omega^c \setminus \Sigma$. Since Ξ is a basis of Ω' modulo $\Im'$, the ideal $\Im'$ of $\wp(\Omega')$ is generated by $x^\delta - \lambda_\delta(x)$, $\delta \in \Omega' \setminus \Xi$, by Fact 1. Similarly, since Σ is a basis of Ω/Ω' modulo $\Im/\Im'$, the ideal $\Im$ of $\wp(\Omega)$ is generated by $\Im'$ and $x^\Delta - \lambda_\Delta(x)$, $\Delta \in \Omega^c \setminus \Sigma$, by Fact 1 again. Therefore, $\Im$ is generated by $x^\Delta - \lambda_\Delta(x)$ for $\Delta \in \Omega^c \setminus \Sigma$ and $x^\delta - \lambda_\delta(x)$ for $\delta \in \Omega' \setminus \Xi$ altogether. By Fact 1 again, the set $\Xi \cup \Sigma$ forms a basis of Ω modulo $\Im$.

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Let Ξ be a basis of Ω' modulo $\Im'$ such that $\Xi \cup \Sigma$ forms a basis of Ω modulo $\Im$. Let $\lambda_\Delta(x) \in \wp(\Xi \cup \Sigma)$ be the $\Xi \cup \Sigma$-reduced form of x^Δ for $\Delta \in \Omega \setminus (\Xi \cup \Sigma)$. By Fact 1, $\Im$ is generated by these $x^\Delta - \lambda_\Delta(x)$. For $\delta \in \Omega' \setminus \Xi$, if $\lambda'_\delta(x)$ is the Ξ-reduced form of x^δ (modulo $\Im'$), then this $\lambda'_\delta(x)$ is also the $\Xi \cup \Sigma$-reduced form of x^δ (modulo $\Im$), since $\lambda'_\delta(x) \in \wp(\Xi) \subseteq \wp(\Xi \cup \Sigma)$ and $x^\delta - \lambda'_\delta(x) \in \Im' \subseteq \Im$. So the $\Xi \cup \Sigma$-reduced form $\lambda_\delta(x)$ of x^δ for $\delta \in \Omega'$ is also its Ξ-reduced form. By Fact 1 again, $\Im'$ is generated by $x^\delta - \lambda_\delta(x)$, $\delta \in \Omega' \setminus \Xi$. So $\Im$ is generated by $\Im'$ and $x^\Delta - \lambda_\Delta(x)$, $\Delta \in \Omega^c \setminus \Sigma$. By Fact 1 again, Σ forms a basis of Ω/Ω' modulo $\Im/\Im'$.

Fact 1 and Theorem 3 are actually valid for any arbitrary *operator* sets Ω/Ω, as can be seen from their proofs above. Since our word sets are inductively defined, we need notions with ordered structures to allow some sorts of induction:

Definition: Let Ω/Ω' be expansion closed word sets.

1. A polynomial $\varphi \in \wp(\Omega) \setminus \wp(\Omega')$ is said to be of Ω/Ω'-*order* n if n is the least integer ≥ 0 such that $\varphi \in \wp(\Omega' \cup \Omega_n)$. That is, the Ω/Ω'-order of $\varphi \in \wp(\Omega) \setminus \wp(\Omega')$ is the least $n \geq 0$ such that all words involoved in φ fall in $\Omega' \cup \Omega_n$. By a leading Ω/Ω'-*word* of $\varphi \in \wp(\Omega) \setminus \wp(\Omega')$, we mean a word $\Delta \in \Omega \setminus \Omega'$ of highest possible order which occurs nontrivially in φ. By the leading Ω/Ω'-part of $\varphi \in \wp(\Omega) \setminus \wp(\Omega')$, we mean the sum of terms of φ which involove leading Ω/Ω'-words of φ. Obviously, $\varphi \in \wp(\Omega) \setminus \wp(\Omega')$ is of Ω/Ω'-order n if and only if all its leading Ω/Ω'-words are of order n. We *postulate* that $\varphi \in \wp(\Omega')$ is of Ω/Ω'-order -1 and has *no* leading Ω/Ω'-words. It is also convenient to set $\Omega_{-1} \stackrel{\text{def.}}{=} \emptyset$. We then have that $\varphi \in \wp(\Omega)$ is of Ω/Ω'-*order* n if n is the least integer ≥ -1 such that $\varphi \in \wp(\Omega' \cup \Omega_n)$.

2. Let $\Im/\Im'$ be ideals of expansion closed word sets Ω/Ω'. A basis Σ of Ω/Ω' modulo $\Im/\Im'$ is said to be *ordered*, if for any given $\varphi \in \wp(\Omega)$, there exists $\psi \in \wp(\Omega' \cup \Sigma)$, whose Ω/Ω'-order is $\leq$ the Ω/Ω'-order of φ, such that $\varphi \equiv \psi$ modulo $\Im$. By the definition of bases of Ω/Ω' modulo $\Im/\Im'$, such ψ is unique up to within equivalence modulo the ideal generated by $\Im'$ in $\wp(\Omega)$.

Again, if $\Omega' = \emptyset$ and $\Im' = 0$, then ordered bases of Ω/Ω' modulo $\Im/\Im'$ defined above reduce to ordered bases of Ω modulo $\Im$ as defined in [**3**]. We have the following simple characterization of ordered bases:

Fact 2. *Let $\Im/\Im'$ be ideals of expansion closed word sets Ω/Ω'. A basis Σ of Ω/Ω' modulo $\Im/\Im'$ is ordered if and only if for each $\Delta \in \Omega \setminus (\Omega' \cup \Sigma)$, the Σ-reduced form of x^Δ has Ω/Ω'-order $\leq$ the order of Δ.*

Proof: Let $\lambda_\Delta(x)$ be the Σ-reduced form of x^Δ for $\Delta \in \Omega \setminus (\Omega' \cup \Sigma)$. The sufficiency ($\Rightarrow$) is obvious. For the necessity ($\Leftarrow$), the Σ-reduced form of $\varphi \in \wp(\Omega)$ is merely the expression obtained by replacing every occurrence of x^Δ in φ by $\lambda_\Delta(x)$ for those $\Delta \in \Omega \setminus (\Omega' \cup \Sigma)$. If the Ω/Ω'-order of $\lambda_\Delta(x)$ is $\leq$ the order of Δ for each $\Delta \in \Omega \setminus (\Omega' \cup \Sigma)$, the resulting reduced form also has the Ω/Ω'-order $\leq$ the order of φ.

In [**3**], ordered bases are analyzed in terms of basic polynomials, which we have recalled above. We define our generalization of simple basic polynomials of [**3**] in the following:

Definition: Let Ω/Ω' be expansion closed word sets and let $\Im$ be an ideal of $\wp(\Omega)$. A basic (g, h)-polynomial $\varphi(x)$ (modulo $\Im$) of Ω/Ω'-order $n > 0$ is called Ω/Ω'-*simple* if it assumes the form

$$\varphi(x) = \sum_{i=1}^{s} a_i x^{\Delta_i} b_i + \rho(x),$$

where $\Delta_i \in \bar{\Omega}_n(g_i, h_i) \setminus \Omega'$ are distinct, where $a_i, b_i \in U^{-1}$ are such that $a_i x^{g_i} - x^g a_i \in \Im$, $x^{h_i} b_i - b_i x^h \in \Im$ for each i, and where $\rho(x)$ is a linear polynomial in $\wp(\Omega' \cup \Omega_{n-1}^+)$ or in $\wp(\Omega' \cup \Omega_{n-1}^-)$ according as $g, h \in \Omega_0^+$ or $g, h \in \Omega_0^-$ respectively. Equivalently, a basic (g, h)-polynomial $\varphi(x)$ (modulo $\Im$) of Ω/Ω'-order $n > 0$ is called Ω/Ω'-simple if it is a linear polynomial in $\wp(\Omega' \cup \Omega_n^+)$ or in $\wp(\Omega' \cup \Omega_n^-)$, according as $g, h \in \Omega_0^+$ or $g, h \in \Omega_0^-$ respectively, and if its leading Ω/Ω'-part assumes the form $\sum_{i=1}^s a_i x^{\Delta_i} b_i$, where $\Delta_i \in \bar{\Omega}_n(g_i, h_i) \setminus \Omega'$ are distinct and where $a_i, b_i \in U^{-1}$ are such that $a_i x^{g_i} - x^g a_i \in \Im$, $x^{h_i} b_i - b_i x^h \in \Im$ for each i. If $\varphi(x)$ happens to be an identity of R, then we call $\varphi(x)$ an Ω/Ω'-simple basic (g, h)-identity modulo $\Im$.

Intuitively, a basic (g, h)-polynomial $\varphi(x)$ is Ω/Ω'-simple if, modulo $\Im$, $\varphi(xy) - x^g \varphi(y) - \varphi(x) y^h$ is of lower Ω/Ω'-order in a simple way. To be precise, we actually define the Ω/Ω'-simplicity so that the following holds:

Fact 3. *Let Ω/Ω' be expansion closed word sets and let $\Im$ be an ideal of $\wp(\Omega)$. For given $\varphi(x) \in \wp(\Omega)$ of Ω/Ω'-order $n > 0$, $\varphi(x)$ is Ω/Ω'-simple basic modulo the ideal $\Im$ if and only if it is Ω/Ω'-simple basic modulo the ideal generated by $\Im \cap \wp(\Omega' \cup \Omega_{n-1})$.*

Proof: The necessity ($\Leftarrow$) is obvious. We show the sufficiency ($\Rightarrow$): Suppose that $\varphi(x)$ is an Ω/Ω'-simple basic (g, h)-polynomial modulo $\Im$ in the form:

$$\varphi(x) = \sum_i a_i x^{\Delta_i} b_i + \rho(x),$$

where $a_i, b_i \in U^{-1}$, $\Delta_i \in \bar{\Omega}_n^c(g_i, h_i) \stackrel{\text{def.}}{=} \bar{\Omega}_n(g_i, h_i) \setminus \Omega'$ are such that $a_i x^{g_i} - x^g a_i \in \Im$, $x^{h_i} b_i - b_i x^h \in \Im$, and where $\rho(x)$ is a linear polynomial in $\wp(\Omega' \cup \Omega_{n-1}^+)$ or in $\wp(\Omega' \cup \Omega_{n-1}^-)$ according as $g, h \in \Omega_0^+$ or $g, h \in \Omega_0^-$ respectively. Set $\tilde{x} \stackrel{\text{def.}}{=} x, \tilde{y} \stackrel{\text{def.}}{=} y$ for $g, h \in \Omega_0^+$ and $\tilde{x} \stackrel{\text{def.}}{=} y, \tilde{y} \stackrel{\text{def.}}{=} x$ for $g, h \in \Omega_0^-$. For each i, we let

$$\psi_{\Delta_i}(x, y) \stackrel{\text{def.}}{=} \pi_{\Delta_i}(x, y) - \tilde{x}^{g_i} \tilde{y}^{\Delta_i} - \tilde{x}^{\Delta_i} \tilde{y}^{h_i}.$$

