行政院國家科學委員會補助專題研究計畫成果報告

※※※※※※※※※※※※※※※※※※※※※※※※※

※ 有限可換群上的自同構群 ※

※※※※※※※※※※※※※※※※※※※※※※※※※

計畫類別：☑個別型計畫　　□整合型計畫

計畫編號：NSC89－2115－M－002－021

執行期間：89 年 08 月 01 日至 90 年 07 月 31 日

計畫主持人：朱樺

共同主持人：

本成果報告包括以下應繳交之附件：

□赴國外出差或研習心得報告一份

□赴大陸地區出差或研習心得報告一份

□出席國際學術會議心得報告及發表之論文各一份

□國際合作研究計畫國外研究報告書一份

執行單位：台大數學系

中　華　民　國　91 年　1　月 9　日

行政院國家科學委員會專題研究計畫成果報告

有限可換群上的自同構群

計畫編號：NSC89－2115－M－002－021
執行期限：89 年 08 月 01 日至 90 年 07 月 31 日
主持人：朱樺　執行機構及單位名稱：台大數學系

一、中文摘要

詳附件。

Abstract

詳附件。.

Keywords: 詳附件。

二、本年進度

詳附件。

三、成果自評

詳附件。

摘要

刻劃所有 p-群是群論的主要課題.建構 p-群 P 的標準方法是:取 P 的最大正規可換子群 A,於是 P/A 可視為 $\mathrm{Aut}(G)$ 的子群.因此對可換群之自同構群的 p-結構之研究是非常重要的.

這研究是要決定一些可換群之自同構群的 Sylow p-子群的結構.在第一節中,我們求可換 p-群之自同構群的 Sylow p-子群的階數. 在第二節中,我們決定一些低階可換 p-群之自同構群的 Sylow p-子群的結構. 在第三節中,我們推廣第二節所得的結果.

關鍵詞: 可換群. p-群.齊循環群. 自同構.

ABSTRACT

To classify all finite p-groups is an important problem in group theory. A standard strategy to construuct a p-group P is as follows: Consider a maximal abelian normal subgroup A of P ,then $P/A \hookrightarrow \mathrm{Aut}(A)$, P/A can be regarded as a p-subgroup of $\mathrm{Aut}(G)$.Hence to investegate the p-strructure of the automorphism groups of abelian groups is important.

In this note, we want to determine the structure of Sylow p-subgroup of $\mathrm{Aut}(G)$ for some abelian groups. This note is organized as follows: In § 1,we find the order of Sylow p-subgroups of automorphism groups of abelian p-groups. In § 2, we detemine the structure of the Sylow p-subgroups of automorphism group of some abelian p-groups with small order. In §3, we generalize the results in §2.

Keywords Abelian groups, p-groups, Homocyclic groups, Automorphism.

AUTOMORPHISMS OF ABELIAN GROUPS

§ 0

To classify all finite p-groups is an important problem in group theory. A standard strategy to construuct a p-group P is as follows: Consider a maximal abelian normal subgroup A of P ,then $P/A \hookrightarrow \mathrm{Aut}(A)$, P/A can be regarded as a p-subgroup of $\mathrm{Aut}(G)$.Hence to investegate the p-strructure of the automorphism groups of abelian groups is important.

As for p'-automorphisms of p-groups, there are some important results in [3,chap.5]:

Theorem 0.1 Let A be a p'-group of automorphisms acting indecomposably on the abelian p-group P. Then P is homocyclic.

Theorem 0.2 Let A be a p'-group of automorphisms of the abelian p-group P.Then

$$P = C_P(A) \times [P, A].$$

Theorem 0.3 Let A be a p'-group of automorphisms of the abelian p-group P which acts trivially on $\Omega_1(P)$. Then $A = 1$.

Other related results about automorphism group of ablian groups can be found in [1,2,5,6,7].

