行政院國家科學委員會補助專題研究計畫成果報告

※※※※※※※※※※※※※※※※※※※※※※※※※
※　　　　　　　　　　　　　　　　　　　　　　　　※
※　　Weyl 體上的不變量理論　　　　　　　　　　　※
※　　Invariant Theory on Weyl Fields　　　　　　※
※　　　　　　　　　　　　　　　　　　　　　　　　※
※※※※※※※※※※※※※※※※※※※※※※※※※

計畫類別：☑個別型計畫　　□整合型計畫

計畫編號：NSC89－2115－M－002－017

執行期間：88 年 08 月 01 日至 89 年 07 月 31 日

計畫主持人：朱樺

本成果報告包括以下應繳交之附件：

- □赴國外出差或研習心得報告一份
- □赴大陸地區出差或研習心得報告一份
- □出席國際學術會議心得報告及發表之論文各一份
- □國際合作研究計畫國外研究報告書一份

執行單位：台大數學系

中 華 民 國 89 年 12 月 31 日

行政院國家科學委員會專題研究計畫成果報告

Weyl 體上的不變量理論（Invariant Theory on Weyl Fields）

計畫編號：NSC89－2115－M－002－017

執行期限：88 年 08 月 01 日至 89 年 07 月 31 日

主持人：朱樺　執行機構及單位名稱: 台大數學系

一、中文摘要

令 k 爲特徵數爲 p 之體，且 G 爲有限線性子群且 $p \mid |G|$。令 $A_1(k)=k[x, y]$，$xy-yx=1$，爲 Weyl 代數。G 衍生一個在 $A_1(k)$上的線性作用，我們求出不變子代數 $A_1(F_q)^{GL_2(F_q)}$ 及 $A_1(F_q)^{SL_2(F_q)}$。令 $\sigma: x \mapsto x, y \mapsto x+y$，且 $G=<\sigma>$是循環群，我們對 p=2 及 p=3 分別求出 $A_1(k)^{<\sigma>}$。

關鍵詞：Weyl 代數，模不變量，線性作用。

Abstract

Let k be a field of charasteristic p and $G \leq GL(k)$ be a finite group such that $p \mid |G|$. Let $A_1(k)=k[x, y]$, $xy-yx=1$, be the Weyl algebra over k. G induces a linear action on $A_1(k)$. We find the invariant subalgebras $A_1(F_q)^{GL_2(F_q)}$ and $A_1(F_q)^{SL_2(F_q)}$. Let $\sigma: x \mapsto x, y \mapsto x+y$, and $G=<\sigma>$ be the cyclic group generated by σ. We find $A_1(k)^{<\sigma>}$ for $p \geq 3$ and p=2, respectively.

Keywords: Weyl algebra, modular invariant, linear action.

Modular Invariants on Weyl Algebras

by

Huah Chu

Let k be a field and $G \leq GL_n(k)$ be a finite subgroup. Let $R = k[X_1, \dots, X_n]$ be the polynomial ring. Then G induces a linear action on R. The invariant subring is $R^G = \{f \in R \mid \sigma f = f \text{ for all } \sigma \in G\}$. If char $k \not| \, |G|$, there are plenty of results about R^G (See e.g. [15], [3], [14]). This is not the case for char $k \mid |G|$.

Let $K = k(X_1, \dots, X_n)$ and char $k \mid |G|$. The invariant subfields $K^{GL_n(\mathbb{F}_q)}$, $K^{SL_n(\mathbb{F}_q)}$ are found in [11]. The invariant subfields $K^{O_n(\mathbb{F}_q)}$, $K^{U_n(\mathbb{F}_{q^2})}$ are found in [4]. The invariant subfield $K^{SP_{2n}(\mathbb{F}_q)}$ is found in [7]. The invariant subrings $R^{GL_n(\mathbb{F}_q)}$ and $R^{SL_n(\mathbb{F}_q)}$ also appear in [11]. $R^{SP_{2n}(\mathbb{F}_q)}$ is settled in [5, 3]. $R^{O_3(\mathbb{F}_q)}$ is found in [10], $R^{O_4(\mathbb{F}_q)}$ is found in [8].

On the other hand, if G is a cyclic group generated by $\sigma : X_1 \mapsto X_1, X_2 \mapsto X_1 + X_2, \dots, X_n \mapsto X_{n-1} + X_n$. Then R^G is constructed in [12, 13] for $n \leq 5$.