So we have $\psi_{\Delta_i}(x, y) \in \wp(\Omega' \cup \Omega_{n-1})$ and $\pi_{\Delta_i}(x, y) = \tilde{x}^{g_i} \tilde{y}^{\Delta_i} + \tilde{x}^{\Delta_i} \tilde{y}^{h_i} + \psi_{\Delta_i}(x, y)$. We compute:

$$\begin{aligned}
&\varphi(xy) - \tilde{x}^g \varphi(\tilde{y}) - \varphi(\tilde{x}) \tilde{y}^h \\
&= \sum_i a_i (xy)^{\Delta_i} b_i + \rho(xy) - \tilde{x}^g \Big(\sum_i a_i \tilde{y}^{\Delta_i} b_i + \rho(\tilde{y})\Big) - \Big(\sum_i a_i \tilde{x}^{\Delta_i} b_i + \rho(\tilde{x})\Big) \tilde{y}^h \\
&= \sum_i a_i (\tilde{x}^{g_i} \tilde{y}^{\Delta_i} + \tilde{x}^{\Delta_i} \tilde{y}^{h_i} + \psi_{\Delta_i}(x, y)) b_i + \rho(xy) \\
&\qquad - \tilde{x}^g \Big(\sum_i a_i \tilde{y}^{\Delta_i} b_i + \rho(\tilde{y})\Big) - \Big(\sum_i a_i \tilde{x}^{\Delta_i} b_i + \rho(\tilde{x})\Big) \tilde{y}^h \\
&= \sum_i (a_i \tilde{x}^{g_i} - \tilde{x}^g a_i) \tilde{y}^{\Delta_i} b_i + \sum_i a_i \tilde{x}^{\Delta_i} (\tilde{y}^{h_i} b_i - b_i \tilde{y}^h) \\
&\qquad + \sum_i a_i \psi_{\Delta_i}(x, y) b_i + \rho(xy) - \tilde{x}^g \rho(\tilde{y}) - \rho(\tilde{x}) \tilde{y}^h.
\end{aligned}$$

Observe that $a_i\tilde{x}^{g_i} - \tilde{x}^g a_i$, $\tilde{y}^{h_i}b_i - b_i\tilde{y}^h$ and $\varphi(xy) - \tilde{x}^g\varphi(\tilde{y}) - \varphi(\tilde{x})\tilde{y}^h$ all fall in $\Im$ by the definition of being Ω/Ω'-simple basic modulo $\Im$. So the expression $\sum_i a_i\psi_{\Delta_i}(x,y)b_i + \rho(xy) - \tilde{x}^g\rho(\tilde{y}) - \rho(\tilde{x})\tilde{y}^h$ also falls in $\Im$ and hence in $\Im \cap \wp(\Omega' \cup \Omega_{n-1})$, since each term in this expression is of Ω/Ω'-order $< n$. But all $a_i\tilde{x}^{g_i} - \tilde{x}^g a_i, \tilde{y}^{h_i}b_i - b_i\tilde{y}^h \in \Im \cap \wp(\Omega_0) \subseteq \Im \cap \wp(\Omega' \cup \Omega_{n-1})$, since $n > 0$. The last equality displayed above says that $\varphi(xy) - \tilde{x}^g\varphi(\tilde{y}) - \varphi(\tilde{x})\tilde{y}^h$ falls in the ideal of $\wp(\Omega)$ generated by the polynomials $a_i\tilde{x}^{g_i} - \tilde{x}^g a_i$, $\tilde{y}^{h_i}b_i - b_i\tilde{y}^h$ and $\sum_i a_i\psi_{\Delta_i}(x,y)b_i + \rho(xy) - \tilde{x}^g\rho(\tilde{y}) - \rho(\tilde{x})\tilde{y}^h$, all of which have been shown to be in $\Im \cap \wp(\Omega' \cup \Omega_{n-1})$. So $\varphi(xy) - \tilde{x}^g\varphi(\tilde{y}) - \varphi(\tilde{x})\tilde{y}^h$ falls in the ideal generated by the set $\Im \cap \wp(\Omega' \cup \Omega_{n-1})$. Also, since $\varphi(x)$ is linear in x, $\varphi(x+y) - \varphi(x) - \varphi(y)$ is the zero polynomial and trivially falls in the ideal generated by $\Im \cap \wp(\Omega' \cup \Omega_{n-1})$. Thus $\varphi(x)$ is an Ω/Ω'-simple basic (g,h)-identity modulo the ideal generated by $\Im \cap \wp(\Omega' \cup \Omega_{n-1})$, as asserted.

The crucial step of our argument is the following generalization of Fact 2 ([**3**], pp. 308):

Fact 4. *Let Σ be an ordered basis of expansion closed word sets Ω/Ω' modulo the ideals $\Im/\Im'$. Let Ξ be a basis of Ω' modulo $\Im'$. If the $\Xi \cup \Sigma$-reduced form of x^g, $g \in \Omega_0$, assumes the form $ux^{\tilde{g}}u^{-1}$ for some $u \in U^{-1}$ and $\tilde{g} \in (\Xi \cup \Sigma) \cap \Omega_0$, then we have the following:*

(0) Any basic polynomial $\varphi(x) \in \wp(\Xi \cup \Sigma)$ modulo $\Im$ of Ω/Ω'-order -1 falls in $\wp(\Xi)$ and is also basic modulo $\Im'$.

(1) Any basic (g,h)-polynomial $\varphi(x) \in \wp(\Xi \cup \Sigma)$ modulo $\Im$ of Ω/Ω'-order 0 assumes the form $a(x^{\tilde{g}}c - cx^{\tilde{h}})b$, where $a, b \in U^{-1}$, $0 \neq c \in U$ and $\tilde{g}, \tilde{h} \in \Sigma_0$ are such that $ax^{\tilde{g}} - x^g a \in \Im$, $x^{\tilde{h}}b - bx^h \in \Im$.

(2) Any basic (g,h)-polynomial $\rho(x) \in \wp(\Xi \cup \Sigma)$ modulo $\Im$ of Ω/Ω'-order > 0 is Ω/Ω'-simple.

Proof: By Theorem 3, $\Xi \cup \Sigma$ forms a basis of Ω modulo $\Im$. We start with two claims:

Claim 1. For $\varphi(x) \in \wp(\Xi \cup \Sigma)$ in x only, if $\varphi(x+y) - \varphi(x) - \varphi(y) \in \Im$, then $\varphi(x)$ is linear in x: Assume on the contrary that $\varphi(x)$ involves a nonlinear term $a_0x^{\Delta_1}a_1 \cdots x^{\Delta_s}a_s$, where $s > 1$, $a_0, a_1, \ldots, a_s \in U$ and $\Delta_1, \ldots, \Delta_s \in \Xi \cup \Sigma$. Then $\varphi(x+y) - \varphi(x) - \varphi(y)$ involoves nontrivially all terms in the form $a_0z_1^{\Delta_1}a_1 \cdots z_s^{\Delta_s}a_s$, where each z_i is either x or y and where there exist at least one z_i equal to x and also at least one z_i equal to y, since all such terms do come from $a_0x^{\Delta_1}a_1 \cdots x^{\Delta_s}a_s$ and they can*not* be cancelled with each other. So $\varphi(x+y) - \varphi(x) - \varphi(y)$ is a nonzero polynomial of $\wp(\Xi \cup \Sigma)$ and thus $\varphi(x+y) - \varphi(x) - \varphi(y) \notin \Im$, since $\Xi \cup \Sigma$ is a basis of Ω modulo $\Im$. But this contradicts the assumption that $\varphi(x+y) - \varphi(x) - \varphi(y) \in \Im$.

Claim 2$^+$. For a linear $\varphi(x) \in \wp(\Xi \cup \Sigma)$, if $\varphi(xy) \equiv \sum_i a_ix^{\delta_i}b_iy^{\delta_i'}c_i$ modulo $\Im$ for some $a_i, b_i, c_i \in U$ and $\delta_i, \delta_i' \in \Omega$, then $\varphi(x) \in \wp(\Xi \cup \Sigma^+)$, where $\Sigma^+ \overset{\text{def.}}{=} \Sigma \cap \Omega^+$: Assume on the contrary that some $\Delta \in \Sigma \cap \Omega_m^-$, with m maximal possible, occurs nontrivially in $\varphi(x)$. Let us consider the case that $\Delta \in \Omega_m(g,h)$, where $m > 0$ and $g, h \in \Omega_0^-$. The case that $\Delta \in \Omega_0^-$ can be treated analogously. Let $uy^{\tilde{g}}u^{-1}$, $vx^{\tilde{h}}v^{-1}$, where $u, v \in U^{-1}$ and $\tilde{g}, \tilde{h} \in (\Xi \cup \Sigma) \cap \Omega_0$, be the $\Xi \cup \Sigma$-reduced forms (modulo $\Im$) of y^g, x^h respectively. By the fact that Σ is an ordered basis of Ω/Ω' modulo $\Im/\Im'$, the $\Xi\cup\Sigma$-reduced form of $(xy)^\Delta$ is equal to

$$uy^{\tilde g}u^{-1}x^\Delta + y^\Delta vx^{\tilde h}v^{-1} + \text{terms of } \Omega/\Omega'\text{-order} < m.$$

In the $\Xi\cup\Sigma$-reduced form of $\varphi(xy)$, terms with y, x occurring in this order must come from the expansion of $(xy)^\delta$ for some $\delta\in(\Xi\cup\Sigma)\cap\Omega^-$. If $\delta\in\Xi\subseteq\Omega'$, then $(xy)^\delta$ falls in $\wp(\Omega')$ and hence its $\Xi\cup\Sigma$-reduced form falls in $\wp(\Xi)$. So such $(xy)^\delta$, where $\delta\in\Xi$, can*not* contribute to give a term with $y^{\tilde g}$, x^Δ occurring in this order. If $\delta\in\Sigma\cap\Omega^-_{m-1}$, then $(xy)^\delta$ also falls in $\wp(\Omega^-_{m-1})$ and its $\Xi\cup\Sigma$-reduced form consists only of terms of Ω/Ω'-order $\le m-1$, since Σ is an ordered basis of Ω/Ω' modulo $\Im/\Im'$. Such $(xy)^\delta$, where $\delta\in\Sigma\cap\Omega^-_{m-1}$, can*not* contribute to give a term with $y^{\tilde g}$, x^Δ occurring in this order either. So we are left with the case $\delta\in\Sigma\cap\Omega^-_m$. For $\delta\in\Sigma\cap\Omega^-_m(g',h')$, the only terms of Ω/Ω'-order m in $(xy)^\delta$ are $y^{g'}x^\delta$ and $y^\delta x^{h'}$. To give terms with $y^{\tilde g}$, x^Δ occurring in this order, this δ must be Δ itself. So it suffices to consider terms involoving Δ. Let $\sum_{j=1}^s a'_jx^\Delta b'_j$ be the sum of all terms of $\varphi(x)$ involving Δ. In the reduced form of $\varphi(xy)$, terms with $y^{\tilde g}, x^\Delta$ occurring in this order must come from $\sum_{j=1}^s a'_j(xy)^\Delta b'_j$. The sum of such terms is thus given by $\sum_{j=1}^s a'_juy^{\tilde g}u^{-1}x^\Delta b'_j$. This must be the zero polynomial, since $\varphi(xy)$ is assumed to be equivalent (modulo $\Im$) to a sum of terms with x^δ, $y^{\delta'}$ occurring in this order and hence the $\Xi\cup\Sigma$-reduced form of $\varphi(xy)$ must also consist entirely of terms with x^δ, $y^{\delta'}$ occurring in this order. We thus have $\sum_{j=1}^s a'_ju\otimes u^{-1}\otimes b'_j=0$, where the tensor product is taken over C. But this implies immediately $\sum_{j=1}^s a'_j\otimes b'_j=0$, a contradiction.