In this note, we want to determine the structure of Sylow p-subgroup of $\mathrm{Aut}(G)$ for some abelian groups. This note is organized as follows: In § 1,we find the order of Sylow p-subgroups of automorphism groups of abelian p-groups. In § 2, we detemine the structure of the Sylow p-subgroups of automorphism group of some abelian p-groups with small order. In §3, we generalize the results in §2.

§ 1

let G be a finitely generated abelian group. Then

$$G = G_0 \oplus G_1 \oplus \cdots \oplus G_m$$

where G_0 is a free abelian group of finite rank l, and G_i are finite p_i−groups. $p_i \neq p_j$ for $i \neq j$. If σ is an automorphism of G,then it is obvious that $\sigma(G_i) = G_i$ for all $i = 0, 1, \ldots, m$. Hence

$$\mathrm{Aut}(G) = \mathrm{Aut}(G_0) \times \mathrm{Aut}(G_1) \times \cdots \times \mathrm{Aut}(G_m).$$

If $G_0 = \mathbf{Z}^l$,then $\mathrm{Aut}(G_0) \cong GL_l(\mathbf{Z})$.Thus we just need to determine $\mathrm{Aut}(G_i)$ for p-group G_i. Thus we may assume that G is a finite p-group.

Notations: For $a \in \mathbf{Z}$,we denote the image of a in $\mathbf{Z}/p^n\mathbf{Z}$ as ${}_na$. If $m > n$,there is a natural epimorphism $\pi : \mathbf{Z}/p^m\mathbf{Z} \to \mathbf{Z}/p^n\mathbf{Z}$, we write $\pi({}_ma) = {}_n({}_ma) = {}_na$ .We also identify ${}_m(ap^n) = ({}_{m-n}a)p^n, m > n$. We denote a cyclic group of order n as C_n. For a group G and $x, y \in G$, we write $x^y := yxy^{-1}$.

Let G be a p-group, write $G = \langle e_1\rangle \oplus \cdots \oplus \langle e_m\rangle$,where $\langle e_i\rangle = C_{p^{n_i}}, n_1 \leq n_2 \leq \cdots \leq n_m$.

If $\sigma \in \mathrm{End}(G)$,we may represent σ as a matrix

$$\sigma = \begin{pmatrix} {}_{n_1}a_{11} & {}_{n_1}a_{12} & \cdots & {}_{n_1}a_{1m} \\ {}_{n_2}a_{21} & {}_{n_2}a_{22} & \cdots & {}_{n_2}a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ {}_{n_m}a_{m1} & {}_{n_m}a_{m2} & \cdots & {}_{n_m}a_{mm} \end{pmatrix}$$

Define $\Omega(G) = \{g \in G | g^p = 1\}$. Then $\Omega(G) = \langle p^{n_1-1}e_1\rangle \oplus \cdots \oplus \langle p^{n_m-1}e_m\rangle$.

Consider the restriction map $\rho : \mathrm{End}(G) \to \mathrm{End}(\Omega(G))$. It is easy to check that ,for $\sigma \in \mathrm{End}(G)$, $\rho(\sigma) = [b_{ij}]$ where

$$b_{ij} = \begin{cases} {}_1a_{ij}p^{n_j-n_i}, & \text{if } i < j; \\ {}_1a_{ij}, & \text{if } i \geq j. \end{cases}$$

