行政院國家科學委員會專題研究計畫成果報告

計畫名稱: Existence of Mean Field Type Equations

計畫編號: NSC 89-2115-M-002-027
執行期限： 89 年 8 月 1 日至 90 年 7 月 31 日
主持人:　陳俊全
執行機構及單位名稱:國立台灣大學 數學系

1 Main Results

In this project, we study the following nonlinear elliptic equation on a compact Riemann surface (M, ds).

$$\Delta_0 u + \rho \left(\frac{h(x)e^u}{\int_M h(x)e^u d\mu} - 1 \right) = 0 \quad \text{on} \quad M, \tag{1.1}$$

where $h(x)$ is a positive $C^{2,\beta}$ function on M, ρ a positive constant, $d\mu$ the volume form and Δ_0 stands for the Beltrami-Laplace operator on (M, ds). Throughout the paper, we normalize the volume $|M|$ of M by

$$|M| = 1. \tag{1.2}$$

Equation (1.1) or its variants often appear in many different disciplines of mathematics. In the conformal geometry, (1.1) is called the Nirenberg problem when (M, ds) is the standard sphere S^2 and $\rho = 8\pi$, or Kazdan-Warner problem in general. The Nirenberg's problem, a subject under extensive study in recent years, is to determine which function $h(x)$ on S^2 can be the Gaussian curvature of a metric which is pointwisely conformal to the standard metric of S^2. For the recent progress made on this problem, see Moser [29], Kazdan and Warner [22], Chang and Yang [8], Chang, Gursky and Yang [9], Chen and Li [14], Cheng and Lin [15, 16] and the references therein. For spheres or bounded domains of $\mathbf{R}^2$, (1.1) can be obtained from the mean field limit of point vortices of Euler flows or spherical Onsager vortex theory, as studied in Caglioti, Lions, Marchioro and Pulvirenti [4, 5], Kiessling [23], Chanillo and Kiessling [6], and the references therein. Recently, it has drawn a lot of attention because it also arises from self-dual condensate solutions from some Chern-Simons-Higgs model when some parameter tends to zero. For recent developments of these subjects or related Liouville systems in more general settings, we refer the readers to Spruck and Yang [36], Caffarelli and Yang [3], Chanillo and Kiessling [6], Chipot, Shafrir and Wolansky [17], Tarantello [39], Nolasco and Tarantello [32, 33], Riciardi and Taranetto [34, 35], Struwe and Tarantello [37], Ding, Jost, Li and Wang [19, 21], Chen and Lin [11], Lin [26], and the references therein.

Clearly, (1.1) is the Euler-Lagrange equation of the nonlinear functional J_ρ

$$J_\rho(\phi) = \frac{1}{2} \int_M |\nabla \phi|^2 d\mu - \rho \log \left(\int_M h e^\phi d\mu \right)$$

for $\phi \in H^1(M)$, where $H^1(M)$ denotes the Sobolev space of L^2 functions with L^2-integrable first derivatives. For $\rho < 8\pi$, $J_\rho(\phi)$ is bounded from below and the infinimum of $J_\rho(\phi)$ can be achieved by the well-known inequality due to Moser and Trudinger. For $\rho \geq 8\pi$, however, the existence problem of (1.1) is more difficult. By using some special variational scheme, Struwe and Tarantello [37] was able to obtain non-trivial solutions of (1.1) for $8\pi < \rho < 4\pi^2$ when $h \equiv 1$ and M is the flat torus with the fundamental domain $[0,1] \times [0,1]$. Also by using the similar approach, Ding, Jost, Li and Wang [20] proved the existence of solutions to (1.1) for $8\pi < \rho < 16\pi$ when M is a compacat Riemann surface with genus $g \geq 1$. For the case $M = S^2$ and $8\pi < \rho < 16\pi$, the second author [26] proved nonvanishing of the Leray-Schauder degree to equation (1.1), and consequently, the existence of solutions follows for the case of genus 0. In spite of the success for solving (1.1) in the range of $(8\pi, 16\pi)$, the existence problem generally remains open for $\rho > 16\pi$. This is the main issue we are going to address in this paper and the subsequent one [13].

In [24], Y.Y. Li initiated to study the existence of solutions to (1.1) by way of computing the Leray-Schauder topological degree. Obviously, equation (1.1) is invariant under adding a constant. Hence, we can seek solutions in the class of functions which are normalized by

$$\int_M u(x) d\mu = 0. \tag{1.3}$$

Li proved, among other things, that for any integer $m \geq 0$ and for any compact set I in $(8m\pi, 8(m+1)\pi)$, solutions of (1.1) which are normalized by (1.3), are uniformly bounded for any positive C^1 function h and $\rho \in I$. Thus, the Leray-Schauder degree $d(\rho)$ of (1.1) can be defined in the space of functions with vanishing mean value for $\rho \neq 8\pi m$. Furthermore, he proved that $d(\rho)$ is independent of the function $h(x)$ and the parameter ρ whenever $\rho \in (8m\pi, 8(m+1)\pi)$, and showed that $d(\rho) = 1$ for $\rho \in (0, 8\pi)$. The main purpose of this and the subsequent project is to complete, among other things, the following theorem.

