行政院國家科學委員會補助專題研究計畫成果報告

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計畫類別：■個別型計畫 □整合型計畫

計畫編號：NSC 89－2115－M－002－022

執行期間： 89 年 8 月 1 日至 90 年 7 月 31 日

計畫主持人：張志中

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■赴國外出差或研習心得報告一份

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□出席國際學術會議心得報告及發表之論文各一份

□國際合作研究計畫國外研究報告書一份

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行政院國家科學委員會專題研究計畫成果報告

交互作用粒子系統的流力極限 (IV)

Hydrodynamic Limit of Interacting Particle Systems (IV)

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一、中文摘要

本計畫中我們導出 2 維格子點空間 Z^2 上對稱互斥過程中一個位置(one site) 被粒子佔據的時間 (occupation time) 的大離差估計 (large deviation estimate)。

關鍵詞：對稱互斥過程、被粒子佔據的時間、大離差估計

Abstract

We prove a large deviation principle for the occupation time of a site in the two dimensional symmetric simple exclusion process.

Keywords: symmetric simple exclusion process, occupation time, large deviation estimate

二、緣由與目的

Consider the nearest neighbor symmetric simple exclusion process on two dimensional lattice Z^2. This is a continuous time Markov process representing the evolution of random walks on Z^2 with a hard core interaction which prevents more than one particle per site. The configurations of this process are denoted by the Greek letter η so that $\eta(x)$ is equal to 1 or 0 if site x in Z^2 is occupied or not for η.

For each α in [0,1], denote by $\nu(\alpha)$ the Bernoulli product measure on the configuration space Ω with marginals given by

$\nu(\alpha)\{\ \eta,\ \eta(x)=1\ \}=\alpha$

for x in Z^2. A simple computation shows that $\{\nu(\alpha),\ 0\leqq\alpha\leqq 1\}$ is a one-parameter family of reversible invariant measures. In this project we study the nearest-neighbor symmetric simple exclusion process accelerated by T starting from the reversible measure $\nu(\alpha)$ for a fixed α in (0,1).

Consider the occupation time of the origin:

$$V_t = \left(\int_0^t \eta_s(0)\, ds \right) / t .$$

A law of large numbers and a central limit theorem for this additive functional of a reversible Markov process can be proved along the lines of [3]. It can be shown that V_t converges in probability to α as $t \to \infty$ and that $c(t)(V_t-\alpha)$ converges in distribution to a non-degenerate mean zero Gaussian variable, where c(t) is equal to the square root of t/log t. Landim [4] proved a large deviations principle for V_t in dimension $d\neq 2$ and that in dimension 2 the correct order is t/log t.

In this project we establish the large deviations of V_t in dimension d=2. The rate function will have a rather explicit form.

三、結果與討論

The results stated in this report are taken from a joint work with C. Landim and T.Y. Lee [1].

Let $\Omega = \{0,1\}^{\mathbb{Z}^2}$ be the configuration space of the exclusion process. Given $T > 0$, the generator L_T of the *accelerated* symmetric simple exclusion process is given by

$$(L_T f)(\eta) = \frac{T}{2} \sum_{x,y \in \mathbb{Z}^d} [f(\sigma^{x,y}\eta) - f(\eta)],$$

where the summation is carried over all edges $\{x, y\}$ of $\mathbb{Z}^2$. In this formula, f is a local function and $\sigma^{x,y}\eta$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(y)$:

$$(\sigma^{x,y}\eta)(z) = \begin{cases} \eta(z) & \text{if } z \neq x,\, y, \\ \eta(x) & \text{if } z = y, \\ \eta(y) & \text{if } z = x\,. \end{cases}$$

For each $0 \le \alpha \le 1$, denote by ν_α the Bernoulli product measure on Ω with marginals given by

$$\nu_\alpha\{\eta,\, \eta(x) = 1\} = \alpha$$

for $x \in \mathbb{Z}^2$. Clearly, $\{\nu_\alpha,\, 0 \le \alpha \le 1\}$ is a one-parameter family of reversible invariant measures. For $0 \le \alpha \le 1$, denote by $\mathbb{P}_\alpha = \mathbb{P}_{\alpha,T}$ the probability on the path space $D(\mathbb{R}_+, \Omega)$ corresponding to the nearest-neighbor accelerated symmetric simple exclusion process generated by L_T starting from ν_α.

Define the occupation time of the origin:

$$V_T = \frac{1}{T}\int_0^T \eta_s(0)\, ds\,.$$

The purpose of this project is to establish the large deviations of V_T under $\mathbb{P}_\alpha = \mathbb{P}_{\alpha,T}$ as $T \to \infty$ in dimension $d = 2$.