We have analogously:

Claim 2$^-$. For a linear $\varphi(x)\in\wp(\Xi\cup\Sigma)$, if $\varphi(xy)\equiv\sum_i a_iy^{\delta_i}b_ix^{\delta'_i}c_i$ modulo $\Im$ for some $a_i,b_i,c_i\in U$ and $\delta_i,\delta'_i\in\Omega$, then $\varphi(x)\in\wp(\Xi\cup\Sigma^-)$, where $\Sigma^-\overset{\text{def.}}{=}\Sigma\cap\Omega^-$.

We remark that if the basis Ξ of Ω' modulo $\Im'$ is ordered, then $\varphi(x)\in\wp(\Omega^+)$ or $\varphi(x)\in\wp(\Omega^-)$ respectively, according as $g,h\in\Omega_0^+$ or $g,h\in\Omega_0^-$ respectively: Let us say $g,h\in\Omega_0^+$. We have already shown $\varphi(x)\in\wp(\Omega'\cup\Omega^+)$ in Claim 2. If some $\Delta\in\Xi\cap(\Omega')^-$ occurs nontrivially in $\varphi(x)$, then we pick such Δ with the maximal possible order and, now using the ordered basis Ξ of Ω' modulo $\Im'$, we argue in the same way as Claim 2^+ of [**3**] (pp. 308) to reach a contradiction.

Assume that $\varphi(x)\in\wp(\Xi\cup\Sigma)$, in the indeterminate $x\in X$ only, is a given basic (g,h)-polynomial modulo $\Im$. Let us assume that $g,h\in\Omega_0^+$. The case that $g,h\in\Omega_0^-$ is treated analogously. Since $\varphi(x+y)-\varphi(x)-\varphi(y)\in\Im$, $\varphi(x)$ is linear by Claim 1. Since $\varphi(xy)\equiv x^g\varphi(y)+\varphi(x)y^h$ modulo $\Im$, $\varphi(x)\in\wp(\Xi\cup\Sigma^+)$ by Claim 2. Let $ux^{\tilde g}u^{-1}$, $vy^{\tilde h}v^{-1}$, where $u,v\in U^{-1}$ and $\tilde g,\tilde h\in(\Xi\cup\Sigma)\cap\Omega_0$, be the $\Xi\cup\Sigma$-reduced forms modulo $\Im$ of x^g, y^h respectively. We divide our argument into three cases according to the Ω/Ω'-order of the given basic (g,h)-polynomial $\varphi(x)$ modulo $\Im$:

Case 0. $\varphi(x)$ is of Ω/Ω'-order -1: By the definition of Ω/Ω'-order -1, we have $\varphi(x)\in\wp(\Xi)$. Let $\varphi(x,y)$ be the Ξ-reduced form of $\varphi(xy)$ modulo $\Im'$. Since $\Im\supset\Im'$, $\varphi(x,y)$ is also the $\Xi\cup\Sigma$-reduced form of $\varphi(xy)$ modulo $\Im$. The $\Xi\cup\Sigma$-reduced form of $\varphi(xy)-x^g\varphi(y)-$

$\varphi(x)y^h$ modulo $\Im$ is thus given by

$$\varphi(x,y) - u^{-1}x^{\tilde{g}}u\varphi(y) - \varphi(x)vy^{\tilde{h}}v^{-1}.$$

This expression is the zero polynomial, since $\varphi(x)$ is (g,h)-basic modulo $\Im$. If $\tilde{g} \in \Sigma$, then $u^{-1}x^{\tilde{g}}u\varphi(y)$ consists of all terms with $x^{\tilde{g}}$, y^{δ}, where $\delta \in \Xi$, occurring in this order. Therefore, $u^{-1}x^{\tilde{g}}u\varphi(y)$ and hence $\varphi(y)$ also are the zero polynomial in this case. Analogously, $\tilde{h} \in \Sigma$ also implies that $\varphi(x)$ is the zero polynomial. Therefore, if $\tilde{g} \in \Sigma$ or $\tilde{h} \in \Sigma$, then $\varphi(x)$ is the zero polynomial and is hence basic modulo $\Im'$ in a trivial way. So we assume that $\tilde{g}, \tilde{h} \in \Xi$. But then we only need identities in $\Im'$ to obtain the above $\Xi \cup \Sigma$-reduced form of $\varphi(xy) - x^g\varphi(y) - \varphi(x)y^h$. That is, $\varphi(xy) - x^g\varphi(y) - \varphi(x)y^h \equiv 0$ modulo $\Im'$. Since $\varphi(x) \in \wp(\Xi) \subseteq \wp(\Omega')$, $\varphi(x)$ is also (g,h)-basic modulo $\Im'$, as asserted.

Case 1. $\varphi(x)$ is of Ω/Ω'-order 0: Write

$$\varphi(x) = \sum_{i=1}^{s}\sum_{j=1}^{s_i} a_{ij}x^{g_i}b_{ij} + \rho(x),$$

where $a_{ij}, b_{ij} \in U$, $g_i \in \Sigma_0^+$ and $\rho(x) \in \wp(\Xi)$. Let $\rho(x,y) \in \wp(\Xi)$ be the Ξ-reduced form modulo $\Im'$ of $\rho(xy)$. Since $\Im' \subseteq \Im$, $\rho(x,y)$ is also the $\Xi \cup \Sigma$-reduced form modulo $\Im$ of $\rho(xy)$. We compute modulo $\Im$:

$$\begin{aligned}
0 \equiv\, & \varphi(xy) - x^g\varphi(y) - \varphi(x)y^h \\
=\, & \sum_{i=1}^{s}\sum_{j=1}^{s_i} a_{ij}x^{g_i}y^{g_i}b_{ij} - x^g\Big(\sum_{i=1}^{s}\sum_{j=1}^{s_i} a_{ij}y^{g_i}b_{ij}\Big) - \Big(\sum_{i=1}^{s}\sum_{j=1}^{s_i} a_{ij}x^{g_i}b_{ij}\Big)y^h \\
& \qquad + \rho(xy) - x^g\rho(y) - \rho(x)y^h \\
\equiv\, & \sum_{i=1}^{s}\sum_{j=1}^{s_i} a_{ij}x^{g_i}y^{g_i}b_{ij} - u^{-1}x^{\tilde{g}}u\Big(\sum_{i=1}^{s}\sum_{j=1}^{s_i} a_{ij}y^{g_i}b_{ij}\Big) - \Big(\sum_{i=1}^{s}\sum_{j=1}^{s_i} a_{ij}x^{g_i}b_{ij}\Big)vy^{\tilde{h}}v^{-1} \\
& \qquad + \rho(x,y) - u^{-1}x^{\tilde{g}}u\rho(y) - \rho(x)vy^{\tilde{h}}v^{-1}.
\end{aligned}$$

This last line above gives the $\Xi \cup \Sigma$-reduced expression of $\varphi(xy) - x^g\varphi(y) - \varphi(x)y^h$ and hence must be the zero polynomial. We first assume that all a_{ij} are picked from a C-basis of U which contains u^{-1}. Note that $\rho(x,y) - u^{-1}x^{\tilde{g}}u\rho(y) - \rho(x)vy^{\tilde{h}}v^{-1}$ can*not* contribute to give terms with x^{g_i}, y^{g_i} occurring in this order, since $g_i \in \Sigma$ but $\rho(x,y), \rho(x), \rho(y) \in \wp(\Xi)$. If $a_{ij} \neq u^{-1}$ or $g_i \neq \tilde{g}$, then the term $a_{ij}x^{g_i}y^{g_i}b_{ij}$ in the first summation must be cancelled by terms in the third summation and, for these terms, we must have $g_i = \tilde{h}$ and $b_{ij} = \beta v^{-1}$ for some $\beta \in C$. We can thus write $\varphi(x) = u^{-1}x^{\tilde{g}}b + ax^{\tilde{h}}v^{-1} + \rho(x)$ for some $a, b \in U$. A direct computation shows that

$$\begin{aligned}
\varphi(xy) - x^g\varphi(y) - \varphi(x)y^h \equiv\, & -u^{-1}x^{\tilde{g}}(ua + bv)y^{\tilde{h}}v^{-1} \\
& + \rho(x,y) - u^{-1}x^{\tilde{g}}u\rho(y) - \rho(x)vy^{\tilde{h}}v^{-1} \qquad \text{modulo } \Im.
\end{aligned}$$

Being the leading Ω/Ω'-words of $\varphi(x)$, both $\tilde{g}$ and $\tilde{h}$ fall in Σ_0. In the $\Xi \cup \Sigma$-reduced form of $\varphi(xy) - x^g\varphi(y) - \varphi(x)y^h$, all terms with x^δ, $y^{\tilde{h}}$, where $\delta \in \Xi$, occurring in this order fall in $\rho(x)vy^{\tilde{h}}v^{-1}$ and conversely, $\rho(x)vy^{\tilde{h}}v^{-1}$ consists of all such terms. So $\rho(x)vy^{\tilde{h}}v^{-1}$ and hence $\rho(x)$ must be the zero polynomial. It then follows that $bv + ua = 0$. Set $c \stackrel{\text{def.}}{=} bv = -ua$. Then $\varphi(x) = u^{-1}x^{\tilde{g}}b + ax^{\tilde{h}}v^{-1} = u^{-1}x^{\tilde{g}}cv^{-1} - u^{-1}cx^{\tilde{h}}v^{-1} = u^{-1}(x^{\tilde{g}}c - cx^{\tilde{h}})v^{-1}$, as asserted.

