行政院國家科學委員會專題研究計畫成果報告

圖之同調理論

計畫類別：☑個別型計畫　　□整合型計畫

計畫編號：NSC 90—2115—M002—003

執行期間：90年 8月 1日至 91年 7月 31日

個別型計畫：計畫主持人：楊樹文
共同主持人：

整合型計畫：總計畫主持人：
子計畫主持人：

註：整合型計畫總報告與子計畫成果報告請分開編印各成一冊，彙整一起繳送國科會。

處理方式：☑可立即對外提供參考
（請打ｖ）□一年後可對外提供參考
□兩年後可對外提供參考
（必要時，本會得展延發表時限）

執行單位：國立台灣大學

中華民國 91年 10月 23日

中文摘要

我們考慮的圖(graph)都是含有一組特殊的點作為圖之頂點的一部分。這些點叫做圖的基點，如此的圖叫做有基的圖。考慮所有這樣的圖可形成一個鏈複體(chain complex)。其同調群恰是某一重要空間的上同調群。而這個結果是與"結理論"中的"零-奇異"(zero-anomaly)相關的.

英文摘要

We consider the graphs whose vertices contain a particular set of points. All of these graphs form a chain complex, the associated homology are called the graph homology. We show that the homology of this chain complex is isomorphic to the cohomology of configuration spaces of distinct points in Euclidean spaces. This result is related the zero-anomaly problem of perturbative Chern-Simons theory.

Graph Homology

Su-Win Yang

Abstract

We define chain complexes of graphs for which a particular set of m points are part of the vertices and show that the homology of the chain complex is isomorphic to the cohomology of configuration spaces of m distinct points in Euclidean spaces. This can lead to some important results of braid and knot invariants.

A graph is an abstract 1-dimensional simplicial complex, that is, a set of vertices and a set of edges, each edge consists of two distinct vertices.

We shall define chain complexes of modules (or vector spaces), each module is freely generated by a set of graphs. Similar to the simplexes in simplicial homology, the graphs need orientations. There are different methods to define the orientation of graph, one of the simplest way is to choose a linear order for the edges, two orders represent the same orientation if they are different by an even permutation. For a graph, a simplicial isomorphism from the graph to itself is said to be an automorphism, if the simplicial isomorphism preserves all structures which are assigned to the graph; the set of all automorphisms forms a group which is called the automophism group of the graph. If there is an automorphism of the graph, which reverses the orientation of the graph, then this graph is said to be non-orientable. The non-orientable graphs are considered as zero in the chain complex, when we use the real number as the coefficients.

Next important thing for chain complexes is the boundary operator. For a graph and an edge of the graph, we can contract this edge to a vertex and get a quotient graph; the summation of all such quotient graphs multiplied by a proper sign is defined as the boundary of the graph. Suppose we use the linear orders of edges as the orientation and Γ is a graph with orientation, then the boundary $\partial(\Gamma) = \sum(-1)^j \partial^{(j)}(\Gamma)$, where $\partial^{(j)}(\Gamma)$ is the quotient graph of Γ by contracting the j-th edge, with orientation the restriction order. Similar to the boundary operator in simplicial homology, $\partial(\partial(\Gamma)) = 0$, it is the only geometric property of a chain complex.

Remark: In Perturbative Chern-Simons theory, the graphs represent differential forms and the boundary operator is exactly part of exterior differentiation of the associative differential forms. Thus the above homology theory of graphs is usually called the graph cohomology as in Bott and Cattaneo []. But, in spirit, it is a homology theory and is dual to the cohomology theory of differential forms by the Stoke's Theorem.

Based graphs

We always assume the graphs with some fixed points as the vertices, such vertices are called the base points of the graphs and the graphs are called the based graphs.

Suppose $x_1, x_2, \cdots, x_m$ are m distinct points. A graph Γ is said to be a graph based on the ordered set $(x_1, x_2, \cdots, x_m)$, if the points $x_1, x_2, \cdots, x_m$ are part of vertices of Γ. The points $x_1, x_2, \cdots, x_m$ are called the base points of Γ.

Notations:

(i) We use Γ, Γ', Γ_1, Γ_2 to denote the graphs.

(ii) For a graph Γ, $V(\Gamma)$ denotes the the set of all vertices of Γ and $\mathcal{E}(\Gamma)$ denotes the set of all edges in Γ. Thus, for any $E \in \mathcal{E}(\Gamma)$, $E = \{v, w\}$, $v, w \in V(\Gamma)$, and $v \neq w$.

(iii) The vertices other than the base points are called the inner vertices of Γ.

Equivalence of based graphs

Suppose Γ_1 and Γ_2 are two graphs based on $(x_1, x_2, \cdots, x_m)$. A bijection $f : V(\Gamma_1) \longrightarrow V(\Gamma_2)$ is an equivalence of based graphs, if $f(x_i) = x_i, i = 1, 2, \cdots, m$, $f(E) \in \mathcal{E}(\Gamma_2)$, for $E \in \mathcal{E}(\Gamma_1)$, and $f^{-1}(E') \in \mathcal{E}(\Gamma_1)$, for $E' \in \mathcal{E}(\Gamma_2)$. (The last two conditions on the edges are the conditions for the bijection f to be a simplicial isomorphism.)

If two graphs are equivalent, it is hard to distinguish one from the other. Thus we need only to choose one graph from each equivalence class of graphs, or just consider the whole equivalence class instead of the particular graph. There are a few assumptions which are crucial to our result. Under these restrictions on the graphs, there are only finite number of equivalence classes of based graphs.

Asssumptios:

(i) **Valency Assumption:** The valency of a vertex in a graph is the number of edges which contain the vertex. Now, we assume that every inner vertices of a based graph are of valency at least 3. For the base points x_i, there is no restriction on the valency, it could be 0 or any positive integer.

(ii) **Order Restriction:** The order of a graph is defined as the number of its edges minus the number of its inner vertices. The defintion of order makes sence only under the above Valency Assumption.

