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一、中文摘要

設R是一個特征不是2之質環，而d是其上的一個導算(derivation)。Chebotar和李白飛教授證明若是$d^3\neq 0$，則$\{[x^d,x]: x\in R\}$產生之加法子群含有非中心之李理想。本文之目的，仍在証明即使$d^3=0$，結論仍然成立。（已被 Comm. Algebra 接受）

A NOTE ON CERTAIN SUBGROUPS OF PRIME RINGS WITH DERIVATIONS

Chen–Lian Chuang and Tsiu–Kwen Lee

Department of Mathematics

National Taiwan University

Taipei 106, Taiwan

E–mail: chuang@math.ntu.edu.tw

E–mail: tklee@math.ntu.edu.tw

Abstract. Let R be a noncommutative prime ring of characteristic not 2 and d a nonzero derivation of R. Chebotar and Lee proved that the additive subgroup of R generated by the subset $\{[x^d, x] \mid x \in R\}$ contains a noncentral Lie ideal of R if $d^3 \neq 0$. In this note we show that the same conclusion holds even if $d^3 = 0$.

Let R be a noncommutative prime ring with center $Z(R)$ and d a nonzero derivation of R. We denote by $G_d(R)$ the additive subgroup of R generated by the subset $\{[x^d, x] \mid x \in R\}$ and by $\overline{G_d(R)}$ the subring of R generated by $G_d(R)$. In [9] Posner proved that $G_d(R) \not\subseteq Z(R)$. Several generalizations of Posner's theorem have been obtained in the literature. In [1] Brešar and Vukman proved that $\overline{G_d(R)}$ contains nonzero one–sided ideals of R if $\operatorname{char} R \neq 2$. In [2] Chebotar generalized Brešar and Vukman's theorem by proving that $\overline{G_d(R)}$ contains a nonzero ideal of R if $\operatorname{char} R \neq 2$. Recently, Chebotar and Lee [3] proved that $G_d(R)$ contains a noncentral Lie ideal of R if $\operatorname{char} R \neq 2$ and $d^3 \neq 0$. They also posed the question whether or not the theorem still holds without the assumption that $d^3 \neq 0$. In this

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note we answer the question in the affirmative. Precisely, we will prove the following

Theorem 1. *Let R be a noncommutative prime ring of characteristic not 2 and d a nonzero derivation of R. If $d^3 = 0$, then the additive subgroup of R generated by the subset $\{[x^d, x] \mid x \in R\}$ contains a noncentral Lie ideal of R.*

Theorem 1 together with [3, Theorem 3] gives the following result.

Theorem 2. *Let R be a noncommutative prime ring of characteristic not 2 and d a nonzero derivation of R. Then the additive subgroup of R generated by the subset $\{[x^d, x] \mid x \in R\}$ contains a noncentral Lie ideal of R.*

As pointed out in [3] we remark that the conclusion is not true any more when $\operatorname{char} R = 2$. In what follows, unless specially stated, let R denote a noncommutative prime ring with extended centroid C, two-sided Martindale quotient ring Q and $\operatorname{char} R \neq 2$. We begin the proof with some elementary observations.

Lemma 1. *Suppose that d is a nonzero derivation of R. Then $G_d(R)$ contains a noncentral Lie ideal of R if and only if there exists a nonzero ideal I of R such that $[I, I]^d \subseteq G_d(R)$.*

Proof. For $x \in R$, by assumption we have $[x^d, x] \in G_d(R)$. Linearizing this relation on x, we obtain

$$(1) \qquad [x^d, y] + [y^d, x] \in G_d(R)$$

for all $x, y \in R$. Since $[x^d, y] = [x, y]^d + [y^d, x]$, by (1) we have

$$(2) \qquad [x, y]^d + 2[y^d, x] \in G_d(R)$$

for all $x, y \in R$.