For the sake of convenience, we write

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_m$$

where $G_i = (C_{p^{n_i}})^{l_i}$ is a homocyclic group.We also assume that $n_1 < n_2 < \cdots < n_m$.For $\sigma \in \mathrm{End}(G)$, σ can be represented as

$$\sigma = \begin{pmatrix} {}_{n_1}A_{11} & {}_{n_1}A_{12} & \cdots & {}_{n_1}A_{1m} \\ {}_{n_2}(p^{n_2-n_1}A_{21}) & {}_{n_2}A_{22} & \cdots & {}_{n_2}A_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ {}_{n_m}(p^{n_m-n_1}A_{m1}) & {}_{n_m}(p^{n_m-n_2}A_{m2}) & \cdots & {}_{n_m}A_{mm} \end{pmatrix}$$

where ${}_{n_i}A_{ij} \in M_{l_i \times l_j}(\mathbf{Z}/p^{n_i}\mathbf{Z})$.Then

$$\rho(\sigma) = \begin{pmatrix} {}_1A_{11} & {}_1(p^{n_2-n_1}A_{12}) & {}_1(p^{n_3-n_1}A_{13}) & \cdots & {}_1(p^{n_m-n_1}A_{1m}) \\ {}_1A_{21} & {}_1A_{22} & {}_1(p^{n_3-n_2}A_{23}) & \cdots & {}_1(p^{n_m-n_2}A_{2m}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ {}_1A_{m1} & {}_1A_{m2} & {}_1A_{m2} & \cdots & {}_1A_{m1} \end{pmatrix}$$

$$= \begin{pmatrix} {}_1A_{11} & 0 & \cdots & 0 \\ {}_1A_{21} & {}_1A_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ {}_1A_{m1} & {}_1A_{m2} & \cdots & {}_1A_{mm} \end{pmatrix}$$

since $n_i < n_j$ for $i < j$.

Now we determine the kernel of the map ρ: Since $\sigma \in \ker\rho$ if and only if ${}_1A_{ij} = 0$ for $i > j$ and ${}_1A_{ii} = I$ for all i, hence we have that

$$\begin{cases} A_{ij} \text{ can be chosen arbitrary,} & \text{if } i < j \ ; \\ A_{ij} \equiv 0 (\text{mod } p\ ), & \text{if } i > j; \\ A_{ii} \equiv I (\text{mod } p\ ). \end{cases}$$

Thus we have

$$|\ker\rho| = (\Pi_{i<j} p^{n_i l_i l_j})(\Pi_{i>j} p^{(n_j-1) l_i l_j})(\Pi_i p^{(n_i-1) l_i^2})$$

It is easy to see that

$$\sigma \in \text{Aut}(G) \text{ if and only if } \rho(\sigma) \in \text{Aut}(\Omega(G)).$$

Moreover, we have

$$\rho(\sigma) \in \text{Aut}(\Omega(G)) \text{ if and only if } {}_1A_{11} \in GL_{l_i}(\mathbf{Z}/p\mathbf{Z}\ ) \text{ for all } i,$$

and

$${}_1A_{ii} \in GL_{l_i}(\mathbf{Z}/p\mathbf{Z}) \text{ if and only if } A_{ii} \in GL_{l_i}(\mathbf{Z}/p^{n_i}\mathbf{Z}).$$

Therefore we have that $\sigma \in$Aut (G) if and only if ${}_1A_{ii} \in GL_{l_i}(\ \mathbf{Z}/p\mathbf{Z})$ .Thus we have

Theorem 1.1

$$|\mathrm{Aut}(G)| = p^{\sum_{i=1}^{m} 2n_i(l_i+l_{i+1}+\cdots+l_m)-\sum_{i=1}^{m}(n_i+1)l_i^2} \times \prod_{i=1}^{m}|GL_{l_i}(\mathbf{Z}/p\mathbf{Z})|.$$

$$|\mathrm{Aut}(G)_p| = p^{\sum_{i=1}^{m} 2n_i(l_i+l_{i+1}+\cdots+l_m)-\sum_{i=1}^{m}(n_i+1)l_i^2+\sum_{i=1}^{m} l_i(l_i-1)/2}.$$

§ 2

Now we want to find the p-structure of the automorphism groups of some abelian p-groups.

