行政院國家科學委員會補助專題研究計畫成果報告

※※※※※※※※※※※※※※※※※※※※※※※※※

※ 具有某些自相似性之隨機過程的 ※

※ 樣本函數解析（IV） ※

※ ※

※※※※※※※※※※※※※※※※※※※※※※※※※

計畫類別：☑個別型計畫 □整合型計畫

計畫編號：NSC89－2115－M－002－023

執行期間：89 年 08 月 01 日至 90 年 07 月 31 日

計畫主持人：謝南瑞

共同主持人：

本成果報告包括以下應繳交之附件：

☑赴國外出差或研習心得報告一份

□赴大陸地區出差或研習心得報告一份

☑出席國際學術會議心得報告及發表之論文各一份

□國際合作研究計畫國外研究報告書一份

執行單位：台大數學系

中 華 民 國 90 年 10 月 6 日

行政院國家科學委員會專題研究計畫成果報告

具有某些自相似性之隨機過程的樣本函數解析（IV）

計畫編號：NSC89－2115－M－002－023

執行期限：89 年 08 月 01 日至 90 年 07 月 31 日

主持人：謝南瑞　執行機構及單位名稱：台大數學系

一、中文摘要

這是同一研究主題第四年的精簡報告。

關鍵詞：分生過程、布朗運動，多重碎形，豪氏維度。

Abstract

This is a brief report of the fourth year of a same research subject.

Keywords: braching processes,Brownian motions, multifractals, Hausdorff dimensions.

二、本年進度

在本年中，我研究分生過程與布朗運動的樣本函數。我考慮〝超臨界〞的 Galton-Watson 分生過程，相對應的樹枝結構有一個〝極限值〞，在此極限集上有一個自然的分生測度。我也考慮布朗運動的樣本函數，在任一水平集與軌跡上，都有自然的測度存在，水平集上者為局部時測度，軌跡上者為佔據測度。我們討論上述諸測度的多重碎形性質。

三、成果自評

對分生測度而言，我們將先前與 S. J. Taylor 所共同獲得之結果（見前一年之成果報告）中，所說的豪氏維度下界的證明，加以更新，原先的論證有若干瑕疵。對布朗運動的碎形性質，我寫下一篇相關論文，並獲邀在維瓦克李維過程國際會議（見附件）宣讀。

附件 1：赴國外出差心得報告

我於 2000 年 12 月 4~9 日赴香港中文大學數學研究所訪問，且出席由彼主辦之碎形與動力系研討會。中文大學數學講座教授 Ka-Sing Lau 為碎形與迭代函數系的專家。此次主辦之研討會，也吸引來自美國與歐洲的相關學者與會。來自大陸之學者約有 20 名。我個人報告了一個進行中之研究，討論迭代函數系與分生過程之關聯。這與一些與會者之興趣相符，故與美國的 D. Mouldin 與德國的 N. Patzschke 就此多所討論。對會中有一北大學者陳培東的隨機動力系的報告，亦感到有所受益。另外來自法國的范愛華教授，也報告了有關樹枝結構上的位勢理論，這與我目前作 GWT 之研究相合。本已安排好他在今（2001）年來台北訪問十天，但因他現在甫獲選為大陸的"長江學者"，在武漢大學有重要學術任務，就不能來台了，甚為可惜。

附件 2：出席國際會議心得報告

（附發表之論文）

我於 2001 年 4 月 1~7 日赴英國維瓦克大學數學研究所出席李維過程國際會議。此會議係由 EU 所支助的會議，由英國與法國之相關學者組織，有約 80 人與會。主要會議為 50 分鐘專題演講與 25 分鐘之成果報告，個人以 Some fractal properties of Brownian Paths（見所附論文）為題，作 25 分鐘之報告。對此，並與相關參加者 S. J. Taylor，P. Morter，Y. Xiao 等人多所討論。有關約 50 分鐘專題演講，主要有 S. J. Taylor 作李維過程的歷史回顧（open lecture），J. F. LeGall 李維過程與分生過程之關聯，Barndorff-Nielson 有關李維過程在經濟與紊流上之應用，以及M. Maejima 有關自相似過程之綜合介紹。對上述諸演講，在與演講者討論後，個人對其發表之主題，能有較深入之了解。

Some Fractal Properties of Brownian Paths

Narn–Rueih SHIEH
Department of Mathematics, National Taiwan University
Taipei, Taiwan
E–mail: shiehnr@math.ntu.edu.tw

Abstract

In this paper, we survey some recent results concerning the fractal structure of Brownian sample paths. The following aspects are discussed:
(1) Average densities of Brownian trails and intersections.
(2) Dimension spectra of Brownian zeroes.
(3) Multifractal properties of Brownian substitutions.