Because the order is invariant under edge contraction, we usually fix the order of graphs in a chain complex.

Proposition Under the valency assumption and the order restriction, we have only finite number of equivalence classes of graphs based on $(x_1, x_2, \cdots, x_m)$.

Proof: First, we show that there are only finite graphs in which the inner vertices are all of valency exact 3.

Assume the order of graph is n.

Let s_i denote the valency of the base point x_i, $i = 1, 2, \cdots, m$, $s = s_1 + s_2 + \cdots + s_m$, r denote the number of inner vertices and k be the number of edges.

Then $s + 3r = 2k$.

Thus $r \leq s + r = 2k - 2r = 2n$,

and hence, $k = n + r \leq 3n$.

Up to equivalence, we may assume that all the graphs have the vertices in the set $\{x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_{2n}\}$. Thus the possible graphs is finite.

From such "trivalent" graphs, we can have all the order n graphs by contracting the edge a finite number of times, and each "trivalent" graph can only produce finite number of graphs. Thus all possible order n graphs are finite.

Chain complex of graphs

All graphs satisfy the valency assumption: the valency of inner vertices are at least three.

We also need a number, the degree of graph. Suppose the vertices of graphs are points of α-dimensional Euclidean space. (We may say that the graphs are on $\mathbb{R}^\alpha$.) Each edge gives a $(\alpha - 1)$-form and each inner vertex can move in $\mathbb{R}^\alpha$. If Γ has k edges and r inner vertices, then the integration of the wedge of the k $(\alpha - 1)$-form on the $r \times \alpha$-dimensional configuration space produces a differential form of degree $k \times (\alpha - 1) - r \times \alpha$. But the degree is only good for cohomology theory. To get a homology theory, we define the degree as $-k \times (\alpha - 1) + r \times \alpha$. For simplicity, our main consideration is the graphs in $\mathbb{R}^2$. Thus we give the following definition.

Definition: (Degree of graph) The degree of a based graph is the double of the number of its inner vertices minus the number of its edges.

Definition: For a positive integer m and an integer i, let C_i^m be the vector space over the real number generated by all oriented degree i graphs based on $(x_1, x_2, \cdots, x_m)$, modulo the following **Orientation-equivalence relation**:

> Two oriented graphs having an orientation preserving equivalence between them are considered as the same element in C_i^m; if the equivalence is orientation reversing, one is equal to the other multiplied by a negative sign.

Orientation systems

There are two different orientation systems, including the linear order of edges mentined previously.

Orientation system (I): (Linear order of edges)

It is the orientation system used in the proof of the theorems in the paper.

Suppose a based graph Γ has k distinct edges $E_1, E_2, \cdots, E_k$. Consider a linear order $(E_1, E_2, \cdots, E_k)$, all the informations of the graph are contained in $(E_1, E_2, \cdots, E_k)$. Thus we still use $(E_1, E_2, \cdots, E_k)$ to denote the oriented graph, and also use the same notation to denote the corresponding element in C_i^m. Interchanging the positions of two edges in a linear order set, we get the negative element. Thus, if $E_j = E_{j'}$, for some $1 \le j < j' \le k$, then $(E_1, E_2, \cdots, E_k) = 0$; ordinarily, we do not meet such an oriented graph, but it does happen on the "degenerate boundary" of a graph discussed in the follwing section.

If $f : V(\Gamma) \longrightarrow V(\Gamma')$ is an equivalence of based graphs from Γ to Γ', then $(f(E_1), f(E_2), \cdots, f(E_k))$ is an oriented graph and

$$(f(E_1), f(E_2), \cdots, f(E_k)) = (E_1, E_2, \cdots, E_k) .$$

When $\Gamma = \Gamma'$, f is an automorphism and $(f(E_1), f(E_2), \cdots, f(E_k))$ is a permutation of $(E_1, E_2, \cdots, E_k)$; if the permutation is odd, then the above equality implies that $(E_1, E_2, \cdots, E_k) = 0$ in C_i^m. In this situation, Γ is said to be non-orientable.

Orientation system (II): (Linear order of vertices together with directions on every edges)

If the graphs are on $\mathbb{R}^3$, we should get this orientation system from the differential forms. (For the graphs on $\mathbb{R}^2$, we get the Orientation system (I).)

For an orientation of Γ, we need to choose a linaer order of the vertices of Γ and directions on every edges of Γ. Two such orientations for Γ are the same , if the total number of changes in the order of vertices and the directions of the edges is an even number. Because the direction of an edge is also an order of the two endpoints, we may get the order by restricting the linear order of vertex set. Thus the linear order $(v_1, v_2, \cdots, v_s)$ of vertices can determine an orientation, denoted by $[v_1, v_2, \cdots, v_s]$, in the Orientation system (II).

What happen when interchanging the positions of two vertices? If $\{v_1, v_2\}$ is an edge of Γ, then

$$[v_2, v_1, \cdots, v_s] = [v_1, v_2, \cdots, v_s] \ ;$$

if $\{v_1, v_2\}$ is not an edge of Γ, then

$$[v_2, v_1, \cdots, v_s] = -[v_2, v_1, \cdots, v_s] \ .$$

We may also have the non-orientable based graph in the Orientation system (II).

Remark: Because the two orientation systems have different non-orientable based graphs for the same numbers m and i, C_i^m can not be the same vector space in the two systems. But the main results of the paper hold in both systems. (The proofs are also completely similar in both systems.)

The boundary operator

To obtain a chain complex, we need a boundary operator from C_i^m to C_{i-1}^m. For both orientation systems, the boundary operator can be defined.

Here we define it only in Orientation system (I). For Orientation system (II), please see Bott and Cattaneo [].

The boundary operator is the sum of edge-contractions, but the edges consisting of two base points can not be contracted, such edges shall be shown to have essential contributions to the graph homology. Thus we need the following definitions.

Definition: (Basic edges) Suppose Γ is a graph based on $(x_1, x_2, \cdots, x_m)$. An edge E is said to be a basic edge, if E consists of two base points, say, x_{j_1}, x_{j_2}, $1 \le j_1 < j_2 \le m$; otherwise, it is called the non-basic edge.