We first consider the case $G_{l,m} := C_{p^l} \oplus C_{p^m} = \langle e_1 \rangle \oplus \langle e_2 \rangle$,$l < m$,which is the abelian group of type (p^l, p^m) . Let $\sigma \in \mathrm{End}(G)$ be an endomorphism. Since $\sigma(e_1)^{p^l} = 1$, so $\sigma(e_1) = e_1^{a_0+a_1p+\cdots+a_{l-1}p^{l-1}} \cdot e_2^{(c_0+c_1p+\cdots+c_{l-1}p^{l-1})p^{m-l}}$.Thus we may write σ as

$$\begin{pmatrix} a_0 + a_1p + \cdots + a_{l-1}p^{l-1} & b_0 + b_1p + \cdots + b_{l-1}p^{l-1} \\ (c_0 + c_1p + \cdots + c_{l-1}p^{l-1})p^{m-l} & d_0 + d_1p + \cdots + d_{m-1}p^{m-1} \end{pmatrix}.$$

Thus $|\mathrm{End}(G)| = p^{3l+m}$.Moreover, σ is an automorphism if and only if $a_0, d_0 \neq 0$.So $|\mathrm{Aut}(G)| = p^{3l+m-2}(p-1)^2$ and $|\mathrm{Aut}(G)_p| = p^{3l+m-2}$,where G_p denotes a Sylow-p subgroup of G.

Suppose that the order of σ is p-power. Since $\sigma \equiv \begin{pmatrix} a_0 & b_0 \\ 0 & d_0 \end{pmatrix} (\mathrm{mod}\ p)$, so $a_0 = d_0 = 1$.Thus there are p^{3l+m-2} elements which are p -elements (including 1),the number is equal to $|\mathrm{Aut}(G)_p|$.Hence there is only one Sylow p-subgroup.

Theorem 2.1 Let $G = C_{p^l} \oplus C_{p^m}, l < m$. Then the Sylow p-subgroup of the automorphism group $\mathrm{Aut}(G)$ is normal in $\mathrm{Aut}(G)$.

Now let $G = G_{1,m}$,then $|\mathrm{Aut}(G)_p| = p^{m+1}$ .We have the following

Theorem 2.2 Let P be a Sylow$p$-subgroup of $\mathrm{Aut}(G_{1,m}), m > 1$.Let

$$x = \begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 \\ p^{m-1} & 1 \end{pmatrix}.$$

Then

$$P = \langle x, y, z | x^{p^{m-1}} = y^p = z^p = 1, yxy^{-1} = zxz^{-1} = x, zyz^{-1} = x^{p^{m-2}}y \rangle.$$

The proof of the Theorem follows by direct verifications.

In this case, the center $Z(G) = \langle x \rangle$,the commutator subgroup $G' = \langle x^{p^{m-2}} \rangle$ .Thus $|Z(G)| = p^{m-1}$, and $|G'| = p$. P has a normal series

$$\langle x \rangle \subset \langle x, y \rangle \subset \langle x, y, z \rangle.$$

Theorem 2.3 Let $G = G_{1,1} \cong (\mathbf{Z}/p\mathbf{Z})^2$.Then $\mathrm{Aut}(G_{1,1}) \cong GL_2(\mathbf{Z}/p\mathbf{Z})$. A Sylow$p$-subgroup P of $\mathrm{Aut}(G_{1,1})$ is cyclic generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Theorem 2.4 Let $G = G_{2,2} \cong (\mathbf{Z}/p^2\mathbf{Z})^2$ and P be a Sylow p- subgroup of $\mathrm{Aut}(G)$.Let

$$x_1 = \begin{pmatrix} 1+p & 0 \\ 0 & 1+p \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}.$$