Running Title: As the title
AMS Subject Classifications(1991): 60G17
Key words:
Brownian motion, self–similarity, Hausdorff dimension, occupation measure, local time measure, intersection measure, average densities, dimension spectra, multifractals.

1 Introductions

Brownian motion is referred to highly irregular motion proceeded by a small particle in some medium, which was firstly observed by British botanist Robert Brown. The mathematical formulation is due to Norbert Wiener, which we describe as follows. Let $B_d = \{B_d(t)\}$ be a stochastic process defined on some probability space (Ω, P) and taking values in $\mathbb{R}^d$. We say that B_d is the *standard d-dimensinal Brownian motion* if it satisfies the following:
(i) the sample path $t \to B_d(t,\omega), \omega \in \Omega$, is continuous;
(ii) the increments $B_d(t_j) - B_d(t_{j-1})$, $1 \le j \le k$, $t_0 = 0$, are stationary and (stochastically) independent for all $k \geq 2$;

(iii) The distribution of $\frac{B_d(t)-B_d(0)}{\sqrt{t}}$ is standard normal in $\mathbb{R}^d$, and
(iv)$B_d(0) = 0$.

The Browian motion process(also named *Wiener process*) described above is *(stochastically) self-similar of index* 1/2 by which it means that, for any $c > 0$, the time-scaled process $\{B_d(ct)\}$ and the space-scaled process $\{\sqrt{c}B_d(t)\}$ are equivalent in the sense of finite-dimensionally distributional equivalence. This self-similar property is central in our study, from which various dimension formulae concerning Brownain paths can be figured out.

Brownian sample paths exhibit highly erratic patterns, despite the continuity; we can appreciate such pictures from many books on stochastic processes. Thus, this should be a rich source of fractal analysis/geometry and be one prevailing topic in nonlinearity(even though the transition density of the process is just heat kernel). We concern with the following fractals

$$\begin{aligned} [B_d] &= \{x: \ x = B_d(t) \ \text{ some } \ t\}, \\ Z &= \{t: \ B_1(t) = 0\}, \\ I_d &= [B_d] \cap [B'_d]. \end{aligned}$$

In the above, B'_d denotes an independent copy of B_d. Thus, the three sets are simply the trail(range), the zero set, and the set of intersections of Brownian paths. Note that the zero set is meaningful only for the 1-dimensinal case, while the trail and the intersection are meaningful only for the multi-dimensional case. These are due to the fact that the 1-dimensional Brownian motion is point-recurrent while it is not so for the multi-dimensional case. These sets are random, since they depend on a particular sample path realization $B_d(t,\omega)$, and so we must interpret any statement about these sets and their associated measures(random, too) as being true "with probability one". Let $\dim K$ denote the Hausdorff dimension of a Borel K. The following results are well-known:

$$\begin{aligned} d &= 1, \quad \dim Z = 1/2, \\ d &\geq 2, \quad \dim[B_d] = 2, \\ d &= 2,3, \quad \dim I_d = d - 2(d-2). \end{aligned}$$

We refer to Taylor[T] for a convenient reference on the theory of random fractals arising from the sample paths of stochastic processes, in which the detailed definitions and properties of Hausdorff and other dimension indices are described.

Nowadays the fractal analysis of measures rather than sets has been focused. In our concern, there are natural measures associated with the above fractals $Z, [B_d]$ and I_d ; they are respectively Brownian *local time measure*, *occupation measure* and *intersection measure*. These measures are regarded as *fractal measures*, since each of them is singularly continuous(non–atomic and supported by a set of Lebesgue measure zero) and exhibits a certain self–similarity which is inherited from the self–similarity of the process. The main difference(and difficulty) from pure analysis is that the self–similarity is now always in the stochastical sense rather than the strict(analytic) sense. Therefore, even though the assertions of the theorems are formulated in almost sure statements, we cannot proceed the reasoning just in an analytic(pathwise) way; instead we have to deliberate our arugments by combining both analysis and probability.