Thus a non-basic edge may consists of two inner vertices, or, a base point together with an inner vertex.

Suppose Γ is a graph based on $(x_1, x_2, \cdots, x_m)$ and $(E_1, E_2, \cdots, E_k)$ is a linear order of the edges of Γ.

For each non-basic edge $E_j, 1 \le j \le k$, let $\pi_j : V(\Gamma) \longrightarrow V(\Gamma)/E_j$ denote the quotient map. Then $(\pi_j(E_1), \pi_j(E_2), \cdots, \pi_j(E_{j-1}), \pi_j(E_{j+1}), \cdots, \pi_j(E_k))$ is a linear order of edges of the quotient graph Γ/E_j.

Convention: If the edge E_j consists of a base point x_l and an inner vertex v, then we identify the quotient point $\pi_j(x_l)(=\pi_j(v))$ with the base point x_l.

Thus the quotient graph Γ/E_j still be a graph based on $(x_1, x_2, \cdots, x_m)$ and $(\pi_j(E_1), \pi_j(E_2), \cdots, \pi_j(E_{j-1}), \pi_j(E_{j+1}), \cdots, \pi_j(E_k))$ is the associated oriented graph; it is the j-th boundary of $(E_1, E_2, \cdots, E_k)$ as in the following notation.

Notation: If E_j is a non-basic edge of Γ,

$$\partial^{(j)}(E_1, E_2, \cdots, E_k) = (\pi_j(E_1), \pi_j(E_2), \cdots, \pi_j(E_{j-1}), \pi_j(E_{j+1}), \cdots, \pi_j(E_k)) \quad .$$

If E_j is a basic edge of Γ, $\partial^{(j)}(E_1, E_2, \cdots, E_k) = 0$.

Now, we can define the boundary operator

$$\partial : C_i^m \longrightarrow C_{i-1}^m$$

by the formula

$$\partial(E_1, E_2, \cdots, E_k) = \sum_{j=1}^{k}(-1)^j \partial^{(j)}(E_1, E_2, \cdots, E_k)$$

For a non-basic edge E_j, $\partial^{(j)}(E_1, E_2, \cdots, E_k)$ usually is not equal to zero except that the graph Γ/E_j is non-orientable or the edge E_j is part of a triangle, as described below.

Degenerate boundary (Triangular edge)

$E_j = \{v, w\}$ is said to be a triangular edge, if there is a vertex u of Γ such that both $\{u, v\}$ and $\{u, w\}$ are edges of Γ. For the edge E_j, we consider the boundary $\partial^{(j)}(E_1, E_2, \cdots, E_k)$. $\pi_j(\{u, v\}) = \pi_j(\{u, w\})$. When interchanging the two edges, we have the equality

$$\partial^{(j)}(E_1, E_2, \cdots, E_k) = -\partial^{(j)}(E_1, E_2, \cdots, E_k) \ .$$

Thus $\partial^{(j)}(E_1, E_2, \cdots, E_k) = 0$.

Therefore, for any positive integer m, we have the chain complex

$$\mathcal{C}^m = \{C_i^m, \quad \partial : C_i^m \longrightarrow C_{i-1}^m, i = \cdots, -2, -1, 0, 1, \cdots\}$$

Our main result is the following theorem.

Theorem A Let $C(m, \mathbb{R}^2) = \{(z_1, z_2, \cdots, z_m) \in \prod_m \mathbb{R}^2 : z_p \neq z_q, \text{ for } p \neq q\}$, the configuration space of m distinct points in the plane. Then

$$H_i(\mathcal{C}^m) \cong H^{-i}(C(m, \mathbb{R}^2), \mathbb{R})$$

where $H^{-i}(C(m, \mathbb{R}^2), \mathbb{R})$ is the cohomology group of the space $C(m, \mathbb{R}^2)$ with the coefficient $\mathbb{R}$ at the dimension $-i$. ∎

Thus $H_i(\mathcal{C}^m)$ is non-zero only at the degree ≤ 0.

Remark: The homotopy structure of $C(m, \mathbb{R}^2)$ is easy to describe, this space is homotopy equivalent to the product space $X_{m-1} \times X_{m-2} \times \cdots \times X_1$, where $X_p = S^1 \vee S^1 \vee \cdots \vee S^1$, the wedge of p copies of S^1, S^1 is the unit circle. Thus we can compute the graph homology easily by this theorem. We give some applications in the following.

Splitting $\mathcal{C}^m$ by the order restriction

As mention before, the order of a graph is the number of edges minus the number of inner vertices and it is invariant under the boundary operator.

Let $\mathcal{C}^{m,n} = \{C_i^{m,n}\}$ be the subchain complex of $\mathcal{C}^m$ generated by the order n oriented graphs based on $(x_1, x_2, \cdots, x_m)$. Then

$$\mathcal{C}^m = \bigoplus_n \mathcal{C}^{m,n}$$

We have shown that there are only finite equivalence claases for a fixed number of base points m and a fixed order n. Thus $\mathcal{C}^{m,n}$ is a vector space of finite rank (finite dimension), $\mathcal{C}^m$ can not be a finite rank vector space.

There are some trivial examples:

(i) For any positive integer m, the unique graph of order 0 is the graph Γ_0 without any edge ($\mathcal{E}(\Gamma_0)$ is empty), it is in degree 0. Thus $C_0^{m,0} = \mathbb{R}$, $C_i^{m,0} = 0$, for $i \neq 0$, and the homology $H_i(\mathcal{C}^{m,0}) = C_i^{m,0}$, for all degree i.

(ii) The graph of order 1 is a graph with one edge and no inner vertex. Thus all the order 1 graph are of degree -1. $H_i(\mathcal{C}^{m,1}) = C_i^{m,1} = 0$, except $i = -1$.