Case 2. $\varphi(x)$ is of Ω/Ω'-order > 0: Set $\Sigma_m \stackrel{\text{def.}}{=} \Sigma \cap \Omega_m$ for $m \geq 0$. Let $\Delta \in \Sigma_n$ be a leading Ω/Ω'-word of $\varphi(x)$. We want to show that terms containing Δ in $\varphi(x)$ assumes the asserted form. Consider a typical term $\tau(x) = ax^\delta b$ of $\varphi(x)$, where $a, b \in U$, $\delta \in \Xi \cup \Sigma$, and define

$$\psi_\tau(x, y) \stackrel{\text{def.}}{=} \tau(xy) - x^g\tau(y) - \tau(x)y^h = a\pi_\delta(x, y)b - x^g ay^\delta b - ax^\delta by^h.$$

We want to find those terms $\tau(x) = ax^\delta b$ such that the $\Xi \cup \Sigma$-reduced forms of $\psi_\tau(x, y)$ modulo $\Im$ involve the given Δ nontrivially. If $\delta \in \Xi \subseteq \Omega'$, then $\psi_\tau(x, y)$ also falls in $\wp(\Omega')$ and hence its Ξ-reduced form modulo $\Im'$ falls in $\wp(\Xi)$. Since $\Im' \subseteq \Im$, the Ξ-reduced form of $\psi_\tau(x, y)$ modulo $\Im'$ is also its $\Xi \cup \Sigma$-reduced form modulo $\Im$. The $\Xi \cup \Sigma$-reduced form of such $\psi_\tau(x, y)$ falls in $\wp(\Xi)$ and *cannot* involve Δ. If $\delta \in \Sigma_{n-1} \subseteq \Omega_{n-1}$, then $\psi_\tau(x)$ also falls in $\wp(\Omega_{n-1})$ and hence its $\Xi \cup \Sigma$-reduced form falls in $\wp(\Xi \cup \Sigma_{n-1})$, since Σ is an ordered basis of Ω/Ω' modulo $\Im/\Im'$. The $\Xi \cup \Sigma$-reduced form of such $\psi_\tau(x, y)$ *cannot* involve Δ either. Since $\varphi(x)$ is of Ω/Ω'-order n, we are left with the case that $\delta \in \Sigma_n$. Since the basis Σ of Ω/Ω' modulo $\Im/\Im'$ is ordered, words of order $< n$ in $\Omega \setminus \Omega'$, when expressed in terms of the basis Σ can only give rise to words in Σ_{n-1} plus some words in Ξ. Therefore, for $\tau(x) = ax^\delta b$, where $\delta \in \Sigma_n$, the only word in $\Sigma_n \setminus \Sigma_{n-1}$ which can possibly occur nontrivially in $\psi_\tau(x, y)$ is δ itself. If $\delta \neq \Delta$, then $\psi_\tau(x, y)$ *cannot* involve Δ nontrivially. So we must have $\delta = \Delta$. Assume that $\Delta \in \Omega(g', h')$. For $\tau(x) = ax^\Delta b$, the sum of terms involoving Δ in $\psi_\tau(x, y)$ is

$$ax^{g'}y^\Delta b + ax^\Delta y^{h'}b - x^g ay^\Delta b - ax^\Delta by^h = (ax^{g'} - x^g a)y^\Delta b + ax^\Delta(y^{h'}b - by^h).$$

Let $\sum_{i=1}^s a_i x^\Delta b_i$ denote the sum of all terms involving Δ in $\varphi(x)$. In $\varphi(xy) - x^g\varphi(y) - \varphi(x)y^h$, the sum of terms involving Δ is then given by

$$\sum_{i=1}^s (a_i x^{g'} - x^g a_i)y^\Delta b_i + \sum_{i=1}^s a_i x^\Delta(y^{h'}b_i - b_i y^h).$$

Let $\mu_i(x)$ and $\nu_i(y)$ be the $\Xi \cup \Sigma$-reduced forms of $a_i x^{g'} - x^g a_i$ and $y^{h'}b_i - b_i y^h$. In the $\Xi \cup \Sigma$-reduced form of $\varphi(xy) - x^g\varphi(y) - \varphi(x)y^h$, the sum of terms involving Δ is then given by

$$(\star) \qquad \sum_{i=1}^s \mu_i(x)y^\Delta b_i + \sum_{i=1}^s a_i x^\Delta \nu_i(y).$$

Since $\varphi(x)$ is a basic identity modulo $\Im$, $\varphi(xy) - x^g\varphi(y) - \varphi(x)y^h$ falls in $\Im$ and hence its $\Xi \cup \Sigma$-reduced form $(\star)$ must be the zero polynomial. Note that $\mu_i(x)$, $\nu_i(y)$ are of Ω/Ω'-order ≤ 0 and that y^Δ, x^Δ are of Ω/Ω'-order $n > 1$ by our case assumption. From the fact that $(\star)$ is the zero polynomial, it follows that both $\sum_i \mu_i(x)y^\Delta b_i$, $\sum_i a_i x^\Delta \nu_i(y)$ are also the zero polynomial. Without loss of generality, we may assume that the expression $\sum_i^s a_i x^\Delta b_i$ is so chosen with s minimal possible. Both the two coefficient sets $\{a_i\}$ and $\{b_i\}$ are then C-independent. With this, from the fact that $\sum_i \mu_i(x)y^\Delta b_i$, $\sum_i a_i x^\Delta \nu_i(y)$ are the zero polynomial, we deduce that all $\mu_i(x)$ and all $\nu_i(y)$ are also the zero polynomial. This implies $a_i x^{g'} - x^g a_i, y^{h'} b_i - b_i y^h \in \Im$. We show that $a_i, b_i \in U^{-1}$: Let the $\Xi \cup \Sigma$-reduced forms of $x^{g'}$, $y^{h'}$ respectively be $u' x^{\tilde{g}'}(u')^{-1}$, $v' y^{\tilde{h}'}(v')^{-1}$ respectively, where $\tilde{g}', \tilde{h}' \in \Xi \cap \Omega_0$ and $u', v' \in U^{-1}$. The $\Xi \cup \Sigma$-reduced form of $a_i x^{g'} - x^g a_i$ is then $a_i u' x^{\tilde{g}'}(u')^{-1} - u x^{\tilde{g}} u^{-1} a_i$ and this must be the zero polynomial, since $a_i x^{g'} - x^g a_i \in \Im$. We hence have $\tilde{g}' = \tilde{g}$ and $a_i = \alpha_i u(u')^{-1}$ for some $\alpha_i \in C$. Similarly, we have $b_i = \beta_i v' v^{-1}$ for some $\beta_i \in C$. But $\{a_i\}$ and $\{b_i\}$ are C-independent by taking s to be minimal. So $s = 1$ and $a_1 x^\Delta b_1$ is the only term involoving Δ in $\varphi(x)$. This complets our proof of Fact 4.

Our proof of Fact 4 above follows pretty much the same line as that of Fact 2 [**3**]. But our Fact 4 does *not* follow immediately from Fact 2 [**3**], since the leading Ω/Ω'-part of a polynomial in $\wp(\Omega)$ is usually different from its leading part (that is, its leading $\Omega/$-part). Also, as shown by the example below, it is *not* always possible to choose a basis Ξ of Ω' modulo $\Im(\Omega')$ so that the basis $\Xi \cup \Sigma$ of Ω modulo $\Im$ is ordered.

Just as in [**3**], we relate our generalized ordered bases to some generalized dependence relation defined below:

Definition: Let Ω/Ω' be expansion closed word sets.

1. Let $\Im_0(\Omega/\Omega')$ be the ideal of $\wp(\Omega)$ generated by $\Im(\Omega')$ and $\Im_0(\Omega)$. (Recall that $\Im_0(\Omega)$ is the ideal generated by all identities in the form $\varphi(x) = x^g u - u x^h$, where $u \in U$ and $g, h \in \Omega_0^+$ or $g, h \in \Omega_0^-$.) For $n \geq 1$, we define inductively $\Im_n(\Omega/\Omega')$ to be the ideal generated by $\Im_{n-1}(\Omega/\Omega')$ and all Ω/Ω'-simple basic identities modulo $\Im_{n-1}(\Omega'/\Omega)$ of Ω/Ω'-order n.

2. Set $\bar{\Omega}_0 \stackrel{\text{def.}}{=} \Omega_0$ and $\bar{\Omega}_n \stackrel{\text{def.}}{=} \Omega_n \setminus \Omega_{n-1}$ for $n \geq 1$. A subset Σ of $\bar{\Omega}_0 \setminus \Omega'$ is said to be *Ω/Ω'-dependent* if $g \sim h$ for two distinct $g, h \in \Sigma$ *or* for some $g \in \Sigma$ and $h \in \Omega_0'$. (Recall that $g \sim h$, if $g, h \in \Omega_0^+$ or $g, h \in \Omega_0^-$ *and* if there exists $u \in U^{-1}$ such that $x^g = u x^h u^{-1}$ for all $x \in R$.) For $n \geq 1$, a subset Σ of $\bar{\Omega}_n \setminus \Omega'$ is said to be *Ω/Ω'-dependent* if there exists an Ω/Ω'-simple basic identity modulo $\Im_{n-1}(\Omega/\Omega')$ with all its leading Ω/Ω'-words in Σ. (This basic identity must hence be of Ω/Ω'-order n.)

3. A subset Σ of $\Omega \setminus \Omega'$ is said to be *Ω/Ω'-dependent* if for some $n \geq 0$, $\Sigma \cap \bar{\Omega}_n$ is Ω/Ω'-dependent. A subset Σ of $\Omega \setminus \Omega'$ is said to be *Ω/Ω'-independent* if it is *not* Ω/Ω'-dependent.

Roughly speaking, a subset of $\Omega \setminus \Omega'$ is Ω/Ω'-independent if it does *not* destroy the independence modulo $\Im(\Omega')$. Again, if $\Omega' = \emptyset$, then the notions above reduce to the corresponding notions of [**3**]. We have the following pleasant analogy of Theorem 1 ([**3**]):

Theorem 4. *Let Ω/Ω' be expansion closed word sets.*

(1) $\Im(\Omega) = \bigcup_{n=0}^{\infty} \Im_n(\Omega/\Omega')$.

(2) A subset Σ of $\Omega \setminus \Omega'$ forms an ordered basis of Ω/Ω' modulo $\Im(\Omega)/\Im(\Omega')$ if and only if Σ is a maximal Ω/Ω'-independent subset of $\Omega \setminus \Omega'$.

We recall the following:

Definition: ([3] pp. 308) Let Ω be an expansion closed word set. An ideal $\Im$ of $\wp(\Omega)$ is said to be *expansion closed* if for any $\varphi \in \Im$ and any indeterminate x involved in φ, by writing $\varphi = \varphi(x)$, we have $\varphi(x+y) \in \Im$ and $\varphi(xy) \in \Im$.

We collect some simple observations below:

Fact 5. *Let Ω be an expansion closed word set and $\Im$, an ideal of $\wp(\Omega)$.*

(1) A sum of basic (g,h)-polynomials modulo $\Im$ is also a basic (g,h)-polynomial modulo $\Im$.

(2) Given $g,h \in \Omega_0^+$ (or $g,h \in \Omega_0^-$), let $u,v \in U^{-1}$ and $g',h' \in \Omega_0^+$ (or $g',h' \in \Omega_0^-$ respectively) be such that $ux^g - x^{g'}x, y^h v - vy^{h'} \in \Im$. If $\varphi(x) \in \wp(\Omega)$ is a basic (g,h)-polynomial modulo $\Im$, then $u\varphi(x)v$ is a basic (g',h')-polynomial modulo $\Im$.

(3) Assume that the ideal $\Im$ is expansion closed. If $\varphi(x) \in \wp(\Omega)$ is equivalent (modulo $\Im$) to a basic (g,h)-polynomial modulo $\Im$, then $\varphi(x)$ is also a basic (g,h)-polynomial modulo $\Im$.