Suppose that $G_d(R)$ contains a noncentral Lie ideal of R. Then there exists a nonzero ideal I of R such that $[I, R] \subseteq G_d(R)$ (see the proof of [5, Lemma 1.3]). In particular, we have $[y^d, x] \in G_d(R)$ for all $x, y \in I$. By (2) we see that $[x, y]^d \in G_d(R)$ for all $x, y \in I$. That is, $[I, I]^d \subseteq G_d(R)$.

Suppose next that there exists a nonzero ideal I of R such that $[I, I]^d \subseteq G_d(R)$. By (2) we have $2[y^d, x] \in G_d(R)$ for all $x, y \in I$. That is, $2[I^d, I] \subseteq G_d(R)$. We note that d may not be a derivation of the prime ring I. However, $[I^d, I]$ still contains a noncentral Lie ideal of R by applying the same argument given in the proof of [3, Proposition 2] together with the following fact: Let J be a nonzero ideal of R and A an additive subgroup of R. Then $(J^2)^{d^2} \subseteq J^d$, $J^{d^2} \neq 0$ and $[A, J] = [\overline{A}, J]$, where $\overline{A}$ denotes the subring of R generated by A. Thus $G_d(R)$ contains a noncentral Lie ideal of R, proving the lemma.

Let F be the algebraic closure of C and set $\widetilde{R} = RC \otimes_C F$. In view of [4], $\widetilde{R}$ is a prime ring with extended centroid F. A standard argument proves that every nonzero ideal of $\widetilde{R}$ intersects nontrivially with R. It is well–known that every derivation of R is uniquely extended to a derivation of Q. If d is a derivation of R such that $C^d = 0$, then d is canonically extended to a derivation of $\widetilde{R}$, denoted by d also, such that $F^d = 0$. We need the following lemma in our proof.

Lemma 2. *Suppose that d is a derivation of R such that $C^d = 0$. If $G_d(\widetilde{R})$ contains a noncentral Lie ideal of $\widetilde{R}$, then $G_d(R)$ contains a noncentral Lie ideal of R.*

Proof. We claim that $G_d(\widetilde{R}) = G_d(R)F$. Denote by G_R ($G_{\widetilde{R}}$) the additive subgroup of R (resp. $\widetilde{R}$) generated by all elements $[x^d, y] + [y^d, x]$ for $x, y \in R$ (resp. $\widetilde{R}$). It is clear that $G_{\widetilde{R}} = G_R F$ since $F^d = 0$. Since $\operatorname{char} R \neq 2$, we see that $2G_d(\widetilde{R}) \subseteq G_{\widetilde{R}} \subseteq G_d(\widetilde{R})$. Since F is an algebraically closed field and $F^d = 0$, it is clear that $G_d(\widetilde{R})$ is a vector space over F. Thus $G_{\widetilde{R}} = G_d(\widetilde{R})$, implying that $G_d(\widetilde{R}) = G_d(R)F$, as claimed.

Suppose next that $G_d(\widetilde{R})$ contains a noncentral Lie ideal of $\widetilde{R}$. In view of Lemma 1, there exists a nonzero ideal I of $\widetilde{R}$ such that $[I, I]^d \subseteq G_d(\widetilde{R})$. Thus $[I \cap R, I \cap R]^d \subseteq G_d(R)F \cap R$. Note that $G_d(R)F \cap R = G_d(R)$ and that $I \cap R$ is a nonzero ideal of R. Thus $I \cap R$ is a nonzero ideal of R such that $[I \cap R, I \cap R]^d \subseteq G_d(R)$, implying that $G_d(R)$ contains a noncentral Lie ideal of R by Lemma 1. This proves the lemma.

The next lemma plays a key role in the sequel.