Now let $A_1(k)$ be the Weyl algebra $k[x, y]$, $xy - yx = 1$ and $D_1(k)$ be the Weyl field $k(x, y)$, $xy - yx = 1$. If $G \leq GL_2(k)$, then G also induces linear actions on $A_1(k)$ and $D_1(k)$. Suppose char $k \not| \, |G|$. [6] had constructed the subalgebra $A_1(k)^G$ by means of $k[X, Y]^G$. [1] had proved that the invariant subfield $D_1(k)^G$ is isomorphic to $D_1(k)$. [2] had proved that if G is a finite abelian group, then $D_n(k)^G \cong D_n(k)$. [9] has proved the result for the case that G is a monomial action on $D_1(k)$.

In this note, we consider the invariant subalgebra $A_1(k)^G$ for char $k \mid |G|$.

§1.

In $A_1(k)$ we difine a filtration

$$k = M_0 \subset M_1 \subset M_2 \subset M_3 \subset \dots$$

where M_n is the k-module generated by all monomials $x^{\alpha_1}y^{\beta_1}x^{\alpha_2}y^{\beta_2}\dots$ $x^{\alpha_l}y^{\beta_l}$, $\sum \alpha_i + \sum \beta_i \le n$. Then we have an associated graded ring

$$gr(A_1(k)) = M_0 \oplus M_1/M_0 \oplus M_2/M_1 \oplus \dots.$$

In our case, $gr(A_1(k)) = k[X, Y]$, the commutative polynomial ring with two variables X and Y. The group G acts linearly on $A_1(k)$ induces a linear action on $k[X, Y]$.

For $f \in A_1(k)$, f can be written as $f = f_n + f_{n-1} + \dots$, f_i is a homogeneous polynomial of degree i (the expression is not uniqne), we define $\Phi(f) = f_n + M_{n-1} \in k[X, Y]$. That is, if $f_n = ax^{\alpha_1}y^{\beta_1}\dots x^{\alpha_l}y^{\beta_l} + \dots$, then $\Phi(f) = aX^{\alpha_1+\dots+\alpha_l}Y^{\beta_1+\dots+\beta_l}$.

Lemma 1.1. If $f \in A_1(k)^G$, then $\Phi(f) \in k[X, Y]^G$.

Proof. Write $f = f_n + g$, $g \in M_{n-1}$. Then, for any $\sigma \in G$, $f = \sigma(f) = \sigma(f_n) + \sigma(g)$, $\sigma(f_n) - f_n = g - \sigma(g) \in M_{n-1}$. Hence $\sigma(\Phi(f)) = \Phi(f)$. □

Theorem 1.2. Let $k[X, Y]^G = k[G_1, \dots, G_m]$. Suppose for any G_i, we have a polynomial $\Psi(G_i) := f_i \in A_1(k)^G$ such that $\Phi(f_i) = G_i$. Then

$$A_1(k)^G = k[f_1, \dots, f_m].$$

Proof. We just need to show that $A_1(k)^G \cap M_n \subset k[f_1, \dots, f_m]$ by induction on n. Suppose $f \in A_1(k)^G \cap M_n$, we may assume that $f \notin$

M_{n-1}. $\Phi(f) \in k[X,Y]_n^G$. So

$$\Phi(f) = \sum \alpha_\lambda G_1^{\lambda_1} \dots G_m^{\lambda_m},$$

$\alpha_\lambda \in k$, $\sum_{i=1}^m \lambda_i \deg(G_i) = n$. Since $f \in A_1(k)^G$ and $\Phi\Psi(G_i) = G_i$, it follows that $f - \sum \alpha_\lambda \Psi(G_1)^{\lambda_1} \cdots \Psi(G_m)^{\lambda_m} \in M_{n-1}$. Apply the induction hypothesis, we prove the result. □

Remark 1.3. If char $k \nmid |G|$, then we can define $\Psi(G_i)$ as follows. We choose a preimage $f \in M_n$ of G_i and define

$$\Psi(G_i) = \frac{1}{|G|} \sum_{\sigma \in G} \sigma(f) \in M_n$$

(See [6]). In this case, if $k[X,Y]^G$ in found, then we can construct $A_1(k)^G$ straightforward. But it is not well-defined if char $k \mid |G|$.

§2

From the definition $xy - yx = 1$, we can easily deduce that

Lemma 2.1. $y^n x = xy^n - ny^{n-1}$, $yx^n = x^n y - nx^{n-1}$.