Proof: (1) Trivial.

(2) Say, $g,h \in \Omega_0^+$. Let $\varphi(x) \in \wp(\Omega)$ be a basic (g,h)-polynomial modulo $\Im$. Then $u\varphi(x+y)v \equiv u(\varphi(x)+\varphi(y))v \equiv u\varphi(x)v + u\varphi(y)v$ modulo $\Im$. Also, $u\varphi(xy)v \equiv u(x^g\varphi(y) + \varphi(x)y^h)v \equiv x^{g'}u\varphi(y)v + u\varphi(x)vy^{h'} + (ux^g - x^{g'}u)\varphi(y)v + u\varphi(x)(y^h v - vy^{h'}) \equiv x^{g'}u\varphi(y)v + u\varphi(x)vy^{h'}$, where the last equivalence follows since $ux^g - x^{g'}u, y^h v - vy^{h'} \in \Im$. It follows that $u\varphi(x)v$ is a basic (g',h')-polynomial modulo $\Im$.

(3) Let $\psi(x)$ be a basic (g,h)-polynomial modulo $\Im$ such that $\psi(x) \equiv \varphi(x)$ modulo $\Im$. Equivalently, $\psi(x) - \varphi(x) \in \Im$. By the expansion closedness of $\Im$, we have $\psi(x+y) - \varphi(x+y) \in \Im$ and $\psi(xy) - \varphi(xy) \in \Im$. So $\varphi(x+y) - \varphi(x) - \varphi(y) \equiv \psi(x+y) - \psi(x) - \psi(y) \in \Im$ modulo $\Im$. Also $\varphi(xy) - \tilde{x}^g\varphi(\tilde{y}) - \varphi(\tilde{x})\tilde{y}^h \equiv \psi(xy) - \tilde{x}^g\psi(\tilde{y}) - \psi(\tilde{x})\tilde{y}^h \in \Im$ modulo $\Im$, where $\tilde{x} \stackrel{\text{def.}}{=} x, \tilde{y} \stackrel{\text{def.}}{=} y$ for $g,h \in \Omega_0^+$ and $\tilde{x} \stackrel{\text{def.}}{=} y, \tilde{y} \stackrel{\text{def.}}{=} x$ for $g,h \in \Omega_0^-$. Thus $\varphi(x)$ is also a basic (g,h)-polynomial modulo $\Im$, as asserted.

We are now ready for

Proof of Theorem 4: We observe first by induction on $n \geq 0$ that $\Im_n(\Omega/\Omega')$ is expansion closed: The case $n = 0$ is obvious. As the induction hypothsis, assume that $\Im_{n-1}(\Omega/\Omega')$ is expansion closed. To show that $\Im_n(\Omega/\Omega')$ is expansion closed, it suffices to check $\varphi(x+y), \varphi(xy) \in \Im_n(\Omega/\Omega')$ for Ω/Ω'-simple basic identities $\varphi(x)$ of Ω/Ω'-order

n modulo $\Im_{n-1}(\Omega/\Omega')$, for these identities together with $\Im_{n-1}(\Omega/\Omega')$ generate $\Im_n(\Omega/\Omega')$. So let $\varphi(x)$ be an Ω/Ω'-simple basic (g,h)-identity of Ω/Ω'-order n modulo $\Im_{n-1}(\Omega/\Omega')$. Then $\varphi(x) \in \Im_n(\Omega/\Omega')$ by the definition of $\Im_n(\Omega/\Omega')$. Since $\varphi(x)$ is linear, $\varphi(x+y) = \varphi(x) + \varphi(y) \in \Im_n(\Omega/\Omega')$. To show $\varphi(xy) \in \Im_n(\Omega/\Omega')$, set $\tilde{x} \stackrel{\text{def.}}{=} x, \tilde{y} \stackrel{\text{def.}}{=} y$ for $g, h \in \Omega_0^+$ and $\tilde{x} \stackrel{\text{def.}}{=} y, \tilde{y} \stackrel{\text{def.}}{=} x$ for $g, h \in \Omega_0^-$. Since $\varphi(xy) - \tilde{x}^g\varphi(\tilde{y}) - \varphi(\tilde{x})\tilde{y}^h \in \Im_{n-1}(\Omega/\Omega')$, we have

$$\varphi(xy) \in \tilde{x}^g\varphi(\tilde{y}) + \varphi(\tilde{x})\tilde{y}^h + \Im_{n-1}(\Omega/\Omega') \subseteq \Im_n(\Omega/\Omega),$$

where the last inclusion follows since $\varphi(\tilde{x}), \varphi(\tilde{y}) \in \Im_n(\Omega/\Omega')$ by the definition of $\Im_n(\Omega/\Omega')$.

Recall that $\bar{\Omega}_0 \stackrel{\text{def.}}{=} \Omega_0$ and $\bar{\Omega}_n \stackrel{\text{def.}}{=} \Omega_n \setminus \Omega_{n-1}$ for $n \geq 1$. For any subset Σ of Ω, we set $\Sigma_n \stackrel{\text{def.}}{=} \Sigma \cap \Omega_n$ and $\bar{\Sigma}_n \stackrel{\text{def.}}{=} \Sigma \cap \bar{\Omega}_n$. For brevity, we define $\tilde{\Omega}_n \stackrel{\text{def.}}{=} \Omega \setminus \Omega_n$. Since we will often deal with complements of Ω', it is convenient here to define $\Omega^c \stackrel{\text{def.}}{=} \Omega \setminus \Omega'$, $\Omega_n^c \stackrel{\text{def.}}{=} \Omega_n \setminus \Omega'$, $\bar{\Omega}_n^c \stackrel{\text{def.}}{=} \bar{\Omega}_n \setminus \Omega'$, $\bar{\Omega}_n^c(g,h) \stackrel{\text{def.}}{=} \bar{\Omega}_n(g,h) \setminus \Omega'$ and $\tilde{\Omega}_n^c \stackrel{\text{def.}}{=} \tilde{\Omega}_n \setminus \Omega'$. (The superscrit c here suggests the "complement" of Ω'.) Throughout the rest of the proof, we fix an ordered basis Ξ of Ω' modulo $\Im(\Omega')$.

Assume that Σ is a maximal Ω/Ω'-independent subset of $\Omega \setminus \Omega'$. By induction on n, we proceed to define a polynomial $\lambda_\Delta(x) \in \wp(\Xi \cup \Sigma_n)$ for each $\Delta \in \bar{\Omega}_n^c \setminus \Sigma$ and to prove the following three assertions:

(**1**) For $\Delta \in \bar{\Omega}_n^c(g,h) \setminus \Sigma$, $n > 0$, the polynomial $x^\Delta - \lambda_\Delta(x)$ is an Ω/Ω'-simple basic (g,h)-identity modulo $\Im_{n-1}(\Omega'/\Omega)$ of Ω/Ω'-order n.

(**2**) The set $\Sigma_n \cup \tilde{\Omega}_n^c$ forms an ordered basis of Ω/Ω' modulo $\Im_n(\Omega/\Omega')/\Im(\Omega')$ with $\lambda_\Delta(x)$ being the $\Sigma_n \cup \tilde{\Omega}_n^c$-reduced form of x^Δ for $\Delta \in \Omega_n^c \setminus \Sigma_n$. Equivalently by Fact 1, the ideal $\Im_n(\Omega/\Omega')$ is generated by $\Im(\Omega')$ and $x^\Delta - \lambda_\Delta(x)$, where $\Delta \in \Omega_n^c \setminus \Sigma_n$.

(**3**) Any basic identity modulo $\Im_n(\Omega/\Omega')$ of Ω/Ω'-order $\leq n$ falls in $\Im_n(\Omega/\Omega')$.

We start with $n = 0$: Let $g \in \Omega_0^c \setminus \Sigma_0$ be given. By the maximality of Σ_0, the set $\Sigma_0 \cup \{g\}$ is *no* longer Ω/Ω'-independent. So $g \sim \sigma$ for some $\sigma \in \Omega_0' \cup \Sigma$ distinct from g. If $\sigma \in \Omega'$, then $\sigma \sim \sigma'$ for some $\sigma' \in \Xi_0$ and, replacing σ by σ', we have $\sigma \in \Xi_0$. So we may always assume that $\sigma \in \Xi_0 \cup \Sigma_0$. Let $u \in U^{-1}$ be such that $x^g - ux^\sigma u^{-1} = 0$ for all $x \in R$ and define $\lambda_g(x) \stackrel{\text{def.}}{=} ux^\sigma u^{-1} \in \wp(\Xi_0 \cup \Sigma_0)$. We have $x^g - \lambda_g(x) = x^g - ux^\sigma u^{-1} = (x^g u - ux^\sigma)u^{-1} \in \Im_0(\Omega)$, since the identity $x^g u - ux^\sigma$ falls in $\Im_0(\Omega)$. This proves the assertion (**1**) for $n = 0$. For convenience, we also set $\lambda_g(x) \stackrel{\text{def.}}{=} x^g$ for $g \in \Xi_0 \cup \Sigma_0$.

Consider an identity of $\Im_0(\Omega/\Omega')$ in the form $x^g c - cx^h$, where $0 \neq c \in U$ and where $g, h \in \Omega_0^+$ or $g, h \in \Omega_0^-$. Let $\lambda_g(x) \stackrel{\text{def.}}{=} ux^\sigma u^{-1}$ and $\lambda_h(x) \stackrel{\text{def.}}{=} vx^\tau v^{-1}$, where $u, v \in U^{-1}$ and $\sigma, \tau \in \Xi_0 \cup \Sigma_0$. We have

$$\begin{aligned} x^g c - cx^h &= (x^g - \lambda_g(x))c - c(x^h - \lambda_h(x)) + \lambda_g(x)c - c\lambda_h(x) \\ &= (x^g - \lambda_g(x))c - c(x^h - \lambda_h(x)) + (ux^\sigma u^{-1}c - cvx^\tau v^{-1}) \\ &= (x^g - \lambda_g(x))c - c(x^h - \lambda_h(x)) + u(x^\sigma u^{-1}cv - u^{-1}cvx^\tau)v^{-1}. \end{aligned}$$