Then

$$P = \langle x_1, x_2, y, z | x_1^p = x_2^p = y^{p^2} = z^p = 1, x_1^{x_2} = x_1^y = x_1^z = 1,$$

$$x_2^y = x_2 y^p, x_2^z = x_2, y^z = x_1^{p-1} x_2^2 y \rangle$$

Proof. It follows from the following identities.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix}, \qquad (1)$$

$$\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix} = \begin{pmatrix} 1-p & 0 \\ 0 & 1+p \end{pmatrix},$$

$$\begin{pmatrix} 1+p & 0 \\ 0 & 1+p \end{pmatrix}^{p-1} \begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix}^2 = \begin{pmatrix} 1-p & 0 \\ 0 & 1+p \end{pmatrix}.$$

In this case, the center $Z(G) = \langle x_1 \rangle$,the commutator subgroup $G' = \langle x^{p-1}x_2^2, y^p \rangle = \langle \begin{pmatrix} 1-p & 0 \\ 0 & 1+p \end{pmatrix}, \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \rangle$ .Thus $|Z(G)| = p$, and $|G'| = p^2$. P has a normal series

$$\langle x_1 \rangle \subset \langle x_1, x_2 \rangle \subset \langle x_1, x_2, y \rangle \subset P.$$

Now we consider the case $G = G_{2,3}$. In this case

$$\sigma = \begin{pmatrix} 1+a_1p & b_0+b_1p \\ (c_1+c_2p)p & 1+d_1p+d_2p^2 \end{pmatrix}$$

and

$$\sigma^n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}$$

where

$$\begin{cases} \alpha_n = 1 + \{\binom{n}{1}a_1 + \binom{n}{2}b_0c_1\}p, \\ \beta_n = nb_0 + \{\binom{n}{2}a_1b_0 + \binom{n}{1}b_1 + \binom{n}{3}b_0^2c_1 + \binom{n}{2}b_0d_1\}p, \\ \gamma_n = nc_1p + \{\binom{n}{2}a_1c_1 + \binom{n}{3}b_0c_1^2 + nc_2 + \binom{n}{2}c_1d_1\}p^2, \\ \delta_n = 1 + f(m,1,n)p + f(m,2,n)p^2 + \binom{n}{2}b_0c_1p + gp^2. \end{cases}$$

and

$$f(m,j,n) = \sum \frac{n!}{i_1!i_2!\dots i_{m-1}!(n-i_1-\cdots-i_{m-1})!} d_1^{i_1}\dots d_{m-1}^{i_{m-1}}$$

where the summation is run over all the tuples $\{i_1,\dots,i_{m-1}\}$ such that $i_1 + 2i_2 + \cdots + (m-1)i_{m-1} = j$.

$$g = \binom{n}{3}a_1b_0c_1 + \binom{n}{2}b_0^2c_1^2 + \binom{n}{2}b_1c_1 + \binom{n}{2}b_0c_2 + 2\binom{n}{3}b_0c_1d_1.$$

Note that all elements in $\mathrm{Aut}(G)_p$ have order at most p^2,because $\binom{p^2}{3} \equiv 0(\mathrm{mod}\ p)$ for all p. In fact ,the center of the Sylow-p subgroup is

$$Z(P) = \begin{pmatrix} 1+a_1p & 0 \\ 0 & 1+a_1p+d_2p^2 \end{pmatrix}$$

which is cyclic of order p^2 generated by $x_1 = \begin{pmatrix} 1+p & 0 \\ 0 & 1+p \end{pmatrix}$.

Theorem 2.5 Let $G = G_{2,3}$ and let P be a Sylow p- subgroup of $\mathrm{Aut}(G)$. Let

$$x_1 = \begin{pmatrix} 1+p & 0 \\ 0 & 1+p \end{pmatrix},\quad x_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix},\quad y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\quad z = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}.$$

Then

$$P = \langle x_1, x_2, y, z | x_1^{p^2} = y^{p^2} = z^{p^2} = 1, x_2^p = x_1^p,$$
$$x_2^y = x_2 y^p, x_2^z = x_2 z^{-p}, y^z = x_1^{p-1} x_2^{2-p} y^{p+1} z^{-p} \rangle$$