Let $P_n = \sum_{i=1}^{n} \binom{n}{i} x^i y^{n-i}$. Note that it is the "formal" expansion of $(x+y)^n$ under the condition $xy = yx$.

Lemma 2.2. (1) $P_n x = xP_n - nP_{n-1}$.

(2) $P_n(x+y) = P_{n+1} - nP_{n-1}$

Proposition 2.3.

$$(x+y)^n = \sum_{i=0}^{[n/2]} (-1)^i 1\cdot 3\cdots(2i-1)\begin{pmatrix} n \\ 2i \end{pmatrix} P_{n-2i}.$$

Proof. By induction on n,

$$(x+y)^{n+1} = \sum_{i=0}^{[n/2]} (-1)^i 1\cdot 3\cdots(2i-1)\begin{pmatrix} n \\ 2i \end{pmatrix} P_{n-2i}(x+y)$$
$$= \sum_{i=0}^{[n/2]} (-1)^i 1\cdot 3\cdots(2i-1)\begin{pmatrix} n \\ 2i \end{pmatrix} \{P_{n-2i+1} - (n-2i)P_{n-2i-1}\}$$
$$= \sum_{i=0}^{[n/2]} (-1)^i 1\cdot 3\cdots(2i-1)\begin{pmatrix} n \\ 2i \end{pmatrix} P_{n+1-2i} - \sum_{i=0}^{[n/2]} (-1)^i 1\cdot 3\cdots(2i-1)$$
$$(n-2i)\begin{pmatrix} n \\ 2i \end{pmatrix} P_{n+1-2(i+1)}$$
$$= P_{n+1} + \sum_{i=1}^{[n/2]} \{(-1)^i 1\cdot 3\cdots(2i-1)\begin{pmatrix} n \\ 2i \end{pmatrix} - (-1)^{i-1} 1\cdot 3\cdots(2i-3)$$
$$(n+2-2i)\begin{pmatrix} n \\ 2i-2 \end{pmatrix}\} P_{n+1-2i} + P'$$
$$= P_{n+1} + \sum_{i=1}^{[n/2]} (-1)^i 1\cdot 3\cdots(2i-3)\tfrac{n!}{(2i-2)!(n-2i)!}\{\tfrac{(2i-1)}{2i(2i-1)} + \tfrac{(n+2-2i)}{(n+2-2i)(n+1-2i)}\}$$
$$P_{n+1-2i} + P'$$
$$= P_{n+1} + \sum_{i=1}^{[n/2]} (-1)^i 1\cdot 3\cdots(2i-3)\tfrac{n!}{(2i-2)!(n-2i)!}\cdot\tfrac{n+1}{2i(n-2i+1)}\cdot P_{n+1-2i} + P'$$
$$= \sum_{i=0}^{[\frac{n+1}{2}]} (-1)^i 1\cdot 3\cdots(2i-1)\begin{pmatrix} n+1 \\ 2i \end{pmatrix} P_{n+1-2i}$$

where $P' = 0$, if n is even; $P' = (-1)^{k+1} 1\cdot 3\cdots(n-2)\cdot nP_1$, if $n = 2k+1$.
□

By direct checking, we have the following

Corollary 2.4. (1)If char $k = 2$, then

$$(x+y)^{2^n} = x^{2^n} + y^{2^n} + 1.$$

(2) if char $k = p > 2$, then

$$(x+y)^{p^n} = x^{p^n} + y^{p^n}.$$

§3.

Now we consider the case of G being a cyclic group $\langle\sigma\rangle$, where

$$\sigma : x \mapsto x$$

$$y \mapsto x + y.$$

In commutative case, it is easy to show that

Lemma 3.1. If char $k = p$, then

$$k[X,Y]^{\langle\sigma\rangle} = k[X, Y^p - X^{p-1}Y].$$

Theorem 3.2. If char $k = p \geq 3$, then

$$A_1(k)^{\langle\sigma\rangle} = k[x, y^p - x^{p-1}y].$$

Proof. By Corollary 2.4. (2)

$$\sigma(y^p - x^{p-1}y) = (x+y)^p - x^{p-1}(x+y)$$
$$= x^p + y^p - x^p - x^{p-1}y = y^p - x^{p-1}y.$$

Hence in Theorem 1.2., $G_1 = X$, $G_2 = Y^p - X^{p-1}Y$, we define $f_1 = \Psi(G_1) = x$ and $f_2 = \Psi(G_2) = y^p - x^{p-1}y$, get the result. □

Note that Theorem 1.2. can not apply to the case of $p = 2$. Since in this case, $G_2 = Y^2 + XY$. Then $\Psi(G_2)$ has the form $y^2 + xy + a$. But

$$\sigma(y^2 + xy) = (x+y)^2 + x(x+y)$$

$$= y^2 + xy + 1$$

by Corollary 2.4. (1), we can not find an element $\Psi(G_2)$. In the following, we will settle this case.