Since $u, v^{-1} \in U$ and since $x^g c - cx^h$, $x^g - \lambda_g(x)$ and $x^h - \lambda_h(x)$ are all identities, so is $x^\sigma u^{-1}cv - u^{-1}cvx^\tau$. The identity $x^\sigma u^{-1}cv - u^{-1}cvx^\tau$ implies $\sigma \sim \tau$. Since $\sigma, \tau \in \Xi_0 \cup \Sigma_0$, we have $\sigma = \tau$ and hence also $u^{-1}cv \in C$. The identity $x^\sigma u^{-1}cv - u^{-1}cvx^\tau$ is thus the zero polynomial. So we have $x^g c - cx^h = (x^g - \lambda_g(x))c - c(x^h - \lambda_h(x))$. The ideal $\Im_0(\Omega/\Omega')$, defined to be the ideal generated by $\Im(\Omega')$ and all identities of the form $x^g c - cx^h$, where $g, h \in \Omega_0^+$ or $g, h \in \Omega_0^-$, is therefore also generated by $\Im(\Omega')$ and identities $x^k - \lambda_k(x)$, $k \in \Omega_0$. But $x^k - \lambda_k(x) = x^k - x^k = 0$ for $k \in \Xi_0 \cup \Sigma_0$ and also $x^k - \lambda_k(x) \in \Im_0(\Omega')$ for $k \in \Omega_0' \setminus \Xi$. The ideal $\Im_0(\Omega/\Omega')$ is thus generated by $\Im(\Omega')$ and identities $x^k - \lambda_k(x)$, $k \in \Omega_0^c \setminus \Sigma_0$. Observe that $\Omega_0^c \setminus \Sigma_0 = \Omega^c \setminus (\Sigma_0 \cup \tilde{\Omega}_0^c)$. By Fact 1, $\Sigma_0 \cup \tilde{\Omega}_0^c$ forms a basis of Ω/Ω' modulo $\Im_0(\Omega/\Omega')/\Im(\Omega')$ with $\lambda_g(x)$, $g \in \Omega_0^c \setminus \Sigma_0$, being the $\Sigma_0 \cup \tilde{\Omega}_0^c$-reduced form of x^g. Since each $\lambda_k(x)$, $k \in \Omega_0^c \setminus \Sigma_0$, has Ω/Ω'-order ≤ 0, the basis $\Sigma_0 \cup \tilde{\Omega}_0^c$ of Ω/Ω' modulo $\Im_0(\Omega/\Omega')/\Im(\Omega')$ is ordered by Fact 2. This proves the assertion (**2**) for $n = 0$.

By Theorem 3, the set $\Xi \cup \Sigma_0 \cup \tilde{\Omega}_0^c$ forms a basis of Ω modulo $\Im_0(\Omega/\Omega')$. Let $\varphi(x)$ be a basic identity modulo $\Im_0(\Omega/\Omega')$ of Ω/Ω'-order ≤ 0. The $\Xi \cup \Sigma_0 \cup \tilde{\Omega}_0^c$-reduced form of $\varphi(x)$ modulo $\Im_0(\Omega/\Omega')$ is still basic modulo $\Im_0(\Omega/\Omega')$ by (3) of Fact 5 and also has Ω/Ω'-order ≤ 0, since $\Sigma_0 \cup \tilde{\Omega}_0^c$ is an ordered basis of Ω/Ω' modulo $\Im_0(\Omega/\Omega')/\Im(\Omega')$. Replacing $\varphi(x)$ by its $\Xi \cup \Sigma_0 \cup \tilde{\Omega}_0^c$-reduced form, we may assume $\varphi(x) \in \wp(\Xi \cup \Sigma_0 \cup \tilde{\Omega}_0^c)$. If $\varphi(x)$ has Ω/Ω'-order -1, then $\varphi(x) \in \wp(\Xi)$ is also basic modulo $\Im(\Omega')$ by Fact 4 and we have $\varphi(x) \in \Im(\Omega') \subseteq \Im_0(\Omega/\Omega')$ by the defining minimality of $\Im(\Omega')$. We hence assume that $\varphi(x)$ has Ω/Ω'-order 0. By Fact 2 again, $\varphi(x)$ assumes the form $a(x^g c - cx^h)b$, where $a, b \in U^{-1}$, where $0 \neq c \in U$ and where $g, h \in \Sigma_0^+$ or $g, h \in \Sigma_0^-$. The identity $a(x^g c - cx^h)b$ implies $g \sim h$. Since $g, h \in \Sigma_0$, we have $g = h$ and hence $c \in C$. So $a(x^g c - cx^h)b$ is merely the zero polynomial and it follows $\varphi(x) \in \Im_0(\Omega/\Omega')$. We have thus shown that $\Im_0(\Omega/\Omega')$ contains all basic identities of Ω/Ω'-order ≤ 0 modulo $\Im_0(\Omega/\Omega')$. This proves the assertion (**3**) for $n = 0$. The induction basis for $n = 0$ is thus completed.

As the induction hypothesis, we assume that $\lambda_\Delta(x)$ has been defined for $\Delta \in \Omega_{n-1} \setminus \Sigma_{n-1}$ so that the assertions (**1**), (**2**) and (**3**) hold for $n-1$. By Theorem 3, the set $\Xi \cup \Sigma_{n-1} \cup \tilde{\Omega}_{n-1}^c$ forms a basis of Ω modulo $\Im_{n-1}(\Omega/\Omega')$. The assertion (**1**) for the induction step n follows from the following

Claim 1. For each $\Delta \in \bar{\Omega}_n^c(g,h) \setminus \bar{\Sigma}_n$, $n \geq 1$, there exists $\lambda_\Delta(x) \in \wp(\Xi \cup \Sigma_n)$ such that $x^\Delta - \lambda_\Delta(x)$ is an Ω/Ω'-simple basic (g,h)-identity modulo $\Im_{n-1}(\Omega/\Omega')$ of Ω/Ω'-order n: Given $\Delta \in \bar{\Omega}_n^c(g,h) \setminus \bar{\Sigma}_n$, the set $\bar{\Sigma}_n \cup \{\Delta\}$, being strictly larger than the maximal Ω/Ω'-independent subset $\bar{\Sigma}_n$ of $\bar{\Omega}_n^c$, is *no* longer Ω/Ω'-independent by the maximality of $\bar{\Sigma}_n$. Therefore, there exists an Ω/Ω'-simple basic identity $\varphi(x)$ modulo $\Im_{n-1}(\Omega/\Omega)$ with all its leading Ω/Ω'-words in $\bar{\Sigma}_n \cup \{\Delta\}$:

$$\varphi(x) \stackrel{\text{def.}}{=} \sum_{i=0}^{s} a_i x^{\Delta_i} b_i + \rho(x),$$

where $a_i, b_i \in U^{-1}$, $\Delta_i \in \bar{\Sigma}_n \cup \{\Delta\}$ and $\rho(x) \in \wp(\Omega' \cup \Omega_{n-1})$ satisfy the defining properties of Ω/Ω'-simple basic polynomials modulo $\Im_{n-1}(\Omega/\Omega')$. If $\Delta \neq \Delta_i$ for all i, then all $\Delta_i \in \bar{\Sigma}_n$, contradicting the Ω/Ω'-independence of $\bar{\Sigma}_n$. So some Δ_i, say Δ_0, must be equal to Δ. But $a_0^{-1}\varphi_\Delta(x)b_0^{-1}$ is also an Ω/Ω'-simple basic identity modulo $\Im_{n-1}(\Omega/\Omega')$. Replacing $\varphi(x)$ by $a_0^{-1}\varphi(x)b_0^{-1}$ by (2) of Fact 5, we may assume $a_0 = b_0 = 1$ to start with. So $\varphi(x)$

assumes the form

$$\varphi(x) \stackrel{\text{def.}}{=} x^{\Delta} + \sum_{i=1}^{s} a_i x^{\Delta_i} b_i + \rho(x),$$

and is hence an Ω/Ω'-simple basic-(g,h) identity of order n modulo $\Im_{n-1}(\Omega/\Omega')$. We may assume that $\rho(x)$ is in the reduced form with respect to the basis $\Xi \cup \Sigma_{n-1} \cup \tilde{\Omega}^c_{n-1}$ by (3) of Fact 5. Since $\rho(x)$ is of Ω/Ω'-order $\leq n-1$ and since $\Sigma_{n-1} \cup \tilde{\Omega}^c_{n-1}$ forms an ordered basis of Ω/Ω' modulo $\Im_{n-1}(\Omega/\Omega')/\Im(\Omega)$, we have $\rho(x) \in \wp(\Xi \cup \Sigma_{n-1})$. Define

$$\lambda_{\Delta}(x) \stackrel{\text{def.}}{=} -\sum_{i=1}^{s} a_i x^{\Delta_i} b_i - \rho(x) \in \wp(\Xi \cup \Sigma_n).$$

Then $x^{\Delta} - \lambda_{\Delta}(x) = \sum_{i=0}^{s} a_i x^{\Delta_i} b_i + \rho(x) = \varphi(x)$ is an Ω/Ω'-simple basic (g,h)-identity modulo $\Im_{n-1}(\Omega/\Omega)$, as asserted.

Claim 2. Any basic (g,h)-identity $\varphi(x)$ of Ω/Ω'-order n modulo $\Im_{n-1}(\Omega/\Omega')$ can be written uniquely in the form

$$\varphi(x) \equiv \sum_{i=1}^{s} a_i(x^{\Delta_i} - \lambda_{\Delta_i}(x))b_i \qquad \text{modulo } \Im_{n-1}(\Omega/\Omega'),$$

where $a_i, b_i \in U^{-1}$, $\Delta_i \in \bar{\Omega}^c_n(g_i,h_i) \setminus \bar{\Sigma}_n$ are such that $a_i x^{g_i} - x^g a_i, x^{h_i} b_i - b_i x^h \in \Im_{n-1}(\Omega/\Omega')$: By (3) of Fact 5, we may assume that $\varphi(x)$ is in the reduced form with respect to the basis $\Xi \cup \Sigma_{n-1} \cup \tilde{\Omega}^c_{n-1}$ of Ω modulo $\Im_{n-1}(\Omega/\Omega')$. Since $\varphi(x)$ is of Ω/Ω'-order n and since $\Sigma_{n-1} \cup \tilde{\Omega}^c_{n-1}$ forms an ordered basis of Ω/Ω' modulo $\Im_{n-1}(\Omega/\Omega')/\Im(\Omega')$, we have $\varphi(x) \in \wp(\Xi \cup \Sigma_{n-1} \cup \bar{\Omega}^c_n)$. By Fact 4, the basic (g,h)-identity $\varphi(x)$ must be Ω/Ω'-simple:

$$\varphi(x) = \sum_{i=1}^{s} a_i x^{\Delta_i} b_i + \rho(x),$$

where $\rho(x) \in \wp(\Xi \cup \Sigma_{n-1})$ and where $a_i, b_i \in U^{-1}$, $\Delta_i \in \bar{\Omega}_n(g_i,h_i)$ are such that $a_i x^{g_i} - x^g a_i, x^{h_i} b_i - b_i x^h \in \Im_{n-1}(\Omega/\Omega')$. Since $\bar{\Sigma}_n$ is Ω/Ω'-independent, *not* all the leading Ω/Ω'-words Δ_i are in $\bar{\Sigma}_n$. By reordering if necessary, we may assume that $\Delta_i \notin \bar{\Sigma}_n$ for $1 \leq i \leq t$ but $\Delta_i \in \bar{\Sigma}_n$ for $t < i \leq s$. For $1 \leq i \leq t$, $x^{\Delta_i} - \lambda_{\Delta_i}(x)$ is an Ω/Ω'-simple basic (g_i,h_i)-identity modulo $\Im_{n-1}(\Omega/\Omega')$ and hence $a_i(x^{\Delta_i} - \lambda_{\Delta_i}(x))b_i$ is an Ω/Ω'-simple basic (g,h)-identity modulo $\Im_{n-1}(\Omega/\Omega')$ by (2) of Fact 5. Being the sum of basic (g,h)-identities modulo $\Im_{n-1}(\Omega/\Omega')$,