(iii) Consider the case $m = 2$. The unique graph of order 1 is the graph with the unique basic edge $\{x_1, x_2\}$. Thus $H_i(\mathcal{C}^{2,1}) = C_i^{2,1} = 0$, for $i \neq -1$, and $H_{-1}(\mathcal{C}^{2,1}) = C_{-1}^{2,1} = \mathbb{R}$.

Consider the "simplest" configuration space of 2 points, $C(2, \mathbb{R}^2)$, it is homotopy equivalent to S^1. By the theorem, $H_i(\mathcal{C}^2) = 0$, for $i \neq -1$, and $H_{-1}(\mathcal{C}^2) = \mathbb{R}$. The results $H_{-1}(\mathcal{C}^{2,1}) = \mathbb{R}$ and $H_0(\mathcal{C}^{2,0}) = \mathbb{R}$ imply that

$H_i(\mathcal{C}^{2,n}) = 0$, for all order $n \geq 2$ and all degree i. This simple fact is related to the problem of zero-anomaly in the perturbative Chern-Simons theory for knot invariant.

Using the same way, we can compute all homology of $\mathcal{C}^{m,n}$ easily. For example, $H_{1-m}(\mathcal{C}^{m,m-1})$ has rank $(m-1)!$, it is the lowest degree in which the homology of $\mathcal{C}^m$ is non-zero, and $H_i(\mathcal{C}^{m,m-1}) = 0$, for $i \neq 1-m$. These are important to the theory of knot invariant.

Proof of Theorem A

We shall define a chain homotopy type linear map $\tau : C_i^m \longrightarrow C_{i+1}^m$ and consider the associated chain map $\lambda = \tau \circ \partial + \partial \circ \tau$ of $\mathcal{C}^m$. We can show that (1) the subchain complex $\mathrm{Ker}(\lambda) = \{x \in \mathcal{C}^m : \lambda(x) = 0\}$, the kernel space of λ, is chain homotopy equivalent to $\mathcal{C}^m$, and (2) $\mathrm{Ker}(\lambda)$ is equal to the tensor product of $\mathcal{C}^{m-1}$ and $\mathcal{E}^{m-1}$, $\mathcal{E}^{m-1}$ is the dual of $H^*(X_{m-1})$, as in Theorem A.

Definition of $\tau : C_i^m \longrightarrow C_{i+1}^m$

For any degree i graph Γ with orientation $(E_1, E_2, \cdots, E_k)$, let $\eta(\Gamma)$ be the graph Γ with an additioal vertex a and an additional edge $E'_{k+1} = \{a, x_1\}$. We assign $\eta(\Gamma)$ the orientation $(E_1, E_2, \cdots, E_k, E'_{k+1})$. Let ϕ be the permutation of the vertex set $V(\Gamma) \cup \{a\}$ interchanging a and x_1, that is, $\phi(a) = x_1, \phi(x_1) = a$, and $\phi(v) = v$, for any other vertices v.

Define $\tau(E_1, E_2, \cdots, E_k) = (-1)^{k+1}(\phi(E_1), \phi(E_2), \cdots, \phi(E_k), \phi(E'_{k+1}))$, the corresponding graph is denoted by $\tau(\Gamma)$. Then $\tau(\Gamma)$ is also a graph based on $(x_1, x_2, \cdots, x_k)$, with an additional inner vertex a and is simplicially isomorphic to $\eta(\Gamma)$. If the valency of x_1 in Γ is at least 2, then the valency of a in $\tau(\Gamma)$ is at least 3, and hence, the associated oriented graph $\tau(E_1, E_2, \cdots, E_k)$ is a qualified element in C_{i+1}^m, i is the degree of Γ; if the valency of x_1 in Γ is equal to 0 or 1, then $\tau(\Gamma)$ can not satisfy the valency assumption and we just define $\tau(E_1, E_2, \cdots, E_k)$ as 0 in C_{i+1}^m.

If Γ' is a graph equivalent to Γ, then $\tau(\Gamma')$ is also equivalent to $\tau(\Gamma)$ and it is straightforwad to prove the remaining well-defined property.

Because in the graph $\tau(\Gamma)$, the valency of x_1 is equal to 1, $\tau(\tau(E_1, E_2, \cdots, E_k))$ is always 0 in C_{i+2}^m. This proves the following lemma.

Lemma 1 The linear homomorphism $\tau \circ \tau : C_i^m \longrightarrow C_{i+2}^m$ is a zero-map.

Lemma 2 Suppose Γ is graph based on $(x_1, x_2, \cdots, x_m)$.
If Γ satisfies the following $\star$-condition:

$(\star)$: every edge containing x_1 is basic and the valency of $x_1 \leq 1$, then

$$(\tau \circ \partial + \partial \circ \tau)([\Gamma]) = 0 \;\; ;$$

if Γ does not satisfy the $\star$-condition, then

$$(\tau \circ \partial + \partial \circ \tau)([\Gamma]) = [\Gamma]$$

where $[\Gamma]$ denotes the graph Γ with some orientation. ∎

We prove Lemma 2 later and use it to prove Theorem A.

At first, we check that the linear homomorphism $\lambda = \tau \circ \partial + \partial \circ \tau$ is a chain map, that is, to show the equality $\lambda \circ \partial = \partial \circ \lambda$ as follows:

$$\lambda \circ \partial = (\tau \circ \partial + \partial \circ \tau) \circ \partial = \tau \circ \partial \circ \partial + \partial \circ \tau \circ \partial = \partial \circ \tau \circ \partial \;,$$

$$\partial \circ \lambda = \partial \circ (\tau \circ \partial + \partial \circ \tau) = \partial \circ \tau \circ \partial + \partial \circ \partial \circ \tau = \partial \circ \tau \circ \partial \;.$$

Let $\mathrm{Ker}(\lambda) = \{x \in \mathcal{C}^m : \lambda(x) = 0\}$, the kernel space of λ, and $\mathrm{Im}(\lambda)$ be the image space of λ. Then both $\mathrm{Ker}(\lambda)$ and $\mathrm{Im}(\lambda)$ are subchain complexes of $\mathcal{C}^m$.

