行政院國家科學委員會補助專題研究計畫成果報告

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※ 二階非線性微分方程有界正解之唯一性 ※

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計畫主持人：張秋俊 台灣大學數學系

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行政院國家科學委員會專題研究計畫成果報告

二階非線性微分方程有界正解之唯一性

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一、中文摘要

考慮廣義 Emden-Fowler 方程 $y''+g(x)y^{\gamma}=0$, $\gamma=2n-1$, $n>1$ 整數，$g(x)>0$。在此論文裡，我們得到在適當條件下滿足 y(a)=0 之有界正解之唯一性及一般解之非震盪性結果。

關鍵詞：非線性方程，正解，唯一性，非震盪性。

Abstract

Consider the generalized Emden-Fowler equation

(*) $y''+g(x)y^{\gamma}=0$ in $[0,\infty)$,

where $\gamma=2n-1$, with $n>1$ an integer and $g(x)>0$. In this article we will give results about uniqueness of bounded positive solution (corollary 10, 11 and the remark) and nonoscillation (Theorem 8 and 9) of solutions.

Keywords: nonlinear equation, uniqueness, nonoscillation.

二、本年進度

此為一年期之計畫。所得結果尚滿意。進一步欲了解文中 $G_y(x)$ 恆大於 0 之條件。

三、成果自評

對唯一性結果如期完成，感滿意。尤其附帶得到一些非震盪性結果。

On uniqueness and nonoscillation of second order nonlinear equation

By

CHIU-CHUN CHANG（張秋俊）

Abstract. Consider the generalized Emden-Fowler equation

$(*)\ y'' + g(x)y^{\gamma} = 0$ in $[0,\infty)$,

where $\gamma = 2n-1$, with $n>1$ an integer and $g(x)>0$. In this article we will give results about uniqueness of bounded positive solution (corollary 10, 11 and the remark) and nonoscillation (Theorem 8 and 9) of solutions.

AMS subject classifications : 34C15

Key words : nonlinear equation, uniqueness, nonoscillation.

1.Introduction

Consider the generalized Emden-Fowler equation

(1) $y'' + g(x)y^{\gamma} = 0, x \in [0, \infty)$,

where $\gamma = 2n - 1 > 1$ and $g(x)$ is assumed to be positive and continuous on $[0, \infty)$.

We also assume that the solutions of (1) are continuable to the entire non-negative real axis ; say under the condition that $g(x)$ is locally of bounded variation [4].

A solution of (1) is nonoscillatory if for every $a > 0$ it has only a finite number of zeros in $[a, \infty)$. Equation (1) is said to be nonoscillatory if every solution is nonoscillatory. As to the nonoscillation conditions of equation (1), we mention some important ones in the follwing

Theorem 1. [1] Equation (1) has a nonoscillatory solution if and only if

(2) $\int_0^\infty xg(x)dx < \infty$

Theorem 2. [1] If $g(x)$ is continuous differentiable and $g'(x) \leq 0$,

(3) $\int_0^\infty x^{\gamma}g(x)dx < \infty$,

then equation (1) is nonscillatory.

Theorem 3. [6] If $x^{\frac{\gamma+3}{2}+\varepsilon}g(x)$ is nonincreasing in x ; for some $\varepsilon > 0$, then equation (1) is nonoscillatory.

Theorem 4. [8] If $(xlogx)^{\frac{\gamma+3}{2}}g(x)$ is nonincreasing in x for sufficient large x, then equation (1) is nonoscillatory.

In [5], Coffman and Wong conjectured that the condition

(4) $\int_0^\infty x^{\frac{\gamma+1}{2}}g(x)dx < \infty$

is necessary and sufficient for equation (1) to be nonoscillatory. In this article, we will show that condition (4) is nearly sufficient for the nonoscillation of equation (1). That is, equation (1) will be nonoscillatory if we have condition (4) together with a mild condition that will be stated below.(Theorem 8)

2. A Pohozaev identity. We use the follwing slightly more general form of Pohozaev identity [3] [8] :

For $a > 0$, y any solution of (1) with $y(a) = 0$, then

(5) $$G_y(s) \equiv (s-a)y'^2(s) - y(s)y'(s) + \frac{2}{\gamma+1}(s-a)g(s)y^{\gamma+1}(s)$$

$$= \frac{2}{\gamma+1}\int_a^s [\frac{\gamma+3}{2} + \frac{(t-a)g'(t)}{g(t)}]g(t)y^{\gamma+1}(t)dt$$

$$\equiv \frac{2}{\gamma+1}\int_a^s Q(t)g(t)y^{\gamma+1}(t)dt.$$

It can be proved by direct differentiation.