The rest of this paper is devided into three sections; each one is devoted to one aspect of Brownian fractal structures. The paper is concise; we merely state the main results together with some explanations on the main idea behind proofs.

2 Average densities

Let μ be a locally finite regular Borel measure in $\mathbb{R}^d$, which is fractal in the sense described in the previous section. Let $\phi(r)$ be a suitable gauge function. Bedford–Fisher[BF] introduced the concept of *average density* of μ, as an analogue of Lebesgue density. The latter one fails to exist for many μ, in view of a famous result of J.M. Marstrand. The average density of *order two* and *order three* of μ at x, w.r.t. ϕ, are defined respectively as

$$\begin{aligned} AD_2(\mu, x) &= \lim_{\epsilon \downarrow 0} \frac{1}{\log(1/\epsilon)} \int_\epsilon^1 \frac{\mu(B(x,r))}{\phi(r)} \frac{dr}{r}, \\ AD_3(\mu, x) &= \lim_{\epsilon \downarrow 0} \frac{1}{\log\log(1/\epsilon)} \int_\epsilon^{1/e} \frac{\mu(B(x,r))}{\phi(r)} \frac{dr}{r \log(1/r)}, \end{aligned}$$

where $B(x,r)$ denotes the closed ball with center x and radius r. Bedford–Fisher introduced and discussed the above definition via the classical summation techniques of Hardy and Riesz. They also discussed some specific cases of the measures associated with middle-third Cantor set, cookie–cutter Cantor set, and Brownian zeroes. Falconer–Xiao[FX] adapted the arguments to prove, among the results, the AD_2 for the occupation measure of B_d, $d \geq 3$, which is the measure $\mu = \mu(\omega)$ defined by

$$\mu(A) = \text{Leb}\,\{t : \quad B_d(t) \in A\}, \quad A \subset \mathbb{R}^d;$$

note that μ is supported by $[B_d]$.

Theorem 2.1 *For Brownian occupation measure μ in $\mathbb{R}^d$, $d \geq 3$, the AD_2 of μ w.r.t. $\phi(r) = r^2$ exists at $\mu - a.e.$ x, and its value is a constant (depending only on the dimension d) multiple of the mean sojourn time of the path in the unit ball.*

Note that the value in the above theorem is non-random(macroscopic, in physical terminology) while the measure is random(microscopic). We also remark that the sojourn time in the above theorem is infinite for the planar Brownian motion, since B_2 is neighborhood-recurrent. Then Mörters[M] proved that the AD_2 of B_2 does not not exist while the AD_3 does exist.

Theorem 2.2 *For planar Brownian occupation measure, the AD_3 of μ w.r.t. $\phi(r) = r^2 \log(1/r)$ exists at $\mu - a.e.$ x, and its value is 2. Moreover, the AD_2 of μ now fails to exist.*

Shieh[S1] considered the average density problem of the intersection of two spatial Brownian motions and proved a partial result. The problem is completely solved, both for the planar and the spatial cases, in Mörters–Shieh[MS]. Let B_d, B'_d be two independent Brownian motions in $\mathbb{R}^d, d = 2, 3$, then the intersection measure is a canonical (random) Borel measure supported by $I_d = [B_d] \cap [B'_d]$ which is expressed heuristically as

$$\mu(A) = \int_A \int_t \int_{t'} \delta_x(B_d(t))\delta_x(B'_d(t'))dtdt'dx;$$

we remark that the rigorous definition of the above measure is somewhat involved; see [MS] for details(Note that we have used a shorter term here rather than a longer one in [MS] for the terminology). It is proved in [MS] that

Theorem 2.3 *Let μ now be the intetsection measure mentioned above. For the spatial case the AD_2 of μ w.r.t. $\phi(r) = r$ exists at $\mu - a.e.$ x, and for the planar case the AD_3 of μ w.r.t. $\phi(r) = r^2 \log^2(1/r)$ esists at $\mu - a.e.$ x. The value in the both cases is $4/\pi$. Moreover, the AD_2 in the planar case again fails to exist.*