By Lemma 2, $\mathrm{Ker}(\lambda)$ contains the linear subspace $\mathcal{D}_1$ of $\mathcal{C}^m$, generated by the set $\{[\Gamma] : \Gamma$ satisfies the $\star$-condition $\}$. We may also consider the linear subspace $\mathcal{D}_2$ of $\mathcal{C}^m$ generated by the set $\{[\Gamma] : \Gamma$ does not satisfy the $\star$-condition $\}$, then $\mathcal{C}^m = \mathcal{D}_1 \oplus \mathcal{D}_2$. Because λ is equal to 0 on $\mathcal{D}_1$ and is equal to the identity map on $\mathcal{D}_2$ (also by Lemma 2), λ is a projection map of $\mathcal{C}^m$, that is, satisfying the equality $\lambda \circ \lambda = \lambda$.

Of course, this leads to the result that $\mathcal{D}_1 = \mathrm{Ker}(\lambda)$ and $\mathcal{D}_2 = \mathrm{Im}(\lambda)$.

By Lemma 1 and a similar computation as above, $\lambda \circ \tau = \tau \circ \lambda$. Thus τ provides a chain homotopy between the identity map and the 0-map in the chain complex $\mathrm{Im}(\lambda)$, and hence, $H_*(\mathrm{Im}(\lambda)) = 0$.

This implies that $H_*(\mathcal{C}^m) \cong H_*(\mathrm{Ker}(\lambda))$.

We summarize the arguments above to the following proposition.

Proposition 3 Suppose $\mathcal{C} = \{C_i, \partial_i : C_i \longrightarrow C_{i-1}, i = \cdots, -2, -1, 0, \cdots\}$ is a chain complex ($\partial_{i-1} \circ \partial_i = 0,$), and $\tau_i : C_i \longrightarrow C_{i+1}, i = \cdots, -2, -1, 0, \cdots,$ are linear maps increasing the grade by 1 which also satisfy the condition of coboundary, $\tau_{i+1} \circ \tau_i = 0$. Furthermore, assume that the associative chain map of $\{\tau_i\}$, $\{\lambda_i = \tau_{i-1} \circ \partial_i + \partial_{i+1} \circ \tau_i : C_i \longrightarrow C_i,\ i = \cdots, -2, -1, 0, \cdots\}$ satisfies the condition of projection map, that is, $\lambda_i \circ \lambda_i = \lambda_i$, for all i.

Then the kernel subchain complex $\mathrm{Ker}(\lambda) = \{\text{kernel of } \lambda_i, \text{ for all } i\}$ has the homologies isomorphic to that of $\mathcal{C}$. ∎

For the different possible basic edge containing x_1, we split $\mathrm{Ker}(\lambda)$ into the subchain complexes which are isomorphic to $\mathcal{C}^{m-1}$.

Let $\mathcal{K}(1)$ be the subchain complex of $\mathrm{Ker}(\lambda)$ generated by all oriented graphs in which the valency of x_1 is 0.

For each j, $2 \leq j \leq m$, let $\mathcal{K}(1,j)$ be the subchain complex of $\mathrm{Ker}(\lambda)$ generated by all oriented graphs in which $\{x_1, x_j\}$ is the unique edge containing x_1.

Then $\mathrm{Ker}(\lambda) = \mathcal{K}(1) \oplus \mathcal{K}(1,2) \oplus \mathcal{K}(1,3) \oplus \cdots \oplus \mathcal{K}(1,m)$.

$\mathcal{K}(1)$ is exactly the chain complex of oriented graphs based on $(x_2, x_3, \cdots, x_m)$, it is canonically isomorphic to $\mathcal{C}^{m-1}$, and for other j, $\mathcal{K}(1,j)$ is isomorphic to $\mathcal{K}(1)$ with the elements decreasing the degree by 1.

To describe the structure precisely, for any positve integer p, let $\mathcal{E}^p$ be the chain complex defined by: for degree 0 and -1, $\mathcal{E}^p_0 = \mathbb{R}$, $\mathcal{E}^p_{-1} = \mathbb{R}^p$; for other degree i, $\mathcal{E}^p_i = 0$. The boundary operator in $\mathcal{E}^p$ are all the zero-map. $\mathcal{E}^p_i \cong H^{-i}(X_p, \mathbb{R})$, for all i.

Then $\mathrm{Ker}(\lambda) \cong \mathcal{E}^{m-1} \otimes \mathcal{K}(1) \cong \mathcal{E}^{m-1} \otimes \mathcal{C}^{m-1}$.

Thus $H_*(\mathcal{C}^m) \cong H_*(\mathcal{E}^{m-1} \otimes \mathcal{C}^{m-1}) \cong \mathcal{E}^{m-1} \otimes H_*(\mathcal{C}^{m-1})$.

By induction, we have

$$H_*(\mathcal{C}^m) \cong \mathcal{E}^{m-1} \otimes \mathcal{E}^{m-2} \otimes \cdots \otimes \mathcal{E}^1 \ ,$$

it is the isomorphism needed in Theorem A.

Proof of Lemma 2

Choose a linear order $(E_1, E_2, \cdots, E_k)$ for the edges of Γ. If Γ satisfies the $(\star)$-condition, then, for the non-basic edge E_j, E_j does not meet x_1 and Γ/E_j also satisfies the $(\star)$-condition. Thus, for the non-basic edge E_j, $\tau(\partial^{(j)}[\Gamma]) = \tau([\Gamma/E_j]) = 0$, and for the basic edge E_l, $\partial^{(l)}[\Gamma]$ is defined as 0: this concludes that

$$\tau(\partial[\Gamma]) = \Sigma_{i=1}^{k}(-1)^j(\tau(\partial^{(j)}([\Gamma]))) = 0 \ .$$

By the valency asoumption, $\tau([\Gamma]) = 0$, and hence, $(\partial \circ \tau + \tau \circ \partial)([\Gamma]) = \partial(\tau([\Gamma])) + \tau(\partial([\Gamma])) = 0$, this proves the first part of the main lemma.