$$\varphi'(x) \stackrel{\text{def.}}{=} \varphi(x) - \sum_{i=1}^{t} a_i(x^{\Delta_i} - \lambda_{\Delta_i}(x))b_i = \rho(x) + \sum_{i=1}^{t} a_i \lambda_{\Delta_i}(x) b_i$$

is also a basic (g,h)-identity of Ω/Ω'-order $\leq n$ modulo $\Im_{n-1}(\Omega/\Omega')$ by (1) of Fact 5. In the expression of $\varphi'(x)$, all $\Delta_1, \ldots, \Delta_t$ are cancelled. Hence words of $\bar{\Omega}^c_n$ occurring nontrivially in $\varphi'(x)$, if any, must be all in $\bar{\Sigma}_n$. If $\varphi'(x)$ were of Ω/Ω'-order n, then all its leading Ω/Ω'-words would be in Σ_n and, since $\varphi'(x)$ is also reduced with respect to the basis $\Xi \cup \Sigma_{n-1} \cup \tilde{\Omega}^c_{n-1}$ of Ω modulo $\Im_{n-1}(\Omega/\Omega')$, the basic identity $\varphi'(x)$ modulo $\Im_{n-1}(\Omega/\Omega')$ must also be Ω/Ω'-simple by Fact 4 again. This contradicts the independence of Σ_n. Therefore, $\varphi'(x)$ is of Ω/Ω'-order $< n$. By the induction hypothesis, $\varphi'(x) \in \Im_{n-1}(\Omega/\Omega')$. Hence $\varphi(x) \equiv \sum_{i=1}^{s} a_i(x^{\Delta_i} - \lambda_{\Delta_i}(x))b_i$ modulo $\Im_{n-1}(\Omega/\Omega')$, as asserted. To show the uniqueness, it suffices to prove that for distinct $\Delta_i \in \tilde{\Omega}^c_n \setminus \Sigma_n$ and for any $a_{ij}, b_{ij} \in U$, $\sum_i \sum_j a_{ij}(x^{\Delta_i} - \lambda_{\Delta_i}(x))b_{ij} \in \Im_{n-1}(\Omega/\Omega')$ implies $\sum_j a_{ij} \otimes b_{ij} = 0$ for each i. This is immediate: The expression $\sum_i \sum_j a_{ij}(x^{\Delta_i} - \lambda_{\Delta_i}(x))b_{ij}$, being an element of $\Im_{n-1}(\Omega/\Omega')$ and also being reduced with respect to the basis $\Xi \cup \Sigma_{n-1} \cup \tilde{\Omega}^c_{n-1}$ of Ω modulo $\Im_{n-1}(\Omega/\Omega')$, must be the zero polynomial. But in the expression $\sum_i \sum_j a_{ij}(x^{\Delta_i} - \lambda_{\Delta_i}(x))b_{ij}$, the sum of terms involving Δ_i is $\sum_j a_{ij}x^{\Delta_i}b_{ij}$. Hence $\sum_j a_{ij} \otimes b_{ij} = 0$ for each i. Claim 2 is thus proved.

The assertion (**2**) for the the induction step n now follows from Claim 2 and the induction hypothesis. So $\Sigma_n \cup \tilde{\Omega}^c_n$ forms an ordered basis of Ω/Ω' modulo $\Im_n(\Omega/\Omega')/\Im(\Omega')$. By Theorem 3, $\Xi \cup \Sigma_n \cup \tilde{\Omega}^c_n$ is a basis of Ω modulo $\Im_n(\Omega/\Omega')$.

Claim 3. The ideal $\Im_{n-1}(\Omega/\Omega')$ is generated by the set $\Im_n(\Omega/\Omega') \cap \wp(\Omega' \cup \Omega_{n-1})$: By the induction hypothesis, $\Im_{n-1}(\Omega/\Omega')$ is generated by $\Im(\Omega')$ and $x^\Delta - \lambda_\Delta(x)$, $\Delta \in \Omega^c_{n-1} \setminus \Sigma_{n-1}$. By the definition of $\Im_n(\Omega/\Omega')$, we have $\Im(\Omega') \subseteq \Im_n(\Omega/\Omega') \cap \wp(\Omega' \cup \Omega_{n-1})$. Also, for $\Delta \in \Omega^c_{n-1} \setminus \Sigma_{n-1}$, we have $x^\Delta - \lambda_\Delta(x) \in \Im_n(\Omega/\Omega') \cap \wp(\Omega' \cup \Omega_{n-1})$. So $\Im_{n-1}(\Omega/\Omega')$ is included in the ideal generated by $\Im_n(\Omega/\Omega') \cap \wp(\Omega' \cup \Omega_{n-1})$. For the other inclusion, it suffices to show $\Im_n(\Omega/\Omega') \cap \wp(\Omega' \cup \Omega_{n-1}) \subseteq \Im_{n-1}(\Omega/\Omega')$: Given $\varphi \in \Im_n(\Omega/\Omega') \cap \wp(\Omega' \cup \Omega_{n-1})$, let φ' be the expression obtained by substituting $\lambda_\Delta(x)$, $\Delta \in \Omega^c_n \setminus \Sigma_n$, for all occurrences of x^Δ in φ. Since $\varphi \in \wp(\Omega' \cup \Omega_{n-1})$, we use only $\lambda_\Delta(x)$ for $\Delta \in \Omega^c_{n-1} \setminus \Sigma_{n-1}$ in obtaining φ'. Since $x^\Delta - \lambda_\Delta(x) \in \Im_{n-1}(\Omega/\Omega')$ for $\Delta \in \Omega^c_{n-1} \setminus \Sigma_{n-1}$, it follows $\varphi \equiv \varphi'$ modulo $\Im_{n-1}(\Omega/\Omega')$. Since $\Im_{n-1}(\Omega/\Omega') \subseteq \Im_n(\Omega/\Omega')$ and since $\varphi' \in \wp(\Omega' \cup \Sigma_{n-1})$, the expression φ' is also the reduced form of φ with respect to the ordered basis $\Sigma_n \cup \tilde{\Omega}^c_n$ of Ω/Ω' modulo $\Im_n(\Omega/\Omega')/\Im(\Omega')$. Since $\varphi \in \Im_n(\Omega/\Omega')$, its $\Sigma_n \cup \tilde{\Omega}^c_n$-reduced form φ' is equivalent to the zero polynomial modulo the ideal generated by $\Im(\Omega')$. That is, φ' falls in the ideal generated by $\Im(\Omega')$. But the ideal $\Im_{n-1}(\Omega/\Omega')$ includes $\Im(\Omega')$. So $\varphi' \in \Im_{n-1}(\Omega/\Omega')$. But then $\varphi \in \Im_{n-1}(\Omega/\Omega')$ also, since $\varphi \equiv \varphi'$ modulo $\Im_{n-1}(\Omega/\Omega')$. This is true for arbitrily given $\varphi \in \Im_n(\Omega/\Omega') \cap \wp(\Omega' \cup \Omega_{n-1})$. The asserted inclusion $\Im_n(\Omega/\Omega') \cap \wp(\Omega' \cup \Omega_{n-1}) \subseteq \Im_{n-1}(\Omega/\Omega')$ follows.

We now prove the assertion (**3**) for the induction step n: Let φ be a given basic identity modulo $\Im_n(\Omega/\Omega')$ of Ω/Ω'-order $\leq n$. Since $\Sigma_n \cup \tilde{\Omega}^c_n$ forms an ordered basis of Ω/Ω' modulo $\Im_n(\Omega/\Omega')/\Im(\Omega')$, the $\Sigma_n \cup \tilde{\Omega}^c_n$-reduced form of φ also has Ω/Ω'-order $\leq n$. By (3) of Fact 5, we may thus assume that φ is in the reduced form with respect to the basis $\Xi \cup \Sigma_n \cup \tilde{\Omega}^c_n$ of Ω modulo $\Im_n(\Omega)$. If the Ω/Ω'-order of φ is -1, then φ falls in $\wp(\Xi)$ and is basic modulo $\Im(\Omega')$ by Fact 4. So $\varphi \in \Im(\Omega') \subseteq \Im_n(\Omega/\Omega')$ in this case. If the Ω/Ω'-order of φ is 0, then φ assumes the form $a(x^g c - cx^h)b$ for some $a, b \in U^{-1}$, $0 \neq c \in U$ and $g, h \in \Sigma_0$. But we have already shown in the induction basis that such identities are in $\Im_0(\Omega/\Omega')$. We hence assume that φ has Ω/Ω'-order > 0. By Fact 4, the basic identity φ must be Ω/Ω'-simple. By Fact 3, φ is also an Ω/Ω'-simple basic identity modulo the ideal generated by $\Im_n(\Omega/\Omega') \cap \wp(\Omega' \cup \Omega_{n-1})$. By Claim 3, the identity φ is therefore Ω/Ω'-simple basic modulo $\Im_{n-1}(\Omega/\Omega')$. If φ is of Ω/Ω'-order n, then φ falls in $\Im_n(\Omega/\Omega')$ by the definition of $\Im_n(\Omega/\Omega')$. If φ is of Ω/Ω'-order $< n$, then $\varphi \in \Im_{n-1}(\Omega/\Omega') \subseteq \Im_n(\Omega/\Omega')$ by the induction hypothesis. This completes our induction step of the assertion (**3**).