Proof. It follows from (1) and the following identities.

$$\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -p^2 & 1+p \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix} = \begin{pmatrix} 1-p & 1 \\ -p^2 & 1+p \end{pmatrix},$$

$$\begin{pmatrix} 1-p & 1 \\ -p^2 & 1+p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1-p} = \begin{pmatrix} 1-p & 0 \\ 0 & 1+p \end{pmatrix}.$$

In this case, the center $Z(G) = \langle x_1 \rangle$,the commutator subgroup $G' = \langle x^{p-1} x_2^{2-p}, y^p, z^p \rangle = \langle \begin{pmatrix} 1-p & 0 \\ 0 & 1+p \end{pmatrix}, \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ p^2 & 1 \end{pmatrix} \rangle$ .Thus $|Z(G)| = p^2$, and $|G'| = p^4$. And P has a normal series

$$\langle x_1 \rangle \subset \langle x_1, x_2 \rangle \subset \langle x_1, x_2, y^p \rangle \subset \langle x_1, x_2, y \rangle \subset \langle x_1, x_2, y, z^p \rangle \subset P$$

In general, Let $G = G_{2,m}, m > 3, P = \mathrm{Aut}(G)_p$ be the Sylow p-subgroup .Then $|G| = p^{m+4}(p-1)^2$ and $|P| = p^{m+4}$. Let

$$x_1 = \begin{pmatrix} 1+p & 0 \\ 0 & 1+p \end{pmatrix}, x_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix}, y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, z = \begin{pmatrix} 1 & 0 \\ p^{m-2} & 1 \end{pmatrix}$$

Then it can be checked that $x_1^{p^{m-2}} \neq 1, x_1^{p^{m-1}} = 1$. It generates the center of P. Moreover,it can be verified

$$P = \langle x_1, x_2, y, z | x_1^{p^{m-1}} = y^{p^2} = z^{p^2} = 1, x_2^p = x_1^p,$$
$$x_2^y = x_2 y^p, x_2^z = x_2 z^{-p}, y^z = x_1^{-p+(\frac{p-1}{2})p^2} y \rangle$$

Let $G = G_{3,m}$ and P the Sylow p-subgroup of $\mathrm{Aut}(G)$ .The element of P has the form

$$\begin{pmatrix} 1+a_1p+a_2p^2 & b_0+b_1p+b_2p^2 \\ (c_{m-3}+c_{m-2}p+c_{m-1}p^2)p^{m-3} & 1+d_1p+\cdots+d_{m-1}p^{m-1} \end{pmatrix}.$$

Let

$$x_1 = \begin{pmatrix} 1+p & 0 \\ 0 & 1+p \end{pmatrix}, x_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix}; y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, z = \begin{pmatrix} 1 & 0 \\ p^{m-3} & 1 \end{pmatrix}.$$

Then we have

$$P = \langle x_1, x_2, y, z | x_1^{p^{m-1}} = y^{p^3} = z^{p^3} = 1, x_2^{p^2} = x_1^{p^2},$$

$$x_2^y = x_2y^p, x_2^z = x_2z^{p^2-p}, y^z = x_1^{-p+(\frac{p-1}{2})p^2}y\rangle$$

§ 3

In this section , we will generalize the result in the above section.

For any $l < m$ and p ,let x_1, x_2, y be as above and $z = \begin{pmatrix} 1 & 0 \\ p^{m-l} & 1 \end{pmatrix}$. We have $x_1^{p^{m-1}} = y^{p^l} = z^{p^l} = 1, x_1^{p^{l-1}} = x_2^{p^{l-1}}$.

$$x_2^y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & p \\ 0 & 1+p \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix}\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} = x_2y^p,$$

$$x_2^z = \begin{pmatrix} 1 & 0 \\ p^{m-l} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -p^{m-l} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -p^{m-l+1} & 1+p \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{p^{m-l+1}}{p+1} & 1 \end{pmatrix} = x_2z^{kp}$$

where $k = -(p^{l-2} - p^{l-3} + \cdots + (-1)^l)$, if $l \neq 1$. If $l = 1$,then $x_2^z = x_2$.