For the second part, assume that the valency of x_1 in Γ is larger than 1, or, the unique edge containing x_1 is equal to $\{x_1, v\}$, for some inner vertex v.

(Case 1): Valency $(x_1) \geq 2$.

In this case, $\tau([\Gamma])$ is non-zero. Consider the orientation $(E_1, E_2, \cdots, E_k)$ for Γ. $\tau([E_1, E_2, \cdots E_k]) = (-1)^{k+1}(\phi(E_1), \phi(E_2), \cdots, \phi(E_k), \phi({E_{k+1}}'))$ (**Note:** ϕ is defined in the definition of τ). Thus

$$(\partial \circ \tau)(E_1, E_2, \cdots E_k) = (-1)^{k+1} \sum_{i=1}^{k+1} (-1)^j \partial^{(j)}(\phi(E_1), \cdots \phi(E_k)).$$

The last term in the summation above,

$$(-1)^{k+1} \cdot (-1)^{k+1} \partial^{(k+1)}(\phi(E_1), \phi(E_2), \cdots, \phi(E_k), \phi(E'_{k+1}))$$

is exactly equal to $(E_1, E_2, \cdots E_k)$, the oriented graph of Γ.

For $1 \leq j \leq k$, we should check that

$$(-1)^{k+1} \cdot (-1)^{k+1} \partial^{(j)}(\phi(E_1), \phi(E_2), \cdots, \phi(E_k), \phi(E'_{k+1}))$$

is equal to $-\tau((-1)^j \partial^{(j)}(E_1, E_2, \cdots, E_k))$. As in the definition of $\partial^{(j)}$, let $\pi_j : V(\Gamma) \to V(\Gamma)/E_j$ denote the quotient map.

$\partial^{(j)}(E_1, E_2, \cdots E_k) = (\pi_j(E_1), \cdots, \pi_j(E_{j-1})), \pi_j(E_{j+1}), \cdots, \pi_j(E_k))$.

Thus $\tau(\partial^{(j)}(E_1, E_2, \cdots E_k) =$
$(-1)^k(\phi(\pi_j(E_1)), \cdots, \phi(\pi_j(E_{j-1})), \phi(\pi_j(E_{j+1})), \cdots, \phi(\pi_j(E_k)), \phi(\pi_j(E'_k)))$.

On the other hand, to study $\partial^{(j)}(\phi(E_1), \phi(E_2), \cdots, \phi(E_{k+1}'))$, let $\overline{\pi_j} : V(\Gamma) \cup \{a\} \longrightarrow (V(\Gamma) \cup \{a\})/\phi(E_j)$ denote the quotient map, where a is the new inner vertex in the definition of $\tau(\Gamma)$.

Then $\partial^{(j)}(\phi(E_1), \cdots, \phi(E_k), \phi(E_{k+1}'))$
$= (\overline{\pi_j}(\phi(E_1)), \cdots, \overline{\pi_j}(\phi(E_{j-1})), \overline{\pi_j}(\phi(E_{j+1})), \cdots, \overline{\pi_j}(\phi(E_k)), \overline{\pi_j}(\phi(E_{k+1}')))$.
It is straightforward to find that $\phi(\pi_j(E_l)) = \overline{\pi_j}(\phi(E_l))$, for $1 \leq l \leq k, l \neq j$, and $\phi(E'_k) = \overline{\pi_j}(\phi(E_{k+1}'))$, which imply the equality
$-\tau((-1)^i\partial_i(E_1, E_2, \cdots, E_k))$
$= (-1)^{k+1}(-1)^j\partial^{(j)}(\phi(E_1), \phi(E_2), \cdots, \phi(E_k), \phi(E_{k+1}'))$,
and hence, we have

$$(\partial \circ \tau)(E_1, E_2, \cdots, E_k) = (E_1, E_2, \cdots, E_k) - (\tau \circ \partial)(E_1, E_2, \cdots, E_k) \ .$$

(Case 2): Valency(x_1) = 1.

There is some edge $E_s = \{x_1, v\}$, for some s, $1 \leq s \leq k$ and for some inner vertex v.

In this situation v can not be a base point, or else, Γ satisfies the ($\star$)-condition.

By the definition of τ, $\tau([\Gamma]) = 0$. For the integer $j \neq s, 1 \leq j \leq k$, x_1 is also of valency 1 in $\partial^{(j)}(\Gamma)$. Thus $\tau(\partial^{(j)}(\Gamma)) = 0$, for $j \neq s$. And it is easy to see that for the particular boundary $\partial^{(s)}(\Gamma)$, its τ-value, $\tau(\partial^{(s)}(\Gamma))$, is equivalent to the original graph Γ. Together with the orientation, we have

$\tau(\partial(E_1, E_2, \cdots, E_k))$
$= \tau((-1)^s(\pi_s(E_1)), \cdots, \pi_s(E_{s-1}), \pi_s(E_{s+1}), \cdots, \pi_s(E_k))$
$= (-1)^k(-1)^s(\phi(\pi_s(E_1)), \cdots, \phi(\pi_s(E_{s-1})), \phi(\pi_s(E_{s+1})), \cdots, \phi(\pi_s(E_k)), \phi(\pi_s(E'_k)))$.

In the equivalence of $\tau(\partial^{(s)}(\Gamma))$ and Γ, the edge $\phi(\pi_s(E'_k))$ is correspondent to E_s. When changing the position of $\phi(\pi_s(E'_k))$ to the original position of

E_s, we get an additional sign $(-1)^{k-s}$.

Thus $\tau(\partial(E_1, E_2, \cdots, E_k))$
$= (-1)^{k+s}(-1)^{k-s}(\phi(\pi_s(E_1)), \cdots, \phi(\pi_s(E_{s-1})), \phi(\pi_s(E'_k)), \phi(\pi_s(E_{s+1})), \cdots, \phi(\pi_s(E_k)))$,
which is exactly equal to $(E_1, E_2, \cdots, E_k)$.