It is known (see [2] for example) that in the investigation of large deviations of Markov processes, a major ingredient is to find the relevant perturbations that create the fluctuations. It turns out that in dimension $d = 2$ the correct perturbations involved in the investigation of the large deviations of the occupation time have an interesting log scale. To describe these perturbations we introduce some notation, mainly for scaling.

Denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^2$: for $x = (x_1, x_2)$, $|x|^2 = x_1^2 + x_2^2$. For each $T > 1$ define the projection $\sigma_T : \mathbb{Z}^2 \to [0,\infty)$ by

$$\sigma_T(x) = \frac{\log|x|}{\log T}\,.$$

We shall see that a scaling involving σ_T is just right for our interest in dimension $d = 2$.

For any $\alpha \in (0,1)$, let $C^2_\alpha(\mathbb{R}_+)$ be the space of twice continuously differentiable functions $\gamma : [0,\infty) \to (0,1)$ such that γ' has a compact support in $(0, \frac{1}{2})$ and such that $\gamma(u) = \alpha$ for $u \ge 1/2$. Thus there exists $0 < \beta < 1$ and $0 < \varepsilon < 1/4$ such that $\gamma(u) = \beta$ for all $\beta \le \varepsilon$, and $\gamma(u) = \alpha$ for all $u \ge \frac{1}{2} - \varepsilon$. For each γ in $C^2_\alpha(\mathbb{R}_+)$, denote by $\Gamma = \Gamma_\gamma$ the function

$$\Gamma(u) = \frac{1}{2}\log\frac{\gamma(u)\,[1-\alpha]}{[1-\gamma(u)]\,\alpha}\,. \tag{3.1}$$

Notice that $\Gamma(\sigma_T(x))$ vanishes for $|x| > T^{(1/2)-\varepsilon}$ for some $\varepsilon > 0$. It turns out that the relevant perturbations in studying the large deviations of the occupation time are the inhomogeneous exclusion process generated by $L_{T,\Gamma}$ of which a particle jumps from x to y at rate $\exp\{\Gamma(\sigma_T(y)) - \Gamma(\sigma_T(x))\}$:

$$(L_{T,\Gamma} f)(\eta) = \frac{T}{2}\sum_{|x-y|=1} \eta(x)\{1-\eta(y)\}$$
$$\times e^{\Gamma(\sigma_T(y)) - \Gamma(\sigma_T(x))}[f(\sigma^{x,y}\eta) - f(\eta)].$$

Let $\mathcal{M}_1$ be the set of all measurable functions from $[0, 1/2]$ to $[0, 1]$ endowed with the weak topology so that a sequence γ_n converges to γ if and only if

$$\lim_{n\to\infty}\langle \gamma_n, G\rangle = \lim_{n\to\infty}\int_0^{1/2}\gamma_n(u)G(u)\,du = \langle\gamma, G\rangle$$

for every continuous function $G : [0, 1/2] \to \mathbb{R}$ which vanishes at the boundary.

For $T > 0$, let μ^T be the empirical measure on $\mathbb{R}^2$ in polar coordinates defined by

$$\mu^T(s, v) = \frac{1}{\log T}\sum_{x\in\mathbb{Z}^2}\frac{\eta_s(x)}{|x|^2}\delta_{\log|x|/\log T,\Theta(x)}(v),$$

where $v = (r,\theta) = (|v|, \Theta(v))$, δ_v is the Dirac measure concentrated on v, and $\Theta(v)$ is the argument of v. Note that if $H: \mathbb{R}_+ \to \mathbb{R}$ depending only on the radius r then

$$\begin{aligned}\langle\langle H, \mu^T\rangle\rangle &\equiv \int_0^\infty \int_{-\pi}^{\pi} H(r)\mu^T(dv) \\ &= \frac{1}{\log T}\sum_{x\in\mathbb{Z}^2} H(\sigma_T(x))\frac{1}{|x|^2}\eta(x).\end{aligned}$$

Recall from (3.1) the function $\Gamma = \Gamma_\gamma$ associated to any $\gamma \in C^2_\alpha(\mathbb{R}_+)$. For f in $\mathcal{M}_1$ and γ in $C^2_\alpha(\mathbb{R}_+)$, let $l(f;\gamma)$ be the bilinear functional defined by

$$l(f;\gamma) = -\langle f, \partial^2\Gamma_\gamma\rangle$$

and let $I_\alpha: \mathcal{M}_1 \to \mathbb{R}_+$ be the rate function defined by

$$I_\alpha(f) = \pi \sup_{\gamma\in C^2_\alpha(\mathbb{R}_+)} \Big\{ l(f;\gamma) - \langle f(1-f), (\partial\Gamma_\gamma)^2\rangle \Big\}.$$

