三維流形之有限型不變量 (2/2)

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主持人: 翁秉仁　　　台灣大學數學系

一、中文摘要

探討使用 Vassiliev 原來的拓樸進路, 定義三維流形有限型不變量之可能性。

關鍵詞: 三維流形, Vassiliev 不變量, 有限型不變量.

Abstract

Investigate the possibility of defining finite type 3 manifold invariants with the Vassiliev's topological approach.

Keyword: three manifolds, Vassiliev's invariants, finite type invariants.

二、計劃緣由與目的

For the last 10 years, Witten's topological-quantum-field-theoretic approach of 3 manifolds + links (knot) invariants dominate the field of 3 manifold invariants and knot invariants[W]. The perturbative version of Witten's invariants made one of the most important progress in 90's, and is now called the Vassiliev invariants or finite type invariants[B][BL][V]. The perturbative 3 manifolds invariants doesn't appear until Ohtsuki's seminar work on finite type invariants defined over **Z** or **Q** homology sphere in 95' [O](Ohtsuki's invariants is proved to be perturbative by Rozansky[R]). Based on the experience of Vassiliev theory, Garoufalidis, Levine, Ohtsuki etc. have made clear the structure of the theory [G][GL1][GL2][GO1][GO2][L][GLin]. And finally the universal perturbative 3 manifold invariant, defined by Le, Murakami and Ohtsuki put in the last puzzle which play the role of Kontsevich integral in Vassiliev theory of knots[Ko][LMO].

The major concern of our project is to fill in the gap of a Vassiliev's (topological) approach of finite type 3 manifolds invariants. Also we want to discuss the interplay between the finite type knot invariants and 3 manifold invariants.

三、結果與討論

1. Topological approach to the 3 manifold invariants

We only consider close 3 manifolds.

The original approach of knot invariants by Vassiliev is topological[V]. He consider the moduli, $\mathcal{K}$, of all knots: smooth embeded S^1 into S^3. And consider the knot invariants as the certain good element in $H^0(\mathcal{K})$. Actually he utilize the infinitely dimensional Alaxander duality to decoded the topology information of knot space into the topology of its complement (the singular knot space or discriminant). So in particular the top dimensional homology of the discriminant (the so-called weight system) produce a new type knot invariants called Vassiliev invariants or finite type invariants. The lower dimensional homology of the discriminant, therefore determine higher dimensional invariants of $\mathcal{K}$, which is the basic works of SW Yang and the author[OY1][OY2][OY3].

In order to apply the Vassiliev approach to the finite type 3-manifold invariants, we face some difficulties. Firstly, knots are homeomorphic to each other, but 3-manifolds are not. So it is a problem to put all 3 manifolds into a grand moduli and then consider the wall crossing. The author tried to use Gromov's distance to give this moduli, $\mathcal{M}$, a metric structure and discussed the wall among the 3 manifolds. The discriminant of $\mathcal{M}$ is rather awkward. But the top dimensional portion of the wall is clear.

- Principal wall of $\mathcal{K}$ consists of the 3 manifolds with one normal singular point.

Here the normal singular point is modeled as the quadratic:

$$x^2 + y^2 + z^2 - w^2 = 0$$

$$x^2 + y^2 - z^2 - w^2 = 0$$

Or put it more topologically, the two manifolds which share the wall are different by one surgery. And just as Vassiliev,

- Manifolds with finite normal singular points contribute to the top dimensional topology of the discriminant.

What make the Vassiliev invariants important is that it is computable. That is because he could decode the top dimensional homology of discriminant into an algebraic-combinatorial structure called "diagrams" (which is proved to be equivalent to the Feymann diagrams used by Bar Nartan in discussing the perturbative Chern Simon functional, the equivalence of the Vassiliev theory[B]). Actually the whole study of the finite type knot invariants is based on various kind of "good" diagrams. Again we faced the difficulty that knots is homeomorphic to each other, but 3 manifolds are not. The very best, if not the unique, choice is the S^3+ framed links description of the 3 manifolds. And

- The wall crossing produce an algebraic and combinatorial object: multiple diagram.

The 4-T relation which is the most fundamental relation among Vassiliev diagrams. is not clear here. We are pretty sure part of them should come from the first order degeneration of the singularity. But it took lots of time to make clear what it means in the multiple diagrams. In spite of the 4-T relations in the multiple diagrams, we still cannot get rid of other combinatorial possibility. And we are not so sure about that they are the only possible degeneration. One way out of this difficulty is going back to algebraic formalism. But we would like to know more about the singularities at this moment.

Besides, we still have several difficulties that are only partially studied:

(1) Unless we give an "absolute" coordinate of $\mathcal{K}$ (i.e. trace all surgery from the very beginning standard S^3, we cannot get a consistent theory of the multiple diagrams.

(2) The correspondence between the components of $\mathcal{K}$ to the set of 3 manifolds is not one to one. Actually it is highly depend on the topology of the diffeomorphism group of the manifolds.

(3) The surgery diagram (i.e. the framed links) is not good enough to make the situation in (2) clear.

That means if one want to make this approach work, a subtler Kirby move criterion is needed. And we have to quotient those equivalence in order to get a meaningful 3 manifolds invariant. This explains why the approach of general finite 3 manifold invariants are more in a combinatorial flavor than in a topological one. The typical example is the universal LMO invariants[] which is defined in the combinatorial level and is quotient by the two Kirby moves[K]. Under such "strong" restriction, the LMO invariant behaves very poorly when the first Betti number of the manifold is large (0 when $b_1(M) > 3$) .

Recently there is another approach of finite type invariants defined by Cochran and Melvin[CM] which is topological and very similar with us (they use the cobordism as the wall). Since they trickily relax the restriction of the wall condition but still capture the complexity discussed above, their invariants are good: it is still "finite type" in their sense; it covers the Ohtsuki's invariants in homological sphere; and it is more complicated when the first Betti number is larger. But the CM invariants are not easy to calculate and are lack a clear algebraic structure. Actually it doesn't fit in some theories generalize the original Ohtsuki's definition.

So as a candidate of Vassiliev's topological, the CK invariants is somewhat still in the middle point of its progress. We would like to study more about it in the future.

2. Others

(1) As described in our annual report[On], all our work on constructing a good description of the image of the finite type 3 manifold invariants into Vassiliev invariants have been overpassed by the work by Kricker[Kr], Spence and Aitchison[KSA].

Our computation is completed although tedious. But the problem is that the Vassiliev diagram is somewhat not that natural as Feymann diagram. Their description as a polynomial algebra generated by the even "wheel" in the Chinese character algebra is indeed provide deeper insight.

But the problem why it is the Alexander-Conway polynomials is still mysterious.

(2) We have studied the "Kontsevich" integral for our or CM invariants for a while but get in vein. The difficulty lies in that even in the knot theory situation it is still not very clear why Kontsevich integral works from purely topological wall crossing point of view. Actually this is the most troublesome problem when Vassiliev

theory appeared. The resolution depend on a good description of the algebraic structure of the diagram algebra. But from this approach, the most natural one is the LMO invariant. As we mentioned above, it cannot be the correct "integral" in the topological approach of the finite type 3 manifold invariants.

四、計劃成果自評

Our goal is roughly done in the sense that we make clear what a topological approach of the finite type 3 manifold invariants should be. But combinatorial difficulties are always around. Besides, the Cochran-Melvin invariants, which is slightly different from ours, provide a good topological theory already.

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