That is,

$$\tau(\partial(E_1, E_2, \cdots, E_k)) = (E_1, E_2, \cdots, E_k) \ .$$

This completes the proof of the lemma.

Embed a subchain complex of $\mathcal{C}^m$ in $\Omega(C(m, \mathbb{R}^{2l}))$

Let $\Omega(C(m, \mathbb{R}^{2l}))$ be the cochain complex of all differential forms on $C(m, \mathbb{R}^{2l})$. We shall define a dual homomorphism sending a degree i oriented graph $[\Gamma]$ in $\mathcal{C}^m$ to a degree $-i$ differential form $\omega([\Gamma])$ in $\Omega(C(m, \mathbb{R}^{2l}))$.

Redefinition of degree: To consider the graph in $\mathbb{R}^{2l}$, we should change the definition of degree to the one depending on the dimension $2l$.
Degree of graph = (number of inner vertices)$\times 2l$ − (number of edges)$\times(2l - 1)$. ∎

In the following, the ordered set $(z_1, z_2, \cdots, z_m)$ is considered as a variable point in $C(m, \mathbb{R}^{2l})$.

Suppose Γ is a graph based on $(x_1, x_2, \cdots, x_m)$. Let $C(\Gamma)$ be the configuration space of all graphs together with equivalences from Γ to them, that is, $\{(g, \Gamma') :\ \Gamma'$ is a graph in $\mathbb{R}^{2l}$ and based on $(z_1, z_2, \cdots, z_m)$, $g : \Gamma \longrightarrow \Gamma'$ is a equivalence of graphs, $g(x_j) = z_j$, $j = 1, 2, \cdots, m.\}$, and $C(\Gamma, z_1, z_2, \cdots, z_m)$ be the subspace os $C(\Gamma)$, $\{(g, \Gamma') \in C(\Gamma) : \Gamma'$ is a graph based on $(z_1, z_2, \cdots, z_m)$ $\}$.

Then $C(\Gamma)$ a a fibre bundle over $C(m, \mathbb{R}^{2l})$ with the fibres $C(\Gamma, z_1, z_2, \cdots, z_m)$.

The element (g, Γ') is completely determined by the map of vertices $V(\Gamma) \longrightarrow \mathbb{R}^{2l}$, for simplicity, we also denote this map by g. And the space $C(\Gamma)$ can be thought as the space of the all injective maps $g : V(\Gamma) \longrightarrow \mathbb{R}^{2l}$.

Assume $E_1, E_2, \cdots, E_k$ are the edges of Γ, $E_j = \{v_j, w_j\}$, $j = 1, 2, \cdots, k$. For each edge E_j of Γ, let $\varphi_{E_j} : C(\Gamma) \longrightarrow S^{2l-1}$ denote the map

$$\varphi_{E_j}(g) = \frac{g(w_j) - g(v_j)}{|g(w_j) - g(v_j)|}$$

Choose a unit volume form ω_0 on S^{2l-1} which is invariant under the anti-podal map of S^{2l-1}. Consider the pull-back of ω_0 by φ_{E_j}, we get the $(2l-1)$-form $\varphi^*_{E_j}(\omega_0)$ on $C(\Gamma)$. The wedge $\wedge_{j=1}^k \varphi^*_{E_j}(\omega_0)$ is a $k(2l-1)$-form on $C(\Gamma)$. For the well-definedness of $\wedge_{j=1}^k \varphi^*_{E_j}(\omega_0)$, we need a linear order of edges, actually, an orientation of Γ. Thus , for an oriented graph $[\Gamma]$, we have a well-defined differential form $\wedge_{j=1}^{k}\varphi_{E_j}^{*}(\omega_0)$.

Let $\omega([\Gamma])$ denote the push-down of $\wedge_{j=1}^{k}\varphi_{E_j}^{*}(\omega_0)$ to the base space $C(m,\mathbb{R}^{2l})$, that is, the fibre integration of $\wedge_{j=1}^{k}\varphi_{E_j}^{*}(\omega_0)$. Thus $\omega(\Gamma)$ is a differential form of degree $k\times(2l-1)-r\times 2l$ on the space $C(m,\mathbb{R}^{2l})$, where r is the number of inner vertices of Γ.

Definition: A based graph is said to be trivalent, if every inner vertices of the graph are of valency 3.

The following proposition is essentially from the work of Bott and Taubes [].

Proposition 4 If Γ is trivalent, then the exterior differentiation of $\omega(\Gamma)$ can be given by the following formula:

$$d\omega([\Gamma])=\sum_{j=1}^{k}(-1)^{j}\omega(\partial^{(j)}[\Gamma])$$

Although the map sending $[\Gamma]$ to $\omega([\Gamma])$ is a linear homomorphism from $\mathcal{C}^m$ to $\Omega(C(m,\mathbb{R}^{2l}))$, it may not be a map preserving the boundary operator except the trivalent graphs.

Let T_i^m be the subspace of C_i^m generated by all degree i trivalent graphs based on $(x_1,x_2,\cdots,x_m)$ and $K_i^m=\{x\in T_i^m:\partial''(x)=0\}$.

Part of boundary operator ∂ For an oriented graph $(E_1,E_2,\cdots,E_k)$, let $\partial'(E_1,E_2,\cdots,E_k)$ be the summation of $(-1)^j\partial^{(j)}(E_1,E_2,\cdots,E_k)$ over the integers j, $1\le j\le k$, for which E_j is a edge containing a base point; and let $\partial''(E_1,E_2,\cdots,E_k)$ be the summation of $(-1)^j\partial^{(j)}(E_1,E_2,\cdots,E_k)$ over the integers j, $1\le j\le k$, for which E_j is a edge consisting of two inner vertices. Thus both ∂' and ∂'' are linear maps from C_i^m to C_{i-1}^m, and $\partial=\partial'+\partial''$. But $\partial'\circ\partial'$ is not a zero-map, even in T_i^m.

The following trivial result is crucial.