Lemma 3.3. Let char $k = 2$, then

$$xy^2 + x^2y + y \quad \text{and} \quad y^4 + x^2y^2 + y^2$$

are invariant under σ.

Proof. $\sigma(xy^2 + x^2y + y) = x(x+y)^2 + x^2(x+y) + x + y$

$$= x(x^2 + y^2 + 1) + x^3 + x^2y + x + y$$
$$= xy^2 + x^2y + y.$$

$$\sigma(y^4 + x^2y^2 + y^2)$$
$$= (x+y)^4 + x^2(x+y)^2 + (x+y)^2$$
$$= x^4 + y^4 + 1 + x^2(x^2 + y^2 + 1) + x^2 + y^2 + 1$$
$$= y^4 + x^2y^2 + y^2.$$

□

Lemma 3.4. Let char $k = 2$, and $G = (Y^2 + XY)^k$, k odd. We can not define an element $\Psi(G) \in A_1(k)^{\langle\sigma\rangle}$.

Proof. We define $\Delta := \sigma - 1$. Then $\Psi(G) \in A_1(k)^{\langle\sigma\rangle}$ if and only if $\Delta(\Psi(G)) = 0$. Note that $\Delta(x^iy^j) = \sigma(x^iy^j) - x^iy^j = x^i\sigma(y)^j - x^iy^j = x^i\Delta(y^j)$. Hence the term y^k appears in $\Delta(x^iy^j)$ only if $i = 0$.

$$\begin{aligned}\Delta(y^j) &= (x+y)^j + y^j \\ &= P_j + \binom{j}{2} P_{j-2} + \binom{j}{4} P_{j-4} + \cdots + y^j,\end{aligned}$$

the term y^j also appears in P_j, hence it is cancelled. Thus the only term y^k appears in $\Delta(y^j)$ is $k \leq j-2$.

Write $\Psi(G) = f_n + f_{n-2} + f_{n-4} + \cdots$, where f_i is a homogeneous polynomial of degree i, $n = 2k$,

$$\begin{aligned} f_n &= \sum_{i=0}^{k} x^i y^{2k-i} \\ &= y^{2k} + x\cdot \text{ (other terms)}, \\ \Delta(y^{2k}) &= (x+y)^{2k} + y^{2k} \\ &= P_{2k} + y^{2k} + \binom{2k}{2} P_{2k-2} + \binom{2k}{4} P_{2k-4} + \cdots. \end{aligned}$$

Since $\binom{2k}{2} = \frac{2k(2k-1)}{2} = k(2k-1)$ is odd, so the term y^{2k-2} appears in Δf_n. On other hand, if y^j appears in any other term Δf_i, $i = n-2, n-4, \cdots$, then $j < i \leq n-2 = 2k-2$. Hence y^{2k-2} can not be cancelled and $\Delta(\Psi(G)) \neq 0$. □

Theorem 3.5. If char $k = 2$, then

$$A_1(k)^{\langle\sigma\rangle} = k[x, xy^2 + x^2y + y, y^4 + x^2y^2 + y^2].$$

Proof. Let $f \in A_1(k)^{\langle\sigma\rangle}$, $\Phi(f) \in [X,Y]^{\langle\sigma\rangle}$ and

$$\Phi(f) = \textstyle\sum_i a_i X^{l_i}(Y^2 - XY)^{k_i}$$

By Lemma 3.4., it is impossible that $l_i = 0$ and k_i is odd. Hence if we have either k_i is odd and $l_i \geq 1$, or k_i is even. Thus we consider

$$\begin{aligned} g = f - \{&\textstyle\sum_{k_i \text{even}} a_i (y^4 + x^2y^2 + y^2)^{\frac{k_i}{2}} \\ &+ \textstyle\sum_{k_i \text{odd}} a_i x^{l_i - 1}(xy^2 + x^2y + y)(y^4 + x^2y^2 + y^2)^{\frac{k_i - 1}{2}}\}. \end{aligned}$$

Then $\Phi(g) \in F_{n-1}$. Then by induction on n, we get the result. □

§4.