The proofs of these results break into two parts; to prove the existence of the density and evaluate its value for a typical point, say $x = 0$, and then prove the results for generic x. To prove the reduction from the generic to the typical, in the case of Theorems 2.1 and 2.2 we can proceed some standard application of Markov property. However, it is not so easy for the intersection case Theorem 2.3, since the t, t' for which $B_d(t) = B'_d(t')$ cannot be realised as stopping times. This difficulty is overcomed, by either using a device of Le Gall on "Brownian loops" or by a more analytic approach based on Palm distributions. To prove the order two case for $x = 0$, we apply Birkoff's ergodic theorem to some suitable scaling flow defined on the space of continuous functions, which is naturally associated with Brownian motions. However this scaling approach does not work for the order three case, since the gauge function ϕ now has a slow varying term involved; the difficulty is overcomed by using a certain crossing-number argument for the path oscillating between the annulus with suitablly chosen small radii. In the planar case we can consider the multiple intersections rather than just the double intersections shown in the above theorem.

3 Dimension spectra

Study on the dimension spectra associated with various fractal measures is now the main concern in fractal analysis/geometry. We refer to Falconer[F, Chapeter 11] for a convenient reference of such *multifractal analysis*, in which some physical background and some detailed notions and properties are described. In particular, it is defined the *upper local dimension*, the *lower local dimension* and the *local dimension* for a locally finite Borel measure μ at a point x. We denote them by $\overline{d}(\mu, x)$, $\underline{d}(\mu, x)$, and $d(\mu, x)$. The definitions are

$$\begin{aligned} \overline{d}(\mu, x) &= \limsup_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log r}; \\ \underline{d}(\mu, x) &= \liminf_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log r}; \\ d(\mu, x) &= \underline{d}(\mu, x) = \overline{d}(\mu, x). \end{aligned}$$

The above definition is a mathematical view of the physical concern of non-uniform local mass concentrations which appears in the case such as oil deposits on a specific region. By (fine) multifractal analysis of μ, we seek for an $f(\alpha)$ curve describing the Hausdorff dimension of the "level set" $d(\mu, \cdot) = \alpha$(or that for $\overline{d}, \underline{d}$), where α is in some range $[\alpha_{\min}, \alpha_{\max}]$. For Brownian occupation measure, its dimension spectra, that is $f(\alpha)$ curve, is trivial, in view of a uniform dimension theorem of Perkins–Taylor. Remark: However, recent works of Dembo–Peres–Rosen–Zeitouni show that it does have non-trivial spectrum for "thick" occupations. For Brownian local time, the situation is completely different. We recall that local time is a canonical measure supported by the Brownian zero set Z(the 1-dimensional case only) which is expressed heuristically as

$$\mu(A) = \int_A \delta_0(B(t))dt, \quad A \subset \mathbb{R}.$$

Hu–Taylor[HT] proved that there exists non–trivial spectrum for the upper $\overline{d}(\mu)$, yet the lower $\underline{d}(\mu)$ has only trivial spectrum. Then Shieh–Taylor[ST1] continued the study to show that, in the latter case, we do have a non–trivial spectrum in which logarithmic order of magnitude plays a crucial role. Since Brownian local time can be viewed as the occupation measure for an 1/2–stable subordinator, they proved the results in terms of any stable subordinator. We state their results only for Brownain local time.

Theorem 3.1 *Let μ be Brownian local time measure. For $\alpha \geq 1/2$, put*

$$A_\alpha = \{t : \overline{d}(\mu, t) = \alpha\}.$$

Then

$$\dim A_\alpha = 1/2\alpha - 1/2, \quad 1/2 \leq \alpha \leq 1,$$

while $A_\alpha = \emptyset$ if α is outside $[1/2, 1]$.
Moreover, the spectrum for the $\underline{d}(\mu)$ is trivial; there is only one value 1/2 happened at $\alpha = 1/2$.