Lemma 5 $\partial'' \circ \partial'' = 0$

Thus $\overline{\mathcal{C}}^m = \{C_i^m, \partial''\}$ is a chain complex. Because an oriented trivalent graph can not be in the ∂''-boundary of any graph, $K_i^m \cap \partial''(C_{i+1}^m) = \{0\}$, and hence, K_i^m is a subspace of $H_i(\overline{\mathcal{C}}^m)$. We hope that K_i^m could be the whole space $H_i(\overline{\mathcal{C}}^m)$. Whether it is true or not, the linear spaces $\{K_i^m$, for all $i\}$ with the boundary operator ∂' form a chain complex, denoted by $\mathcal{K}^m$.

Conjecture 6: $H_i(\overline{\mathcal{C}}^m) = K_i^m$, for all i. ∎

The main purpose of the conjectue is that it could imply the following conjecture.

Conjecture 7: $H_*(\mathcal{K}^m) \cong H_*(\mathcal{C}^m)$. ∎

This implies that using the differential forms $\omega([\Gamma])$ produced from the elements in K_i^m, we have a subcochain complex of $\Omega(C(m, \mathbb{R}^{2l}))$, whose cohomology is equal to that of $C(m, \mathbb{R}^{2l})$. And this is the fundamental theorem to get the braid invariant by K. T. Chen's theory of iterated integral.

Theorem C If $H_i(\overline{\mathcal{C}}^m) = K_i^m$, for all i, then $H_i(\mathcal{K}^m) \cong H_i(\mathcal{C}^m)$, for all i.∎

The proof depends on a spectral sequeence argument. At first, we define a double sequence of spaces.

Suppose Γ is a based graph and v is an inner vertex of Γ. The deficit number of v is defined as the number (3 $-$ the valency of v) and the total deficit number of Γ is the sum of deficit numbers of every inner vertices. For example, the total deficit number of trivalent graph is zero.

Definition: Let $C_{p,q}^m$ is the linear subspace of C_i^m, $i = p + q$, generated by all oriented graphs with the total deficit number q.

It is also convenient to call the number q the inner degree and the number p the base degree. Then the base degree of Γ is equal to $(3-2l)\times$ order$(\Gamma) - s$, where s is the sum of valency numbers of every base points in Γ. When the order is fixed, the total valency number s is an essential part of the base degree.

Proposition The boundary operator ∂'' sends $C^m_{p,q}$ into $C^m_{p,q-1}$. ∎

Proof: Suppose $[\Gamma]$ is an oriented graph. $\partial''([\Gamma])$ is the summation of oriented quotiet graphs $[\Gamma/E]$ over the edges E consisting of two inner vertices. Thus these graphs Γ/E have the same base degree as that of Γ. Thus $\partial''([\Gamma])$ is an element in $C^m_{p,q-1}$. This completes the proof.

Therefore, we have the chain complex $\overline{\mathcal{C}}^m_p = \{C^m_{p,q}, \partial''\}$, (Note: $C^m_{p,0} = T^m_p$, $C^m_{p,q} = 0$, for $q > 0$.) and its top dimensional homology $H_0(\overline{\mathcal{C}}^m_p)$ is the space K^m_p. The other boundary operator ∂' are only defined for the top dimensional space $C^m_{p,0}$, that is, $\partial' : C^m_{p,0} \longrightarrow C^m_{p-1,0}$. For other spaces $C^m_{p,q}$, $q < 0$, ∂' sends it into the direct sum of $C^m_{p-1,q}$, $C^m_{p-2,q+1}$, $\cdots$. We can summarize thses in the following diagram.

$$
\begin{array}{cccccccc}
K^m_p & \cdots & \to & C^m_{p,0} & \xrightarrow{\partial''} & C^m_{p,-1} & \xrightarrow{\partial''} & \\
\downarrow{\scriptstyle \partial'} & & & \downarrow{\scriptstyle \partial'} & & & & \\
K^m_{p-1} & \cdots & \to & C^m_{p-1,0} & \xrightarrow{\partial''} & C^m_{p-1,-1} & \xrightarrow{\partial''} & \\
\downarrow{\scriptstyle \partial'} & & & \downarrow{\scriptstyle \partial'} & & & & \\
K^m_{p-2} & \cdots & \to & C^m_{p-2,0} & \xrightarrow{\partial''} & C^m_{p-2,-1} & \xrightarrow{\partial''} & \\
\downarrow{\scriptstyle \partial'} & & & \downarrow{\scriptstyle \partial'} & & & &
\end{array}
$$

From the diagram we find that for any integer p, the spaces $C^m_{p',q}$, $p' \leq p$, form a subchain complex $F_p(\mathcal{C}^m)$ of $\mathcal{C}^m$, that is the subspace generated by all oriented graphs with the base degree less than or equal to p. The degree i space of $F_p(\mathcal{C}^m)$ is the direct sum of $C^m_{p',i-p'}$, for all $p' \leq p$. Then all these subcomplexes $F_p(\mathcal{C}^m)$, $p = \cdots, -1, 0, 1, \cdots$, form a increasing filtration of $\mathcal{C}^m$ and $\overline{\mathcal{C}}^m_p$ is exactly equal to the quotient chain complex $F_p(\mathcal{C}^m)/F_{p-1}(\mathcal{C}^m)$.

Proof of Theorem C:

The assumption of the theorem that $H_i(\overline{\mathcal{C}}^m) = K_i^m$ for all i implies that $H_q(\overline{\mathcal{C}}_p^m) = 0$ for all $q \neq 0$. Thus, the spectral sequence associated with the filtration $F_p(\mathcal{C}^m)$, $p = \cdots, -1, 0, 1, \cdots$, is degenerae, which implies the conclusion of the theorem that $H_i(\mathcal{K}^m) \cong H_i(\mathcal{C}^m)$, for all i.

References

[1]

[2]

Department of Mathematics
National Taiwan University
Taipei, Taiwan
E-mail: swyang@math.ntu.edu.tw