Lemma 3. *Let R be a prime ring of characteristic not 2, I a nonzero ideal of R and $0 \neq a \in Q$. Then H, the additive subgroup of R generated by all elements $xay + yax$ for $x, y \in I$, contains a nonzero ideal of R.*

Proof. Choose a nonzero ideal J of R such that $aJ + Ja \subseteq I$. Let $x, y, z \in J$. Then, by assumption, we have

$$2xayaz = [(xay)az + za(xay)] - [(zax)ay + ya(zax)] + [(yaz)ax + xa(yaz)] \in H.$$

Thus we see that $2JaJaJ \subseteq H$. Note that $2JaJaJ$ is a nonzero ideal of R. The lemma is thus proved.

Lemma 4. *Suppose that d is a derivation of R defined by $d(x) = ax - xa$ for $x \in R$, where $0 \neq a \in Q$ and $a^2 = 0$. Then $G_d(R)$ contains a noncentral Lie ideal of R.*

Proof. Define $f(x) = [x^d, x] = ax^2 - 2xax + x^2a$ for $x \in R$. Choose a nonzero ideal K of R such that $aK \subseteq R$. Then $f(ax) = axaxa \in G_d(R)$ for $x \in K$. In view of Lemma 3, the additive subgroup of R generated by the subset $\{xax \mid x \in K\}$ contains a nonzero ideal I of R. This implies that $aIa \subseteq G_d(R)$. We may also assume that $aI \subseteq R$ as $a \in Q$.

Let $x \in I$; then $ax \in R$. Then $f(ax + x) - f(ax) - f(x) \in G_d(R)$. This means that

$$a(ax^2 + xax) - 2axax - 2xa^2x + (ax^2 + xax)a \in G_d(R). \tag{3}$$

That is, $-[a, xax] + ax^2a \in G_d(R)$. But $ax^2a \in aIa \subseteq G_d(R)$, we have $[a, xax] \in G_d(R)$ for all $x \in I$. In view of Lemma 3, there exists a nonzero ideal J of R such that $J \subseteq \{xax \mid x \in I\}$ and so $[a, J] \subseteq G_d(R)$, that is, $J^d \subseteq G_d(R)$. By Lemma 1, $G_d(R)$ contains a noncentral Lie ideal of R, proving the lemma.

Recall that a derivation d of R is called *X–outer* in the sense of Kharchenko (see [6] and [7]) if d is not Q–inner. To prove Theorem 1 we need to investigate $\overline{R^d}$, the subring of R generated by R^d.

Theorem 3. *Let R be a prime ring and d an X-outer derivation of R. Then $\overline{R^d}$ contains a nonzero ideal of R unless* $\operatorname{char} R = 2$ *and* $d^2 = 0$.

Proof. Let $x, y \in R$. Then $(xy^d)^d = x^d y^d + xy^{d^2} \in \overline{R^d}$, implying that $xy^{d^2} \in \overline{R^d}$. That is, $Rx^{d^2} \subseteq \overline{R^d}$ and, analogously, $y^{d^2}R \subseteq \overline{R^d}$. Therefore, we see that $Rx^{d^2}y^{d^2}R \subseteq \overline{R^d}$ for all $x, y \in R$. We are done if $x^{d^2}y^{d^2} \neq 0$ for some $x, y \in R$. Suppose now that $x^{d^2}y^{d^2} = 0$ for all $x, y \in R$. The aim is to prove that $\operatorname{char} R = 2$ and $d^2 = 0$.

If $\operatorname{char} R \neq 2$, then d^2 is a regular word and hence, Kharchenko's theorem [7] implies that $xy = 0$ for all $x, y \in R$. This derives a contradiction from the primeness of R. Thus $\operatorname{char} R = 2$. Then d^2 is also a derivation of R. Suppose that $d^2 \neq 0$. Thus the annihilator of R^{d^2} is zero. In particular, $R^{d^2}R^{d^2} \neq 0$, a contradiction. Hence, $d^2 = 0$ follows. This proves the theorem.

We are now ready to give the proof of Theorem 1.