Theorem A. *Let $8m\pi < \rho < 8(m+1)\pi$ and $d(\rho)$ be the Leray-Schauder* *degree for equation (1.1). Then*

$$d(\rho) = \begin{cases} \frac{1}{m!}(-\chi(M)+1)\dots(-\chi(M)+m) & \text{for } m > 0 \\ 1 & \text{for } m = 0, \end{cases}$$

where $\chi(M)$ is the Euler characteristic of M.

As a consequence of Theorem A, equation (1.1) always posseses a solution for $\rho \neq 8m\pi$ whenever the Euler characteristic $\chi(M) \leq 0$. The complete proof of Theorem A will be given in [13], the second part of this series of papers.

Set $d_m^+ = d(\rho)$ as $\rho \downarrow 8m\pi$ and $d_m^- = d(\rho)$ as $\rho \uparrow 8m\pi$. One of the main steps in the proof of Theorem A is to calculate the gap $d_m^+ - d_m^-$ for any integer $m \geq 1$. Once it is known, $d(\rho)$ can be computed inductively on m. Clearly, the gap of $d_m^+ - d_m^-$ is due to the occurrence of blowup solutions when ρ tends to $8m\pi$, that is, there are a sequence of solutions u_k of (1.1) and (1.3) with $\rho = \rho_k$ such that $\max\limits_M u_k \to +\infty$, and ρ_k tends to $8m\pi$. Thus, one of fundamentally important questions is to determine the sign of $\rho_k - 8m\pi$. Our main result of this article is to answer this question. After adding a constant c, we consider a sequence of blowup solutions u_k (still denoted by u_k) of with $\rho = \rho_k$ and $\lim\limits_{k\to+\infty} \rho_k = 8m\pi$. Then by a result of Y.Y. Li, u_k blows up at exact m points $\{p_1, \dots, p_m\}$. Let δ_0 be a small positive number such that the distance $d(p_j, p_l) > 4\delta_0$ for any $j \neq l$. Set $p_{k,j}$ to be the local maximum point of u_k near p_j,

$$\lambda_{k,j} = u_k(p_{k,j}) = \max_{\bar{B}(p_{k,j},\delta_0)} u_k(x) \tag{1.4}$$

Then our main theorem is the following.

Theorem 1.1. *Let h be a positive C^2 function on M and u_k be a sequence of blowup solutions with $\rho = \rho_k$. Assume $8m\pi = \lim\limits_{k\to+\infty} \rho_k$. Then*

$$\begin{aligned} \rho_k - 8m\pi &= \frac{2}{m}\sum_{j=1}^{m} h^{-1}(p_{k,j})[(\Delta_0 \log h(p_{k,j}) + 8m\pi \\ &\quad -2K(p_{k,j})]\lambda_{k,j}e^{-\lambda_{k,j}} + O(e^{-\lambda_k}) \end{aligned} \tag{1.5}$$

where $\lambda_k = \max_{1 \le j \le m} \lambda_{k,j}$, $\lambda_{k,j}$ *is the local maximum of* u_k *in (1.4) and* K *denotes the Gaussian curvature.*

Clearly, Theorem 1.1 implies

Corollary 1.2. *Suppose* $h(x)$ *is a* C^2 *positive function and satisfies*

$$\Delta_0 \log h(x) + 8m\pi - 2K(x) > 0 \quad \text{for } x \in M. \tag{1.6}$$

Then for any compact interval $I \subset (8(m-1)\pi, 8m\pi]$, *there exists a constant* $C > 0$ *such that*

$$|u(x)| \le C \quad \text{for } x \in M \tag{1.7}$$

for any solution u *of (1.1) with* $\rho \in I$.

When $\rho < 8\pi$, the nonlinear function J_ρ has a minimizer by the Moser-Trudinger inequality. As $\rho \uparrow 8\pi$, Ding, Jost, Li and Wang [18] and Nolasco and Tarantello [31], independently, proved that minimizers u_ρ are uniformly bounded in M as $\rho \uparrow 8\pi$ provided that

$$\Delta_0 \log h(p) + 8\pi - 2K(p) > 0$$

for all maximum point p of h. However, their results do not have the explicit expression of (1.5) even for $m = 1$. Note that for the case of the standard sphere S^2, $\Delta_0 \log h(p) + 8\pi - 2K(p) = \Delta_0 \log h(p)$. Thus, their result can not be applied in this case with $h \equiv 1$. For S^2 and $\rho = 8\pi$, the uniform bound of (1.7) has been proved by Chang, Gursky and Yang [9].