Denote by $A\mu^T \in \mathcal{M}_1$ the average of μ^T over the angle and the unit time interval, namely

$$(A\mu^T)(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \int_0^1 ds\mu^T(s;r,\theta).$$

Here we state a basic super-exponential estimate which will be needed later. In fact, such estimates in much more complicated forms are needed in deriving the large deviations upper and lower bounds of $A\mu^T$.

Theorem 3.1 (super-exponential estimate) *For any $\delta > 0$ and $t > 0$,*

$$\limsup_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{\log T}{T} \log \mathbb{P}_\alpha\Big[\Big|\int_0^t ds W(\eta_s, T^\varepsilon)\Big| > \delta\Big] = -\infty, \tag{3.2}$$

where

$$W(\eta_s, T^\varepsilon) = \eta_s(0) - \frac{1}{|B(T^\varepsilon)|}\sum_{y\in B(T^\varepsilon)} \eta_s(y),$$

and $B(r)$ is the ball of radius r centered at the origin.

We establish the large deviations estimate for $A\mu^T$ first. Below is the large deviations upper bound result.

Theorem 3.2 *For any closed set C of $\mathcal{M}_1$,*

$$\limsup_{T\to\infty}\frac{\log T}{T}\log \mathbb{P}_\alpha[A\mu^T \in C] \le -\inf_{f\in C} I_\alpha(f).$$

In fact, $I_\alpha(f)$ is precisely computable for smooth f, as stated in the following lemma.

Lemma 3.3

$$I_\alpha(f) = \frac{\pi}{4}\int_0^{1/2} \frac{f'(r)^2\, dr}{f(r)[1-f(r)]},$$

where the right hand side is ∞ whenever it does not exist or $f(1/2) \neq \alpha$.

With the help of the above lemma, we can derive the large deviations lower bound of $A\mu^T$.

Theorem 3.4 *For any open neighborhood O of a given $\rho \in M_1$,*

$$\limsup_{T\to\infty}\frac{\log T}{T}\log \mathbb{P}_\alpha(A\mu^T \in O) \ge -I_\alpha(\rho).$$

Once having the large deviation result of $A\mu^T$, we may apply the contraction principle to deduce the large deviations estimate for the occupation time of the origin $V_T = T^{-1}\int_0^T \eta_s(0)\, ds$.

Theorem 3.5 *Consider the two dimensional symmetric simple exclusion process $\mathbb{P}_{\alpha,T}$ as $T \to \infty$. The occupation time V_T obeys a large deviation principle of order $T/\log T$, and the rate function Υ is given by*

$$\Upsilon(\beta) = \frac{\pi}{4}\Big[\sin^{-1}(2\beta-1) - \sin^{-1}(2\alpha-1)\Big]^2,$$

which is ∞ when $\beta < 0$, or $\beta > 1$.

Proof: This follows from Theorem 3.1, the contraction principle, and the explicit calculation that

$$\inf_{\gamma\in C^2_\alpha(\mathbb{R}_+),\gamma(0)=\beta} I(\gamma) = \Upsilon(\beta).$$

□

四、計畫成果自評

The large deviation results obtained in this project are, in my opinion, quite nice and the paper [1] will be submitted to the best journal.

五、參考文獻

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附錄

國外差旅心得報告

在國科會的補助下，本人於三月三十日出發前往馬里蘭大學訪問李宗祐教授。三月三十一日抵達華盛頓機場時，李宗祐教授已在機場迎接，盛情令人感激。

週末我們先就這段期間彼此想過的、做過的計算、先檢討一番，以確定大致討論的方向。雖然馬里蘭大學並未放春假，但接下來的一週，在李宗祐教授上課之外，我們皆密集地討論。一方面我們就進行中的論文，尤其是 super-exponential estimates 之中的細節加以補充、確認。主要的困難是 2 維格子點空間中，建構 2 個格子點之間的路徑時，自然要使用 L1-norm，但 super-exponential estimates 中卻使用的是 L2-norm。在兩者的銜接上我們花了許多心力。此外我們研讀了一些論文，主要是爲後續的研究問題做些準備。總結一週的討論令我獲益匪淺。

四月八日本人離開馬里蘭大學，由華盛頓經紐約轉機返回台灣。