Proof of Theorem 1. Suppose first that d is X-outer. Note that $d^2 \neq 0$ as $\operatorname{char} R \neq 2$. It follows from the proof of [3, Theorem 3] that

$$4[\overline{R^d}(x^d y^d)^d \overline{R^d}, R] \subseteq G_d(R) \tag{4}$$

for all $x, y \in R$. In view of Theorem 3, $\overline{R^d}$ contains a nonzero ideal I of R. Thus $4[I(x^d y^d)^d I, R] \subseteq G_d(R)$ for all $x, y \in R$. Suppose that $(x^d y^d)^d = 0$ for all $x, y \in R$. Then $x^{d^2}y^d + x^d y^{d^2} = 0$ for all $x, y \in R$. Since $\operatorname{char} R \neq 2$ and d is X-outer, d and d^2 are distinct regular words (see [6] and [7]). By Kharchenko's theorem [7], $xy + uv = 0$ for all $x, y, u, v \in R$, a contradiction. Thus there exist $x, y \in R$ such that $(x^d y^d)^d \neq 0$. Then $4[I(x^d y^d)^d I, R]$ is a noncentral Lie ideal of R contained in $G_d(R)$. We are done in this case.

Suppose next that d is Q-inner. Thus there exists $a \in Q$ such that $x^d = ax - xa$ for $x \in R$. Suppose for the moment that $\operatorname{char} R > 3$. Since $d^3 = 0$, it follows from [8, Corollary 1] that there exists $\lambda \in C$ such that $(a - \lambda)^2 = 0$. Since $x^d = [a, x] = [a - \lambda, x]$ for $x \in R$, by replacing a with $a - \lambda$ we may assume that $a^2 = 0$. By

Lemma 4, $G_d(R)$ contains a noncentral Lie ideal of R. Therefore, we assume that $\operatorname{char} R = 3$. Then $a^3 \in C$ as $d^3 = 0$. Denote by F the algebraic closure of C and set $\widetilde{R} = RC \otimes_C F \subseteq Q \otimes_C F$. Note that $Q \otimes_C F$ is contained in the two-sided Martindale quotient ring of $RC \otimes_C F$. Then we can choose $\lambda \in F$ such that $(a - \lambda)^3 = 0$. By Lemma 2, it suffices to prove that $G_d(\widetilde{R})$ contains a noncentral Lie ideal of $\widetilde{R}$. Replacing a by $a - \lambda$, we may assume that $a^3 = 0$. If $a^2 = 0$, then we are done by Lemma 4. Thus we always assume that $a^2 \neq 0$.

Set $f(x) = [x^d, x]$ for $x \in \widetilde{R}$. Then, using $\operatorname{char} \widetilde{R} = 3$, we have

$$f(x) = ax^2 + xax + x^2a \in G_d(\widetilde{R}) \tag{5}$$

for all $x \in \widetilde{R}$. Choose a nonzero ideal K of $\widetilde{R}$ such that $Ka^2 + a^2K \subseteq \widetilde{R}$. Replacing x by xa^2 for $x \in K$ in (5), we see that $f(xa^2) = axa^2xa^2 \in G_d(\widetilde{R})$. Analogously, we have $a^2xa^2xa \in G_d(\widetilde{R})$. In view of Lemma 3, the additive subgroup of $\widetilde{R}$ generated by the subset $\{xa^2x \mid x \in K\}$ contains a nonzero ideal I of $\widetilde{R}$. Thus we have

$$aIa^2 + a^2Ia \subseteq G_d(\widetilde{R}).$$

Choose a nonzero ideal I_0 of $\widetilde{R}$ such that

$$I_0 + I_0a + I_0a^2 + aI_0 + a^2I_0 + aI_0a + aI_0a^2 + a^2I_0a \subseteq I.$$

Thus $I_0aI_0 + I_0a^2I_0 \subseteq II_0 \subseteq I$. In particular, we have

$$aI_0aI_0a^2 + aI_0a^2I_0a^2 \subseteq G_d(\widetilde{R}). \tag{6}$$

Let $x \in I_0$. Replacing x by $x(a + a^2) \in I$ in (5) and using $a^3 = 0$ and (6), we obtain

$$axaxa + axa^2xa + xa^2xa + xa^2xa^2 + xaxa^2 + xa^2xa^2 \in G_d(\widetilde{R}). \tag{7}$$