Theorem 1.1 is the crucial result for us when we come to compute the Leray-Schauder degree. As mentioned before, the gap $d_m^+ - d_m^-$ is independent of h. Thus, we might choose some C^2 positive function h_m satisfying the condition (1.6) for all $x \in M$. Then for such h_m, by Theorem 1.1, $d(8m\pi)$ is also well-defined, and $d_m^- = d(8m\pi)$. Therefore, the gap $d_m^+ - d_m^-$ is equal to the sum of Morse index of all possible blowup solutions of (1.1) with the parameter ρ tending to $8m\pi$ from the above. Our job in the subsequent paper [13] is, first, to construct solutions with exact m blowup points and compute its Morse index, and secondly, we should be able to prove that all possible blowup solutions have been already constructed in our previous process. In order to complete the last step, we have to show a sharp estimate for the error term, which is the difference of solutions u_k and its approximation. For the nature of our problems here, we approximate our bubbling solution u_k by using different forms in two different regions of M. By a result of Li, u_k converges to a sum of Green's functions with their singularities at $p_1, \ldots, p_m \in M$. Choose a small $\delta_0 > 0$ and $\bar{B}_{\delta_0}(p_j)$ is the ball for radius δ_0 and center p_j in M. For $x \notin \bigcup_{j=1}^{m} \bar{B}_{\delta_0}(p_j)$, those Green's functions are smooth. Hence we approximate u_k by this smooth limiting function. Inside of each ball $\bar{B}_{\delta_0}(p_j)$, u_k should be approximated by the standard bubbles, although those bubbles should be carefully chosen. In $\bar{B}_{\delta_0}(p_j)$, we denote $\eta_{k,j}$ to be the error of u_k and the approximated bubble. Then it is very important to prove the following pointwise estimates for $\eta_{k,j}$.

Theorem 1.4. *Let $R_{k,j} = \sqrt{\frac{\rho_k h(p_{k,j})}{8} e^{\lambda_{k,j}}}$. Then the error term $\eta_{k,j}$ between the standard bubble and the solution satisfies*

$$\begin{aligned}\eta_{k,j}(x) &= -\frac{8}{\rho_k h(p_{k,j})}[\Delta \log h(p_{k,j}) + 8m\pi - 2K(p_{k,j})]e^{-\lambda_{k,j}}[\log(R_{k,j}|x|+2)]^2 \\ &\quad + O(\log(R_{k,j}|x|+2))e^{-\lambda_k}\end{aligned}$$

on $B_{\delta_0}(p_{k,j})$, where $|x|$ stands for the distance of x and $p_{k,j}$.

References

[1] H. Brezis & F. Merle, *Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions*, Comm. Partial Differential Equation 16(1991), 1223-1254.

[2] H. Brezis, Y.Y. Li and I. Shafrir, *A sup + inf inequality for some nonlinear elliptic equations involving exponential nonlinearities*, J. Functional Analysis 115(1993), 344-358.

[3] L. Caffarelli and Y. Yang, *Vortex condensation in the Chern-Simons Higgs model: an existence theorem*, Comm. Math. Phys. 168 (1995), 321-336.

[4] E. Caglioti, P.L. Lions, C. Marchioro, *A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description*, Comm. Math. Phys. 143 (1992), 501-525.

[5] E. Caglioti, P.L. Lions, C. Marchioro, and M. Pulvirenti, *A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description, part II*, Comm. Math. Phys. 174 (1995), 229-260.

[6] S. Chanillo and M. Kiessling, *Conformally invariant systems of nonlinear pde of Liouville type*, Geom. Func. Anal. 5 (1995), 924-947.

[7] S. Chanillo and M. Kiessling, *Rotational symmetry of solutions of some nonlinear problems in statistical mechanics and in geometry*, Comm. Math. Phys., 160 (1994), 217-238.

[8] S.Y. Chang and P. Yang, *Prescribing Gaussian curvature on S^2*, Acta Math. 159 (1987), 214-259.

[9] S.Y. Chang, M. Gursky and P. Yang, *The scalar curvature equation on* 2- *and* 3-*spheres*. Calc. Var. Partial Differential Equations 1 (1993), no. 2, 205–229.

[10] C.C. Chen and C.S. Lin, *A sharp sup+inf inequality for a nonlinear equation in* $\mathbf{R}^2$, Comm. Anal. Geom. 6 (1998), 1-19.

[11] C.C. Chen and C.S. Lin, *On the symmetry of blowup solutions to a mean field equation*, Ann. Inst. Poincare Anal. Nonlinéaire, to appear.

[12] C.C. Chen and C.S. Lin, *Singular limits of a nonlinear eigenvalue problem in two dimension*, preprint.