Since the sequence of ideals $\Im_n(\Omega/\Omega')$ is increasing, the set $\hat{\Im} \stackrel{\text{def.}}{=} \bigcup_{n\geq 0} \Im_n(\Omega/\Omega')$ forms an ideal of $\wp(\Omega)$. Assume that φ is a basic identity modulo $\hat{\Im}$. To verify this, we only need finitely many identities of $\hat{\Im}$. So the identity φ is also basic modulo $\Im_n(\Omega/\Omega')$ for n large enough. We may let $n \geq$ the Ω/Ω'-order of φ. Then $\varphi \in \Im_n(\Omega/\Omega') \subseteq \hat{\Im}$ by the assertion (**3**). We have thus shown that $\hat{\Im}$ contains all basic identities modulo $\hat{\Im}$ itself. By the defining minimality of $\Im(\Omega)$, $\hat{\Im} \supseteq \Im(\Omega)$ or $\bigcup_{n=0}^{\infty} \Im_n(\Omega/\Omega') \supseteq \Im(\Omega)$. On the other hand, we prove inductively that $\Im(\Omega) \supseteq \Im_n(\Omega/\Omega')$ for all $n \geq 0$: We have already shown $\Im(\Omega) \supseteq \Im(\Omega')$ in the proof of Theorem 1. Let $\varphi(x) = x^g u - u x^h$, where $u \in U$ and $g, h \in \Omega_0^+$, be an identity of R. We compute directly $\varphi(x+y) - \varphi(x) - \varphi(y) = 0$ and $\varphi(xy) - x^g\varphi(y) - \varphi(x)y^h = 0$. Hence $\varphi(x)$ is a basic (g,h)-identity modulo the zero ideal of $\wp(\Omega)$. A similar computation shows that an identity of the form $\varphi(x) = x^g u - u x^h$, where $u \in U$ and $g, h \in \Omega_0^-$, is also simple basic modulo the zero ideal of $\wp(\Omega)$. Hence all such identities fall in $\Im(\Omega)$. But these identities together with $\Im(\Omega')$ generate $\Im_0(\Omega/\Omega')$. So $\Im(\Omega) \supseteq \Im_0(\Omega/\Omega')$. For $n \geq 1$, assume, as the induction hypothesis, that $\Im(\Omega) \supseteq \Im_{n-1}(\Omega/\Omega')$. But then all identities basic modulo $\Im_{n-1}(\Omega/\Omega')$ are also basic modulo $\Im(\Omega)$ and hence must fall in $\Im(\Omega)$ by its defining closure property. Particularly, all Ω/Ω'-simple basic identities modulo $\Im_{n-1}(\Omega/\Omega')$ of Ω/Ω'-order n must fall in $\Im(\Omega)$. Since such identities together with $\Im_{n-1}(\Omega/\Omega')$ generate $\Im_n(\Omega/\Omega')$, we have $\Im(\Omega) \supseteq \Im_n(\Omega/\Omega')$. So it follows that $\Im(\Omega) \supseteq \bigcup_{n=0}^{\infty} \Im_n(\Omega/\Omega')$. Combining these two inclusions, we have $\Im(\Omega) = \bigcup_{n=0}^{\infty} \Im_n(\Omega/\Omega')$, as asserted.

Since each $\Im_n(\Omega/\Omega')$ is generated by $\Im(\Omega')$ and $x^\Delta - \lambda_\Delta(x)$ for $\Delta \in \Omega_n^c \setminus \Sigma_n$, the ideal $\Im(\Omega) = \bigcup_{n=0}^{\infty} \Im_n(\Omega/\Omega')$ is generated by $\Im(\Omega')$ and all $x^\Delta - \lambda_\Delta(x)$, where $\Delta \in \Omega^c \setminus \Sigma$. By Fact 1, the maximal Ω/Ω'-independent set Σ forms a basis of Ω/Ω' modulo $\Im(\Omega)/\Im(\Omega')$ with $\lambda_\Delta(x)$ being the Σ-reduced form of x^Δ for $\Delta \in \Omega^c \setminus \Sigma$. But $\lambda_\Delta(x)$ has Ω/Ω'-order $\leq$ the order of Δ for $\Delta \in \Omega^c \setminus \Sigma$. The basis Σ of Ω/Ω' modulo $\Im(\Omega)/\Im(\Omega')$ is hence ordered by Fact 2. We have thus shown that any maximal Ω/Ω'-independent subset of Ω^c forms an ordered basis of Ω/Ω' modulo $\Im(\Omega)/\Im(\Omega')$.

We finally show that any ordered basis of Ω/Ω' modulo $\Im(\Omega)/\Im(\Omega')$ must be a maximal Ω/Ω'-independent subset of Ω: Let Σ be an ordered basis of Ω/Ω' modulo $\Im(\Omega)/\Im(\Omega')$. First, assume on the contrary that Σ is *not* Ω/Ω'-independent. That is, Σ_n is Ω/Ω'-dependent for some n. If $n = 0$, then $g \sim h$ for two distinct $g, h \in \Sigma$ or for some $g \in \Sigma$, $h \in \Omega_0'$, and we set $\varphi(x) \stackrel{\text{def.}}{=} x^g u - u x^h$, where $u \in U^{-1}$ satisfies $x^g - u x^h u^{-1} = 0$ for all $x \in R$. If $n \geq 1$, there exists an Ω/Ω'-simple basic identity $\varphi(x)$ modulo $\Im_{n-1}(\Omega/\Omega')$ with all its leading Ω/Ω'-words in Σ_n. In either case, there always exists a linear basic identity φ modulo $\Im(\Omega)$ with all its leading Ω/Ω'-words in Σ. Since the leading Ω/Ω'-words of φ are already in Σ, the $\Xi \cup \Sigma$-reduced form of $\varphi(x)$ modulo $\Im(\Omega)$ has the same leading Ω/Ω'-part as φ and *cannot* be the zero polynomial. But the identity φ, being basic modulo $\Im(\Omega)$, falls in $\Im(\Omega)$ by the defining closure of $\Im(\Omega)$ and its $\Xi \cup \Sigma$-reduced form must hence be the zero polynomial, a contradiction. The ordered basis Σ is thus Ω/Ω'-independent.

We now assume that Σ is *not* a maximal Ω/Ω'-independent set. We extend Σ to a *maximal* Ω/Ω'-independent set Σ'. Pick arbitrarily $\Delta \in \Sigma' \setminus \Sigma$ and let $\lambda_\Delta(x) \in \wp(\Xi \cup \Sigma)$ be the $\Xi \cup \Sigma$-reduced expression of x^Δ modulo $\Im(\Omega)$. Then $x^\Delta - \lambda_\Delta(x) \in \Im(\Omega)$. But Σ', being a maximal Ω/Ω'-independent set, must also form an ordered basis of Ω/Ω' modulo $\Im(\Omega)/\Im(\Omega')$ and hence the set $\Xi \cup \Sigma'$ also forms a basis of Ω modulo $\Im(\Omega)$ by Theorem 3. The expression $x^\Delta - \lambda_\Delta(x)$ is obviously *not* the zero polynomial of $\wp(\Xi \cup \Sigma')$, a contradiction to the fact that $x^\Delta - \lambda_\Delta(x) \in \Im(\Omega)$. Therefore, any ordered basis of Ω/Ω' modulo $\Im(\Omega)/\Im(\Omega')$ must be a maximal Ω/Ω'-independent subset of $\Omega \setminus \Omega'$. The proof of Theorem 4 is thus completed.

Theorem 2 now follows easily from Theorems 3 and 4:

Proof of Theorem 2: The union of an increasing sequence of Ω/Ω'-independent subsets of $\Omega \setminus \Omega'$ obviously remains Ω/Ω'-independent. By Zorn's lemma, there exists a maximal Ω/Ω'-independent subset Σ of $\Omega \setminus \Omega'$. By Theorem 4, this maximal Ω/Ω'-independent subset Σ forms an ordered basis of Ω/Ω' modulo $\Im(\Omega')/\Im(\Omega)$. By Theorem 3, for any basis Ξ of Ω' modulo $\Im(\Omega')$, the set $\Xi \cup \Sigma$ forms a basis of Ω modulo $\Im(\Omega)$.

One might expect that any *ordered* basis of Ω' modulo $\Im(\Omega')$ could also be extended to an *ordered* basis of Ω modulo $\Im(\Omega)$. This turns out to be false as shown in the following somewhat trivial example:

Example 1: Let d, δ be two distinct symbols, both of which designate the *same* outer derivation of a prime ring R. Define

$$\Omega_0 \stackrel{\text{def.}}{=} \{1\}, \quad \Omega_1 \stackrel{\text{def.}}{=} \{1, \delta\} \quad \text{and} \quad \Omega \stackrel{\text{def.}}{=} \Omega_2 \stackrel{\text{def.}}{=} \{1, \delta, d\},$$

where 1 is the identity automorphism of R. Set $\Omega' \stackrel{\text{def.}}{=} \Omega \setminus \{\delta\} = \{1, d\}$. Both Ω and its subset Ω' are expansion closed word sets. The set Ω' is an independent subset of Ω' itself and hence forms an ordered basis of Ω' modulo $\Im(\Omega')$. But the set $\Omega'_2 = \{d\}$, though being an independent subset of Ω', is *not* an independent subset of Ω, since $x^d - x^\delta$ is a simple basic identity modulo $\Im(\Omega)$ with the leading word d. So Ω' is *not* an independent subset of Ω and can*not* be extended into an ordered basis of Ω modulo $\Im(\Omega)$.

The trouble comes from the fact that the word d above, designating a derivation, should have been reasonably put in Ω_1 instead of Ω_2. However, the following is a more complicate example:

Example 2: Suppose that δ_1, δ_2 are ordinary derivations of a prime ring R such that δ_1, δ_2 and $\delta_3 \stackrel{\text{def.}}{=} [\delta_1, \delta_2] \stackrel{\text{def.}}{=} \delta_1\delta_2 - \delta_2\delta_1$ are C-independent modulo inner derivations defined by elements of U. Define

$$\Omega_0 \stackrel{\text{def.}}{=} \{1\}, \quad \Omega_1 \stackrel{\text{def.}}{=} \{1, \delta_1, \delta_2, \delta_3\}, \quad \Omega \stackrel{\text{def.}}{=} \Omega_2 \stackrel{\text{def.}}{=} \{1, \delta_1, \delta_2, \delta_3, \delta_1\delta_2, \delta_2\delta_1\},$$

where 1 is the identity automorphism of R. Set $\Omega' \stackrel{\text{def.}}{=} \Omega \setminus \{\delta_3\} = \{1, \delta_1, \delta_2, \delta_1\delta_2, \delta_2\delta_1\}$. Both Ω and its subset Ω' are expansion closed word sets. Note that the set Ω' is an independent subset of Ω' itself and hence forms an ordered basis of Ω' modulo $\Im(\Omega')$. But the set $\Omega'_2 = \{\delta_1\delta_2, \delta_2\delta_1\}$, though being an independent subset of Ω', is *not* an independent subset of Ω, since $x^{\delta_1\delta_2} - x^{\delta_2\delta_1} - x^{\delta_3}$ is a simple basic identity modulo $\Im(\Omega)$ with the leading words $\delta_1\delta_2$ and $\delta_2\delta_1$. So Ω' is *not* an independent subset of Ω and can*not* be extended into an ordered basis of Ω modulo $\Im(\Omega)$.

This time, all derivation words are appropriately put in the right Ω_i. But Ω' is lack of $\delta_3 \stackrel{\text{def.}}{=} [\delta_1, \delta_2]$ to destroy the independence of $\delta_1\delta_2$ and $\delta_2\delta_1$. Example 2 also shows a slight difference between Kharchenko's theory and ours here: In order to apply Kharchenko's theory [**6**] to identities in $\wp(\Omega')$ above, we must first extend Ω' to Ω and then consider regular words with respect to the basis $\{\delta_i : i = 1, 2, 3\}$ ordered by $\delta_1 < \delta_2 < \delta_3$, say. But our theory for identities in $\wp(\Omega')$ completely resides in the word set Ω' itself. This slight generality might sometimes simplify applications of the theory to analyze a particular identity.

The problem on extension of ordered bases to ordered bases will be analyzed in our later work. Let us conclude this paper with the following remark: Although *ordered* bases are very useful in proving things by induction on orders, bases in general (*not* necessarily ordered) are actually more intrinsic in the sense that results stated in terms of bases of Ω will automatically imply the corresponding results for expansion closed word subsets Ω' by Theorem 2.

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