In the following, we assume

(6) For each sufficient large a, the corresponding $Q(t) = \frac{\gamma+3}{2} + \frac{(t-a)g'(t)}{g(t)}$ is eventually of one sign. Hence $Q(t)$ is eventually negative under (4). For bounded positive solution y with $y(a) = 0$, we have $G_y(x) > 0$ for all x in (a, ∞).

Remark. Under (4) and that $Q(t)$ is eventually negative, then $G_y(x)$ will be positive in (a, ∞) if Q is decreasing from taking positive to negative values as examples (a) and (b) considered below.

Corollary 5. If $y_2(t)$ is an oscillatory solution of (1) with $y_2(a_\nu) = 0, \nu = 3, 4 \cdots$. Then $\{(a_\nu - a)y_2'^2(a_\nu)\}$ is decreasing from some ν on. Specially, $lim_{\nu\to\infty} y_2'(a_\nu) = 0$

proof: By (5) and (6), G_{y_2} is decreasing from some s_0 on. And $G_{y_2}(a_\nu) = (a_\nu - a)y_2'^2(a_\nu)$.

Theorem 6. Assume (4) and (6) are satisfied. Let y be a bounded positive solution of (1) in (a, ∞) with $y(a) = 0$. Then $w(x) = \frac{(x-a)y'(x)}{y(x)}$ is decreasing to zero.

Proof. By (6), we have

$$(s-a)y'^2(s) - y'(s)y(s) + \tfrac{2}{\gamma+1}(s-a)g(s)y^{\gamma+1}(s) > 0 \text{ in } (a, \infty).$$

So, $D_y(s) \equiv (s-a)y'^2 - y'(s)y(s) + (s-a)g(s)y^{\gamma+1}(s) > 0$ in (a, ∞).

We have

$$w'(s) = -\frac{D_y(s)}{y^2(s)} < 0 \text{ in } (a, \infty) ,$$

so that w is decreasing.

It is well known that y is increasing and $(x-a)y'(x) = (x-a)\int_x^\infty g y^\gamma \le y^\gamma(\infty)$ $\int_x^\infty g(s) \cdot s ds$. The last term tends to zero by (2) as $x \to \infty$. Hence the theorem is proved.

3. A comparison theorem

We assume that y_1 is a bounded positive solution of

(7) $y_1'' + g_1(x)y_1^\gamma = 0$, $y_1(a) = 0$,

where $0 < g_1(x) \le g(x)$ and $g_1'(x) \le g'(x) < 0$ in $a \le x < \infty$.
(These imply $\frac{g_1'(x)}{g_1(x)} \le \frac{g'(x)}{g(x)}$.)

Theorem 7. Let y_1 be as in (7), $y_1'(a) < y'(a)$, where y is a bounded positive solution indicated in (6). Then

(8) $G_{y_1}(x) \equiv (x-a)y_1'^2(x) - y_1'(x)y_1(x) + \frac{2}{\gamma+1}(x-a)g_1(x)y_1^{\gamma+1}(x) < \frac{g_1(x)}{g(x)}(\frac{y_1}{y})^{\gamma+1}(x) \cdot G_y(x)$ as long as $x > a$ and $(\frac{y_1}{y})'(x) > 0$.

Proof. Consider

(9) $L(t) = G_{y_1}(t) - (\frac{g_1}{g})(t)(\frac{y_1}{y})^{\gamma+1}(t)G_y(t)$.

Then $L(a) = 0$ and

$$(10)\quad L'(t) = \tfrac{2}{\gamma+1}\{(\tfrac{\gamma+3}{2} + \tfrac{(t-a)g_1'}{g_1})g_1 y_1^{\gamma+1} - (\tfrac{\gamma+3}{2} + \tfrac{(t-a)g'}{g})g_1 y_1^{\gamma+1}\}$$

$$-\{\tfrac{g_1}{g}[(\gamma+1)(\tfrac{y_1}{y})^{\gamma}\cdot\tfrac{y_1'y-y'y_1}{y^2}]G_y(t) + \tfrac{g_1'g-g'g_1}{g^2}(\tfrac{y_1}{y})^{\gamma+1}G_y(t)\}$$

$$= -\{\tfrac{g_1}{g}(\gamma+1)(\tfrac{y_1}{y})^{\gamma+1}[\tfrac{y_1'}{y_1} - \tfrac{y'}{y}]G_y(t)\}$$

$$-(\tfrac{g'}{g} - \tfrac{g_1'}{g_1})\{\tfrac{2}{\gamma+1}(t-a)g_1 y_1^{\gamma+1} - \tfrac{g_1}{g}(\tfrac{y_1}{y})^{\gamma+1}G_y(t)\}.$$