Theorem 3.2 *Let μ be Brownian local time measure. For $\alpha \geq 0$, put*

$$B_\alpha = \{t : \limsup_{r \downarrow 0} \frac{\mu(t-r, t+r)}{c\sqrt{r \log(1/r)}} = \alpha\},$$

where the specified constant $c = 2$. Then

$$\dim B_\alpha = (1/2)(1 - \alpha^2), \quad 0 \leq \alpha \leq 1,$$

while $B_\alpha = \emptyset$ if α is outside $[0, 1]$.

The proofs of these results again break into two parts. The easier part is the upper bound estimate for the dimension, in which we apply some Markov property to a certain particular cover of the concerned set. The difficult part is the lower bound estimate, we need to construdct a certain Cantor–like random set contained in A_α, B_α by which there is some measure ν supported. Moreover we need to estimate the energy integral of this ν with respect to some potential kernel. It was pointed out that there is a computational error in the lower bound proof; the corrections and addenda are taken in Shieh–Taylor[ST2]. Furthermore, it is studied in Shieh–Taylor[ST3] that the same scenario can hold for the branching measure on a Galton–Watson tree.

We should remark that Theorems 3.1 and 3.2 are very different from "standard" theory of multifractal analysis. The latter one is mainly based on a certain thermodynamical formalism. Whenever the formalism is indeed true, as it is the case of some self-similar measure associated with an interated function system, there is no distinction for the upper and the lower local dimensions, that is the assertion is true for $d(\mu, x)$, and there is usually no logarithmic factor involved. Thus, it seems that Theorems 3.1 and 3.2 are specific to the effect of random fluctuations.

4 Brownian multifractals

Brownian paths have rich fractal structures, as we have seen from the previous two sections. However, the path is usually qualified as a monofractal, in view that the Hölder exponent of the path is everywhere 1/2(the variations of the regularity are only of a logarithmic order of magnitude). Thus, it is not perfect to use Brownian path as a curve fitting to those data exhibiting the intermittence. The latter one is very important for the study of, say, turbulences. Mandelbrot introduced the concept of Brownian multifractals in his works on finance theory, and a proposed mathematical theory is proceeded recently by Riedi[R]. Let $B(t)$ be a real–valued Brownian motion(or a fractional Brownian motion, if one likes to count the long range dependence), and let $M(t)$ be an increasing process(that is a process which is pathwise increasing in t). Assume that B and M are totally independent(quite rough from the viewpoint of practical applications). The composite $t \to B(M(t))$ is termed *Brownian motion in multifractal time.* The path of the new process indeed has some multifractal(=imtermittent) structure and some dimension spectrum can be computed. In case that M is a subordinator, then the resulting process is a Lévy process. This case is also known in probability as *Brownian (time) substitution.* We recall that a Lévy process is a stochastic process(real–valued or vector–valued) with stationary and independent increments, and that a subordinator is a real–valued Lévy process with increasing paths. Jaffard[J] proved that the paths of "most" Lévy processes are multifractals and he also determined their spectrum of Hölder exponents.

We specify the general works of Riedi and Jaffard as follows. Let $B_d(t) = (B^1(t), \cdots, B^d(t))$ be d-dimensional Brownian motion and let $\theta^j(t), 1 \leq j \leq d$, be d stable subordinators with stability index $\beta_j, 0 < \beta_j < 1$, that is the process $\theta^j(t)$ is such that it is β_j-stable distributed for all t. Then we have the composed process $X(t) = (B^1(\theta^1(t)), \cdots, B^d(\theta^d(t))$. It is Lévy whenever $\theta^j, 1 \leq j \leq d$, are totally independent on B_d. We have the following three cases to consider.

1° That $\beta_j = \beta, \forall j$, and $(\theta^1, \cdots, \theta^d)$ is a d-dimensional β-stable process. Then X is a d-dimensional 2β-stable Lévy process(Note that β is necessarily < 1).

2° That θ^j among themselves are independent. Then X is of independent components. Such process have been termed as a *process with stable components* in Pruitt–Taylor[PT].

3° That we go beyond the first two cases by only assuming that X is a *d–dimensional self–similar* process with a vector ss index $H = (H_1, \cdots, H_d)$. See Shieh[S2] for the precise definition, where the term *dilation–stable* Lévy process is used. Let γ_t be a diagonal transformation in $\mathbb{R}^d$ of which diagonal entries are $t^{H_1}, \cdots, t^{H_d}$. Then the stochastic structure of a dilation–stable Lévy porcess is determined by the following theorem, which can be seen in Hudson–Mason[HM].