[13] C.C. Chen and C.S. Lin, *Topological degree for a mean field equation on Riemann surfaces*, preprint.

[14] W. Chen and C. Li, *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J., 63 (1991), 615-623.

[15] K.S. Cheng and C.S. Lin, *Conformal metrics with prescribed nonpositive Gaussian curvature on* R^2, Calc. Var. Partial Differential Equations 11 (2000), no. 2, 203–231.

[16] K.S. Cheng and C.S. Lin, *Multiple solutions of conformal metrics with negative total curvature*, Adv. Differential Equations 5 (2000), no. 10-12, 1253–1288.

[17] M. Chipot, I. Shafrir and G. Wolansky, *On the solutions of Liouville systems*, J. Diff. Eqs. Appl. 140 (1997), 59-105.

[18] W. Ding, J. Jost, J. Li and G. Wang, *The differential equaiton* $\Delta u = 8\pi - 8\pi h e^u$ *on a compact Riemann surface*, Asian J. Math. 1 (1997), 230-248.

[19] W. Ding, J. Jost, J. Li and G. Wang, *An analysis of two-vortex case in Chern-Simons Higgs model*, Calc. Var. and Partial Differential Equations 7 (1998), 87-97.

[20] W. Ding, J. Jost, J. Li and G. Wang, *Existence results for mean field equations*, Ann. Inst. H. Poincare Anal. Nonlinéaire 16 (1999), no. 5, 653–666.

[21] W. Ding, J. Jost, J. Li and G. Wang, *Multiplicity results for the two-vortex Chern-Simons Higgs model on the two-sphere.* Comment. Math. Helv. 74 (1999), no. 1, 118–142.

[22] J. Kazdan and F. Warner, *Curvature functions for compact 2-manifolds*, Ann. of Math. 99 (1974), 14-47

[23] M.H.K. Kiessling, *Statistical mechanics of classical particles with logarithmic interaction*, Comm. Pure Appl. Math. 46 (1993), 27-56.

[24] Y.Y. Li, *Harnack type inequality: the method of moving planes*, Comm. Math. Phy., 200(1999), 421-444.

[25] Y.Y. Li and I. Shafrir, *Blowup analysis for solutions* $-\Delta u = Ve^u$ *in dimension two*, Indiana Univ. Math. J. 43 (1994), 1255-1270.

[26] C.S. Lin, *Topological degree for mean field equations on* S^2, Duke Math. J. 104 (2000), no. 3, 501–536.

[27] C.S. Lin, *Uniqueness of solutions to the mean field equations for the spherical Onsager vortex*, Arch. Ration. Mech. Anal. 153 (2000), no. 2, 153–176.

[28] C.S. Lin, *Uniqueness of conformal metrics with prescribed total curvature in R^2*, Calc. Var. Partial Differential Equations 10 (2000), no. 4, 291–319.

[29] J. Moser, *On a nonlinear problem in differential geometry, in : Dynamic systems* (ed. : M. Peixote), Academic Press.

[30] K. Nagasaki and T. Suzuki, *Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearity*, Asymptotic Analysis 3(1990), 173-188.

[31] M. Nolasco and G. Tarantello, *On a sharp type inequality on two dimensional compact manifolds*, Arch. Rational Mech. Anal. 145 (1998) 161-195.

[32] M. Nolasco and G. Tarantello, *Double vortex condensates in the Chern-Simons-Higgs theory*, Calc. Var. Partial Differential Equations 9 (1999), no. 1, 31-94.

[33] M. Nolasco and G. Tarantello, *Vortex condensates for the* SU(3) *Chern-Simons theory*, Comm.Math. Phys. 213 (2000), no. 3, 599–639.

[34] T. Riciardi and G. Tarantello, *On a periodic boundary value problem with exponential nonlinearities*, Differential and Integral Equations 11 (1998), no. 5, 745-753.

[35] T. Riciardi and G. Tarantello, *Vortices in the Maxwell-Chern-Simons theory*, Comm. Pure Appl. Math. 53 (2000), no. 7, 811–851.

[36] J. Spruck and Y. Yang, *Topological solutions in the self-dual Chern-Simons theory: existence and approximation*, Ann. Inst. H. Poincáre Anal. Non Linéaire 12(1995), 75-97.

[37] M. Struwe and G. Tarantello, *On multivortex solutions in Chern-Simons Gauge theory*, Boll. Unione Math. Ital. Sez. B Artic. Ric. Mat. (8) 1 (1998), 109-121.

[38] T. Suzuki, *Global analysis for a two-dimensional elliptic eigenvalues problem with the exponential nonlinearity*, Ann. Inst. Henri Poincaŕe, Anal. Nonlinéaire 9(1992), 367-398.

[39] G. Tarantello, *Multiple condensate solutions for the Chern-Simons-Higgs theory*, J. Math. Phys. 37(1996), 3769-3796.