$$- y^z = \begin{pmatrix} 1 & 0 \\ p^{m-l} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -p^{m-l} & 1 \end{pmatrix} = \begin{pmatrix} 1-p^{m-l} & 1 \\ -p^{2m-2l} & 1+p^{m-l} \end{pmatrix}.$$

If $m \geq 2l$, then we have

$$y^z = \begin{pmatrix} 1 & 1 \\ 0 & 1+p^{m-l} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1+p^{m-l} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = x_1^{kp^l} y$$

where k is an integer such that $(1+p)^{kp^l} \equiv 1+p^{m-l} (\mathrm{mod}\ p^m)$. Note that in $\mathbf{Z}/p^m\mathbf{Z}$,the set $\{1+pa | a \in \mathbf{Z}/p^m\mathbf{Z}\}$ is a subgroup,and it is cyclic generated by $1+p$.Hence such an integer k exists.

If $l < m < 2l$, setting $h := m-l < l$ and $k := 1+p^h+p^{2h}+\cdots+p^{[\frac{m}{h}]h}$,and note $k(1-p^h) \equiv 1 (\mathrm{mod}\ p^m)$,then

$$\begin{pmatrix} 1-p^h & 1 \\ -p^{2h} & 1+p^h \end{pmatrix} \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p^{2h}(1-p^h) & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-p^h & 0 \\ -p^{2h} & k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p^{2h}(1-p^h) & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-p^h & 0 \\ 0 & 1+p^h+\cdots+p^{[\frac{m}{h}]h} \end{pmatrix},$$

since $k(1-p^h) \equiv 1 (\mathrm{mod}\ p^m)$.Thus we have proved the following

Theorem 3.1 Let $G = G_{l,m} = C_{p^l} \oplus C_{p^m}, l < m$, and P be the Sylow p-subgroup of $\mathrm{Aut}(G)$. Let

$$x_1 = \begin{pmatrix} 1+p & 0 \\ 0 & 1+p \end{pmatrix}, x_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix}, y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, z = \begin{pmatrix} 1 & 0 \\ p^{m-l} & 1 \end{pmatrix}.$$

Then
(i)$l \neq 1$ and $m \geq 2l$:

$$P = \langle x_1, x_2, y, z | x_1^{p^{m-1}} = y^{p^l} = z^{p^l} = 1, x_2^{p^{l-1}} = x_1^{p^{l-1}},$$

$$x_2^y = x_2 y^p, x_2^z = x_2 z^{k_1 p}, y^z = x_1^{k_2 p^l} y \rangle$$

where $k_1 = -(p^{l-2} - p^{l-3} + \cdots + (-1)^l)$, and k_2 satisfies $(1+p)^{k_2 p^l} \equiv 1 + p^{m-l} (\mathrm{mod}\ p^m)$

(ii)$l \neq 1$ and $l < m < 2l$:

$$P = \langle x_1, x_2, y, z | x_1^{p^{m-1}} = y^{p^l} = z^{p^l} = 1, x_2^{p^{l-1}} = x_1^{p^{l-1}},$$

$$x_2^y = x_2 y^p, x_2^z = x_2 z^{k_1 p}, y^z = x_1^{k_3} x_2^{k_4} y^{k_5} z^{-p^h(1-p^h)} \rangle$$

where $h = m - l, k_5 = 1 + p^h + p^{2h} + \cdots + p^{[\frac{m}{h}]h}$, k_3 satisfies $(1+p)^{k_3} \equiv 1 - p^h (\mathrm{mod}\ p^l)$,k_4 satisfies $(1+p)^{k_4} \equiv (1-p^h)^2 (\mathrm{mod}\ p^m)$.