Theorem 4.1 *The characteristic function of $X(1)$ of the above dilation–stable Lévy process X is determined by*

$$E\exp(i(z, X(1))) = \exp\Big[\int_0^\infty \int_S (\exp(i(z, \gamma_r x)) - 1 - i(z, \gamma_r x)1_D(\gamma_r x))\frac{\lambda(dx)dr}{r^2}\Big],$$

where $\lambda(dx)$ is a finite Borel measure on $S = \{x : |x| = 1\}$ and $D = \{x : |x| \leq 1\}$. Moreover X is of independent components if and only if the mesaure $\lambda(dx)$ is concentrated on the coordinate axes.

Multi–dimensional stable Lévy processes are non–Gaussian counterpart of the Gaussian, that is Brownian, case and have been well studied. Processes with stable components arise from the study of the collisions of two independent stable processes and they are also good examples for showing some significant gaps between the stable and the general Lévy processes. Dilation–stable Lévy processes arise from the study of stochastic flows in which the independent–components assumption is too strong for the purpose. Shieh[S2] proved a dimension formula for the multiple points of dilation–stable Lévy processes; the work shows that for such processes we need to proceed some considerations more complicated than the Brownian and the stable cases.

References

[BF] Bedford, T. and Fisher, A.M.(1992). *Analogues of the Lebesgue density theorem for fractal sets of reals and integers.* Proc. London Math. Soc. **64** 95-124.

[F] Falconer, K.(1997). *Techniques in Fractal Geometry.* Wiley.

[FX] Falconer, K. and Xiao, Y.M.(1995). *Average densities of the image and zero set of stable processes.* Stoch. Proc. Appl. **55** 271-283.

[HM] Hudson, W.N. and Manson, J.D.(1981). *Operator–stable laws.* J. Multivariate Anal. **11** 434-447.

[HT] Hu, X.Y. and Taylor, S.J.(1997). *The multifractal spectrum of stable occupation measure.* Stoch. Proc. Appl. **66** 283-299.

[J] Jaffard, S.(1999). *The multifractal nature of Lévy processes.* Prob. Theory Rel. Fields **114** 207-227.

[M] Mörters, P.(1998). *The average density of the path of planar Brownian motion.* Stoch. Proc. Appl. **74** 133-149.

[MS] Mörters, P. and Shieh, N.R.(1999) *Small scale limit theorems for the intersection local times of Brownian motion.* Electronic J. Prob. **4** paper no. 9, 1-23.

[PT] Pruitt, W.E. and Taylor, S.J.(1969). *Sample paths properties of processes with stable components.* Z. Wahrsch. Verw. Geb. **12** 267-289.

[R] Riedi, R.H.(1999). *Multifractal Processes.* Tech. Report, Rice Univ. TR 99-06.

[S1] Shieh, N.R.(1997). *A growth condition for Brownian intersection points.* In: Trends in Probability Theory and Related Analysis, Proceedings of SAP'96(ed. N. Kono and N.R. Shieh) 265-272. World Scientific.

[S2] Shieh, N.R.(1998). *Multiple points of dilation–stable Lévy processes.* Ann. Probab. **26** 1341-1355.

[ST1] Shieh, N.R. and Taylor, S.J.(1998). *Logarithmic multifractal spectrum of stable occupation measure.* Stoch. Proc. Appl. **75** 249-261.

[ST2] Shieh, N.R. and Taylor, S.J.(1999). *Method of Calculating the Dimension Spectrum of a Borel measure when the multifractal formalism fails.* In: Trends in Probability Theory and Related Analysis, Proceedings of SAP'98(ed. N. Kono and N.R. Shieh). World Scientific.

[ST3] Shieh, N.R. and Taylor, S.J.(1999). *Multifractal Spectra of branching measure on a Galton–Watson tree.* In preparation.

[T] Taylor, S.J.(1986). *The measure theory of random fractals.* Math. Proc. Camb.Phi. Soc. **100** 383-406.