Write

$$\alpha(s) = \tfrac{2}{\gamma+1}(s-a)g(s)y^{\gamma+1}(s) - G_y(s).$$

Then $\alpha(a) = 0, \alpha(\infty) = 0$ and

$$(11)\quad \alpha'(s) = \tfrac{2}{\gamma+1}\{gy^{\gamma+1} + (s-a)g'y^{\gamma+1} + (\gamma+1)(s-a)gy^{\gamma}y'\} - \tfrac{2}{\gamma+1}(\tfrac{\gamma+3}{2} + \tfrac{(s-a)g'}{g})gy^{\gamma+1}$$

$$= \tfrac{2}{\gamma+1}gy^{\gamma+1}\{1 + \tfrac{(s-a)g'}{g} + (\gamma+1)\tfrac{(s-a)y'}{y} - (\tfrac{\gamma+3}{2} + \tfrac{(s-a)g'}{g})\}$$

$$= \tfrac{2}{\gamma+1}gy^{\gamma+1}\{(\gamma+1)w - \tfrac{\gamma+1}{2}\}(w(s) = \tfrac{(s-a)y'(s)}{y(s)})$$

$$= gy^{\gamma+1}(2w-1).$$

Since by Theorem 6, w is decreasing from 1 to zero, we have $\alpha(s) > 0$ in (a, ∞) and the last term in bracket of (10) is positive.

By, assumption, $(\frac{g'}{g} - \frac{g_1'}{g})$ is positive, hence

$L' < 0$ if $(\frac{y_1'}{y_1} - \frac{y'}{y}) > 0$. Hence $L(s) < 0$ for $s > a$ and as long as $\frac{y_1}{y}$ is increasing.

4. The main Theorem

Theorem 8. We assume (4) and (6) and that $g'(x)$ is eventually non-increasing. Then equation (1) is nonoscillatory.

Proof. Let y_2 be any solution of (1), we want to show that y_2 is nonoscillatory. Suppose there are infinite many zeros of y_2. Assume $g(t)$ and $g'(t)$ are non-increasing for $t \geq t_0$. We consider those zeros larger than t_0 and labeled consecutively as $a_3, a_4, \cdots a_\nu, a_{\nu+1}\cdots$. Condition (4) implies the existence of bounded positive solution y_ν of (1) in $[a_\nu, \infty)$ with $y_\nu(a_\nu) = 0$. From [2][7] we have

$$lim_{x\to\infty} y_\nu(x) = C_\nu > 0,\ y_\nu'(x) = \int_x^\infty g y_\nu^\gamma ds$$

and hence $lim_{x\to\infty} x y_\nu'(x) = 0$ by (4) or (2).

Therefore

$$G_{y_\nu}(s) = (s-a_\nu)y_\nu'^2(s) - y_\nu(s)y_\nu'(x) + \frac{2}{\gamma+1}(s-a_\nu)g(s)y_\nu^{\gamma+1}(s)$$

tends to zero as s tends to ∞.

Under the conditions stated we shall show that

(12) $y'_{\nu+1}(a_{\nu+1}) \geq y'_\nu(a_\nu),\ \nu = 3, 4, \cdots$

Otherwise, we assume $y'_{\nu+1}(a_{\nu+1}) < y'_\nu(a_\nu)$. Notice that

$$y''_{\nu+1}(x) + g(x)y^\gamma_{\nu+1}(x) = 0 \text{ in } [a_{\nu+1}, \infty).$$

Let $Y(x) = y_{\nu+1}(x+\delta),\ \delta = a_{\nu+1} - a_\nu$. Then $Y''(x) + g_1(x)Y^\gamma(x) = 0$, with $g_1(x) = g(x+\delta)$. For simplicity, denote $a_\nu = a$, then with $y_1(x) \equiv Y(x),\ y(x) \equiv y_\nu(x)$ in Theorem 7, we have $y_1(a) = 0,\ y(a) = 0,\ y'_1(a) < y'(a)$ and

(13) $L(t) = G_{y_1}(t) - \frac{g_1}{g}(\frac{y_1}{y})^{\gamma+1}(t)G_y(t) < 0$,

for $t > a$ and as long as $(\frac{y_1}{y})$ is increasing. This is true initially because from the equation

(14) $(y'_1 y - y' y_1)(x) = \int_a^x y_1 y(g y^{\gamma-1} - g_1 y_1^{\gamma-1})ds$,

we have $g > g_1$ and $y > y_1$ initially as $y'_1(a) < y'(a)$. We shall show that $(\frac{y_1}{y})' > 0$ all in (a, ∞). If otherwise, let x_0 be the first point that $(\frac{y_1}{y})' = 0$, that is

(15) $\frac{y'_1}{y_1}(x_0) = \frac{y'}{y}(x_0)$.