By (7) and $\operatorname{char} R = 3$, we have

$$axaxa + axa^2xa + xa^2xa + xaxa^2 - xa^2xa^2 \in G_d(\widetilde{R}). \tag{8}$$

Choose a nonzero ideal I_1 of $\widetilde{R}$ such that $I_1 + I_1 a + a I_1 \subseteq I_0$.

Let $x \in I_1$. Then $x(1+a) \in I_0$. Replacing x by $x(1+a)$ in (8) and using $a^3 = 0$ to expand it, we see that

$$
\begin{aligned}
(9)\qquad & a(xa + xa^2)(xa + xa^2) + axa^2(xa + xa^2) + xa^2(xa + xa^2) \\
& + (xa + xa^2)xa^2 - xa^2xa^2 \in G_d(\widetilde{R}).
\end{aligned}
$$

Note that $axaxa^2, axa^2xa^2 \in aIa^2$. The difference of (9) and (8), together with $aIa^2 \subseteq G_d(\widetilde{R})$, yields

$$
(10)\qquad a(xa^2x)a - (xa^2x)a^2 \in G_d(\widetilde{R})
$$

for all $x \in I_1$. By Lemma 3 again, there exists a nonzero ideal I_2 of $\widetilde{R}$ such that $I_2 \subseteq I_1$ and

$$
(11)\qquad aya - ya^2 \in G_d(\widetilde{R})
$$

for all $y \in I_2$. By symmetry, we may also assume that

$$
(12)\qquad aya - a^2y \in G_d(\widetilde{R})
$$

for all $y \in I_2$.

Let $x \in I_2$. Recall that $aIa^2 + a^2Ia \subseteq G_d(\widetilde{R})$. Thus $(a - a^2)x^2a^2 \in G_d(\widetilde{R})$ as $a^2x^2a^2 = a(ax^2)a^2 \in aIa^2$. Replacing x by $(1-a)x(1+a+a^2)$ in (5) and using (6) and $a^3 = 0$ to expand it, we see that

$$
(13)\qquad (a - a^2)x^2(1 + a) + (1 - a)xax(1 + a + a^2) + (1 - a)x^2(a + a^2) \in G_d(\widetilde{R}).
$$

By (5), (6), (11) and (12), we have $-a^2x^2 + x^2a^2, a^2x^2a, ax^2a^2, axaxa^2 \in G_d(\widetilde{R})$. Thus we reduce (13) to

$$
(14)\qquad xaxa + (xax)a^2 - axax - a(xax)a \in G_d(\widetilde{R}).
$$

Choose a nonzero ideal I_3 of R such that $I_3 + aI_3 \subseteq I_2$. Let $x \in I_3$; then $xax \in I_2$. By (12), we have $(xax)a^2 - a(xax)a \in G_d(\widetilde{R})$ and hence, (14) is reduced to

$$(xax)a - a(xax) \in G_d(\widetilde{R}) \tag{15}$$

for all $x \in I_3$. By Lemma 3 again, the additive subgroup of $\widetilde{R}$ generated by the subset $\{xax \mid x \in I_3\}$ contains a nonzero ideal J of $\widetilde{R}$. It follows from (15) that $[a, J] \subseteq G_d(\widetilde{R})$. In view of Lemma 1, $G_d(\widetilde{R})$ contains a noncentral Lie ideal of $\widetilde{R}$ and hence, by Lemma 2, $G_d(R)$ contains a noncentral Lie ideal of R. This proves the theorem.

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