In this section, we consider the invariants of $GL_2(\mathbb{F}_q)$ and $SL_2(\mathbb{F}_q)-$actions. The invariant polynomials in commutative polynomial ring were found by Dickson [11]. Let $D_1 = XY^q - X^qY$ and $D_2 = \frac{X^{q^2-1}-Y^{q^2-1}}{X^{q-1}-Y^{q-1}}$, then

Lemma 4.1.

$$\mathbb{F}_q[X,Y]^{GL_2(\mathbb{F})_q} = \mathbb{F}_q[D_1^{q-1}, D_2],$$

$$\mathbb{F}_q[X,Y]^{SL_2(\mathbb{F})_q} = \mathbb{F}_q[D_1, D_2].$$

If q is odd, by Corollay 2.4 (2), $xy^q - x^qy$ and $(xy^q - x^qy)^{q-1}$ are invariants. For D_2, we must define some elements. Let

$$Q_n := \sum_{i=0}^{n}(-1)^{n-i}\begin{pmatrix} n \\ i \end{pmatrix} x^{i(p-1)}y^{(n-i)(p-1)}.$$

Note that it is the "Formal expansion" of $(x^{p-1} - y^{p-1})^n$ under the condition "$xy = yx$". We define

$$\Psi(D_2) = \sum_{j=0}^{q+1} x^{j(q-1)} \cdot y^{(q+1-j)(q-1)} +$$
$$\sum_{j=1}^{(q-1)/2} 1 \cdot 3 \cdots (2i-1) \begin{pmatrix} q-1 \\ 2i \end{pmatrix} x^{q-1+i(q-2)} Q_{q-1-2i} y^{i(q-2)}.$$

To show that $\Psi(D_2)$ is an invariant, we just note that $SL_2(\mathbb{F}_q)$ is generated by $\sigma_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $\tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $GL_2(\mathbb{F}_q)$ is generated by

$SL_2(\mathbb{F}_q)$ and $\rho_a = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $a \in \mathbb{F}_q^\times$. By direct computation, it can be shown that $\Psi(D_2)$ is an invariant. Hence we have

Theorem 4.2. Let q be an odd prime power.

$$A_1(\mathbb{F}_q)^{GL_2(\mathbb{F}_q)} = \mathbb{F}_q[(xy^q - x^qy)^{q-1}, \Psi(D_2)]$$

$$A_1(\mathbb{F}_q)^{SL_2(\mathbb{F}_q)} = \mathbb{F}_q[xy^q - x^qy, \Psi(D_2)].$$

References

[1] Alev, J. and Dumas, F., Invariants du corps de Weyl sous làction de groups finis, Comm. in Alg., 25(1997), 1655-1672.

[2] Alev, J. and Dumas, F., Sur les invariants des algebres de Weyl et de leurs corps de fractions, preprint.

[3] Benson, D. J. Polynomial Invariants of Finite Groups London Math. Soc. LN 190, 1993.

[4] Carlisle, D. and Kropholler P. H., Rational invariants of certain orthogonal and unitary groups, Bull. London Math. Soc., 24(1992), 57-60.

[5] Carlisle, D. and Kropholler P. H., Modular invariants of finite symplectic groups, preprint.

[6] Chiang, L, Chu, H. and Kang, M-c, Generation of Invariants, J. of Algebra, 221(1999), 232-241.

[8] Chu, H., Polynomial invariants of four dimensional orthogonal groups, to appear in Comm. in Algebra.

[9] Chu, H., The invariants of monomial actions on Weyl fields, preprint.

[10] Cohen, S. D., Rational functions invariant under an orthogonal group, Bull. London Math. Soc. 22(1990), 217-221.

[11] Dickson, L. E., A fundamental system of invariants of the general modular linear group with a solution of the form problem, Trams. Amer. Math. Soc., 12(1911), 75-98.

[12] Neusel, N. D., Invariants of some abelian p-groups in characteristic p, Proc. Amer. Math. Soc. 125(1997), 1921-1931.

[13] Shank, R. J., S. A. G. B. I. bases for rings of formal modular seminvariants, Comment. Math. Helv. 73(1998), 548-565.

[14] Smith, L., Polynomial Invariants of Finite Groups, Wellesley, 1995.

[15] Springer, T. A, Invariant Theory, LNM 585, Springer, 1977.