(iii)$l = 1$:

$$P = \langle x_2, y, z | x_2^{p^{m-1}} = y^p = z^p = 1, yxy^{-1} = zxz^{-1} = x, zyz^{-1} = x^{p^{m-2}} y \rangle.$$

Let $l = m$. Since $\ker\rho$ is a p-group and $\rho(\mathrm{Aut}(G)) = GL_l(\mathbf{Z}/p\mathbf{Z})$, so we may choose P to be the preimage $\rho^{-1}(\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle)$. Define $z = \begin{pmatrix} 1 & 0 \\ p^{m-l} & 1 \end{pmatrix}$.Then

$$x_2^z = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -p^2 & 1+p \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -kp^2 & 1 \end{pmatrix} = x_2 z^{kp}$$

where $k = 1 - p + p^2 + \cdots + (-1)^{l-1} p^{l-3}$.

$$y^z = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix} = \begin{pmatrix} 1-p & 1 \\ -p^2 & 1+p \end{pmatrix},$$

$$\begin{pmatrix} 1-p & 1 \\ -p^2 & 1+p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ kp^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -(1+p) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & 1+p \end{pmatrix} = x_1^{-1} x_2^2.$$

Thus we have proved the following

Theorem 3.2 Let $G = G_{l,l} = (C_{p^l})^2$, and P be the Sylow p-subgroup of $\mathrm{Aut}(G)$. Let

$$x_1 = \begin{pmatrix} 1+p & 0 \\ 0 & 1+p \end{pmatrix}, x_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix}, y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, z = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}.$$

Then
(i)$l > 2$:

$$P = \langle x_1, x_2, y, z | x_1^{p^{l-1}} = x_2^{p^l-1} = y^{p^l} = z^{p^l-1} = 1,$$

$$x_2^y = x_2 y^p, x_2^z = x_2 z^{kp}, y^z = x_1^{-1} x_2^2 y^{1+p} z^{-kp} \rangle$$

where $k = 1 - p + p^2 + \cdots + (-1)^{l-1} p^{l-3}$.

(ii)$l = 1$:

$$P = \langle y | y^p = 1 \rangle$$

(iii)$l = 2$:

$$P = \langle x_1, x_2, y, z | x_1^p = x_2^p = y^{p^2} = z^p = 1, x_1^{x_2} = x_1^y = x_1^z = 1,$$

$$x_2^y = x_2 y^p, x_2^z = x_2, y^z = x_1^{p-1} x_2^2 y \rangle$$

Using the same arguments as given in the proof of Thoerem 2.1,we can generalized as follows.

Theorem 3.3 Let $G = C_{p^{n_1}} \oplus \cdots \oplus C_{p^{n_m}}$ be an abelian p-group with $n_1 < n_2 < \cdots < n_m$.Then the Sylow p-subgroup of $\mathrm{Aut}(G)$ is normal in $\mathrm{Aut}(G)$.

REFERENCES

[1]P.Dubuque, Sur les automorphismes des p-groupes,Mat. Sbornik N.S. 18(60),281-298(1946).

[2]A.Frohich,The representartion of a finite group as a group of automorphisms on a finite abelian group.Quart.J.Math.,Oxford Ser.(2) 1,270-283(1950).

[3]D.Gorenstein,*Finite Groups*.

[4]N. Jacobson, *Basic Algebra*, vol.1.

[5]J.Levine:R.R.Korfhage,Automorphisms of abelian groups induced by involutory matrices,Duke Math. J.29(1962).631-645;30(1963),161-170:31(1964).631-653.

[6]F.A.Lewis,On the group of isomorphisms of Abelian group of order n^m and type (1,1,... ,1),Amer.Math.Monthly 48(1941)199-201.

[7]W,Liebert,Charakterisierrung der Endomorphismenringe endlicher abelscher Gruppen,Arch.Math.18(1967)128-135.