Then (with the notation in Theorem 7), $L'(t) < 0$ in (a, x_0) and we have

(16) $G_{y_1}(x_0) = (x_0 - a){y'_1}^2(x_0) - y'_1(x_0)y_1(x_0) + \frac{2}{\gamma+1}(x_0 - a)g_1(x_0)y_1^{\gamma+1}(x_0)$

$< \frac{g_1(x_0)}{g(x_0)}(\frac{y_1}{y})^{\gamma+1}\ (x_0)\{(x_0 - a){y'}^2(x_0) - y'(x_0)y(x_0) + \frac{2}{\gamma+1}(x_0 - a)g(x_0)y^{\gamma+1}(x_0)\}$,

that is

(17) $(x_0 - a){y'_1}^2(x_0) - y'_1(x_0)y_1(x_0) < \frac{g_1(x_0)}{g(x_0)}(\frac{y_1}{y})^{\gamma+1}(x_0)[(x_0 - a){y'}^2(x_0) - y'(x_0)y(x_0)]$.

We already know $y'_1 y - y' y_1$ is positive at beginning. $(y'_1 y - y' y_0)(x_0) = 0$ would imply by (14) that

(18) $g(x_0)y^{\gamma-1}(x_0) \leq g_1(x_0)y_1^{\gamma-1}(x_0)$,

that is $\frac{g_1}{g}(x_0)(\frac{y_1}{y})^{\gamma-1}(x_0) \geq 1$. (14), (15) and (17) imply that

(19) $(x_0 - a){y'_1}^2(x_0) - y'_1(x_0)y_1(x_0) < \frac{g_1}{g}(x_0)(\frac{y_1}{y})^{\gamma-1}(x_0)[(x_0 - a){y'_1}^2(x_0) - y'_1(x_0)y_1(x_0)]$.

But this is a contradiction because $\frac{g_1}{g}(x_0)(\frac{y_1}{y})^{\gamma-1}(x_0) \geq 1$ and it is well known that $(x_0 - a)y'_1(x_0) < y_1(x_0)$ and $y'_1 > 0$. Therefore we have proved that

(20) $G_{y_1}(x) \leq \frac{g_1}{g}(x)(\frac{y_1}{y})^{\gamma+1}(x)G_y(x),\ a \leq x < \infty$.

We will postpone the proof and state some corollaries and remarks up to now we can obtain.

We often need to use the case $g_1(x) \equiv g(x)$. Then the results up to now can be stated as

Corollary 9. Let y be the bounded positive solution indicated in (6), y_1 another solution of (1) with $y_1(a) = 0$, $y_1'(a) < y'(a)$. Then

(21) $G_{y_1}(x) \leq (\frac{y_1}{y})^{\gamma+1}(x)G_y(x)$,

as long as $(\frac{y_1}{y})'(x) > 0$.

Proof: Theorem 7.

Corollary 10. The above inequality (21) holds for all x in (a, ∞) and in fact $\frac{y_1}{y}$ is increasing all the way.

Proof. By theorem 7 and equations (13) through (20) with $g_1(x) \equiv g(x)$.

With a similar proof we have

Corollary 11. y as in corollary 9, y_1 another solution of (11) with $y_1(a) = 0$, $y_1'(a) > y'(a)$. Then

(22) $G_{y_1}(x) \geq (\frac{y_1}{y})^{\gamma+1}(x)G_y(x)$,

as long as $y_1(x) > 0$ and $(\frac{y_1}{y})'(x) < 0$.

Remark. It is in the above character that we can prove the uniqueness of the bounded positive solution as in [3]. However, in Theorem 1 of [3], the statement and proof was incomplete. It should follow corollary 11.

5. Completion of the proof of the main theorem

Refer the poof of (20), with $y_{\nu+1} = y_1$ and $y_\nu = y$, we have $G_{y_1(x)} < \frac{g_1}{g}(x)(\frac{y_1}{y})^{\gamma+1}$ $G_y(x)$, $a \leq x < \infty$ if $y_{\nu+1}'(a_{\nu+1}) < y_\nu'(a_\nu)$. But this leads to a contradition since $G_{y_1}(\infty) = 0 = G_y(\infty)$ as y_1, y are bounded positive solutions.

Hence we have

(23) $y_{\nu+1}' \geq y_\nu'(a_\nu), \nu = 3, 4, \cdots$

Now, at the beginning of the proof, we assumed that y_2 is oscillatory and $a_3, a_4, \cdots$ are the zeros of y_2. corollary 5 asserted that $y_2'(a_v)$ tends to zero as $\nu \to \infty$. Hence for some ν_0 we would have

(24) $y_2'(a\nu) < y_\nu'(a_\nu)$

because of (23).

But then by corollary (10), on $[a_\nu, \infty)$, $(\frac{y_2}{y_\nu})$ is increasing all the way. That is from some ν_0 on, y_2 will not oscillate. Therefore the Theorem is proved.

We will give a more general result about nonoscillation.

Theorem 9. Under (4) and (6). Then equation (1) is nonocillstory.

Proof. y_2 and y_ν as in Theorem 8. Then since $g' < 0$ eventually, we may assume $y_2'(a_m) < y_2'(a_\nu)$ if $m > \nu$. (as is well known $y_2'^2(s) + \frac{2}{\gamma+1}g(s)y_2^{\gamma+1}(s)$ is decreasing)

Since $Q \leq 0$ eventually, we have $G_{y_2}(x)$ is eventually decreasing and positive.

Hence $y_2'(a_\nu)a_\nu^{\frac{1}{2}} \leq C$, $\nu = 3,\ 4,\ \cdots$

From the equation

(25) $y_\nu(x) = y_\nu(a_\nu) + \int_{a_\nu}^{x}(s-a_\nu)g(s)y_\nu^\gamma(s)ds + (x-a_\nu)y_\nu'(x)$

we have

(26) $y_\nu(2a_\nu) = 0 + \int_{a_\nu}^{2a_\nu}(s-a_\nu)g(s)y_\nu^\gamma(s)ds + a_\nu y_\nu'(2a_\nu)$

(27) $\int_{a_\nu}^{2a_\nu}(s-a_\nu)g(s)y_\nu^\gamma(s)ds \leq y_\nu(2a_\nu)\int_{a_\nu}^{2a_\nu}(s-a_\nu)g(s)[y_\nu'(a_\nu)(s-a_\nu)]^{\gamma-1}ds$

$= y_\nu(2a_\nu)\int_{a_\nu}^{2a_\nu}(s-a_\nu)^{\frac{\gamma+1}{2}}g(s)[y_\nu'(a_\nu)(s-a_\nu)^{\frac{1}{2}}]^{\gamma-1}ds$

$\leq y_\nu(2a_\nu)\cdot[y_\nu'(a_\nu)(2a_\nu-a_\nu)^{\frac{1}{2}}]^{\gamma-1}\cdot\int_{a_\nu}^{2a_\nu}(s-a_\nu)^{\frac{\gamma+1}{2}}g(s)ds.$

(26) divided by $y_\nu(2a_\nu)$, we have

$1 \leq [y_\nu'(a_\nu)(a_\nu)^{\frac{1}{2}}]^{\gamma-1}\int_{a_\nu}^{\infty}g(s)s^{\frac{\gamma+1}{2}}ds + \frac{1}{2}$ or

$\frac{1}{2} \leq C\cdot\varepsilon = \varepsilon_1$, as ν large.

This contradiction prove the theorem.

Example [3]

(a) $y'' + (x\operatorname{csch}^2 x)^2 y^3 = 0.$
$Q(x) = 3 + 2[1 - 2(x-a)\coth x] - \frac{2a}{x}$
It is easy to show that (4) and (6) are satisfied and the equation is nonoscillatory.

(b) $\triangle u + \frac{1}{1+r^2}u^\gamma = 0, 1 < \gamma < \frac{n+2}{n-2}.$
Consider the radial solution $u(r) = u(\mid x \mid)$ and let $y(s) = su(r), s = r^{n-2}$, then y satisfies

$$y'' + \frac{1}{(n-2)^2}\frac{\frac{r^2}{1+r^2}}{s^{1+\gamma}}y^\gamma(s) = 0.$$

Conditions (4) and (6) are satisfied and the equation is nonoscillatory.

(c) $\triangle u + u^{\frac{n+2}{n-2}} = 0, u(r) = u(\mid x \mid)$. As in (b),

$$y'' + \frac{1}{s^{2+\frac{2}{n-2}}}y^{\frac{n+2}{n-2}} = 0.$$

$g(s) = s^{-(2+\frac{2}{n-2})}$ does not satisfy (4) and $Q(s)$ is eventually positive. It has oscillatory solutions. [7] .

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