行政院國家科學委員會補助專題研究計畫成果報告

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※　　樣本函數解析（V）　　※

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計畫類別：☑個別型計畫　　☐整合型計畫

計畫編號：NSC90－2115－M－002－004

執行期間：90 年 08 月 01 日至 91 年 07 月 31 日

計畫主持人：謝南瑞

共同主持人：

本成果報告包括以下應繳交之附件：

☑赴國外出差或研習心得報告一份

☐赴大陸地區出差或研習心得報告一份

☑出席國際學術會議心得報告及發表之論文各一份

☐國際合作研究計畫國外研究報告書一份

執行單位：台大數學系

中　華　民　國　91 年　9　月 26　日

行政院國家科學委員會專題研究計畫成果報告

具有某些自相似性之隨機過程的樣本函數解析（V）

計畫編號：NSC90－2115－M－002－004

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一、中文摘要

這是同一研究主題第五年的精簡報告。

關鍵詞：分生過程、布朗運動，多重碎形，豪氏維度。

Abstract

This is a brief report of the fifth year of a same research subject.

Keywords: braching processes,Brownian motions, multifractals, Hausdorff dimensions.

二、本年進度

在本年中，我研究分生過程與布朗運動的樣本函數。我考慮〝超臨界〞的 Galton-Watson 分生過程，相對應的樹枝結構有一個〝極限值〞，在此極限集上有一個自然的分生測度。我也考慮布朗運動的樣本函數，在任一水平集與軌跡上，都有自然的測度存在，水平集上者為局部時測度，軌跡上者為佔據測度。我們討論上述諸測度的多重碎形性質。

三、成果自評

對分生測度而言，我們將先前與 S. J. Taylor 所共同獲得之結果（見前一年之成果報告）中，所說的豪氏雜度下界的證明，加以推廣。我們考慮了所謂對數型多重碎形譜。對隨機 Gibbs 測度，我有與 FAN AI-HUA 的一篇共同著作，並獲邀在巴西里約熱內盧數學物理國際會議上（見附件）宣讀。

附件1：赴國外出差心得報告

我於2001年3月28日~4月5日赴大陸武漢大學數學學院與中國科學院武漢物理與數學研究所訪問。這是應該兩單位的特聘教授(長江學者)FAN AI-HUA的邀請而成行。我與范教授深入了討論樹枝結構上的分析與機率問題，且安排上述兩單位之年青且研究表現好的工作者到台北作博士後或客座專家，期能有共同研究成果。我也做了兩場報告，有關Galton-Watson樹枝上的多重分析與有關布朗運動的碎形研究。

附件 2：出席國際會議心得報告

（附發表之論文）

我於 2002 年 8 月 15 ~ 30 日赴巴西里約熱內盧 IMPA 研究所，出席由該所主辦之物理數學國際研討會。此會議也獲得美國 NSF 補助，主要是美國與巴西當地工作者，也有一些由歐洲（特別是法國）學者與會。我在會議中報告了有關隨機 Gibbs 測度的多重問題，甚引起主持人 V. Sidokowish 的興趣，並就他所專長的格子點滲流（percolation）的碎形問題，交換心得。會中另有多人提到隨機薛丁格算子的問題，這也是近年我國一些分析、機率工作者甚重視的主題。

Multifractal Spectra of Certain Random Gibbs Measures

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Abstract

We consider a random Gibbs measure $\mu(d\eta, \omega)$ generated by a certain sequence of random functions $g_n(\eta, \omega)$ on the configuration space of one-dimensional system of lattice particles. Under concrete conditions, we prove that, for almost sure ω, $\mu(d\eta, \omega)$ has a non-random non-trivial multifractal spectrum. The basic idea is to relate our situation to random matrix products discussed in Ruelle(1979).

KEY WORDS : Gibbs measure; pressure function; multifractal spectrum

1 Introduction

We consider a one-dimensional system of lattice particles. The configuration space is then

$$\Gamma = \{\eta = (\eta_1, \eta_2, \cdots) : \quad \eta_i = 0, 1\},$$

and Γ is topologized by the metric

$$d(\eta, \zeta) = 2^{-\inf\{n: \eta_n \neq \zeta_n\}}.$$

In this paper, we shall consider a random Gibbs measure $\mu(d\eta,\omega)$ on Γ, which is regarded as an infinite product defined by a certain sequence of random generating functions $g_n(\eta,\omega)$. Under some concrete conditions, we prove that, for almost sure ω, $\mu(d\eta,\omega)$ has a non-random non-trivial multifractal spectrum, and this spectrum can be calculated explicitly. The basic idea is to relate our situation to random matrix products discussed in Ruelle(1979). We remark that the mathematical rigid derivation of multifractal spectra of (deterministic) Gibbs measures is still a challenging problem. Our results in this paper illustrate fruitfully to tackle this problem via random aspects and methods. We refer the readers to Ruelle(1978) for the general background of Gibbs measures, and to Brown–Michon–Peyrière(1992) and Fan(1994,1997) for the multifractal analysis related to Gibbs measures.

The paper is organized as follows. In §2, we state some general theory for random Gibbs measure and its multifracal analysis. In §3, we prove the main theorem which describes the multifracal spectrum of the random Gibbs measure for which the generating g's depend on finite sites. In §4, we proceed some close calculations in the case where g's depend only on two sites (the Ising model). In concluding remarks, we mention some aspects on multifractal analysis of random measures which are potentially fruitful for the future investigation.

Acknowledgements The main results of this paper were obtained while the first author visited National Taiwan University, under a grant from National Science Council of Taiwan.

2 General theory

We begin with some notation. The configuration space Γ and its defining metric d are those defined in §1. An interval $I_n(\eta)$ in Γ containing $\eta \in \Gamma$ is

$$I_n(\eta) = \{\zeta \in \Gamma : \quad \zeta_i = \eta_i, \quad i = 1, \cdots, n\}.$$

The shift transformation on Γ is denoted by Θ,

$$(\Theta\eta)_n = \eta_{n+1}, \quad n = 1, 2, \cdots.$$

The reference measure on Γ, denoted by μ_0, is the direct product of symmetric Bernoulli distributions on each site,

$$\mu_0(d\eta) = \prod_{n=1}^{\infty} \Big(\frac{1}{2}\delta_{\eta_n=0} + \frac{1}{2}\delta_{\eta_n=1}\Big).$$

Fix an $\alpha : 0 < \alpha < 1$, we consider the class H_+^α of all functions $g(\eta)$ on Γ satisfying

$$g(\eta) \geq c > 0 \quad \forall \eta \in \Gamma,$$

$$|g(\eta) - g(\zeta)| \leq C(d(\eta, \zeta))^\alpha \quad \forall \eta, \zeta \in \Gamma,$$

where c, C are constants(may depend on g). We consider the following randomization. Let a functional $\Psi(x)$, $x \in \mathbf{R}$, take its values in H_+^α and $\{X_n\}_{n\geq 1}$ be a sequence of random variables defined on a probability space (Ω, P). A sequence of random functions is then defined by

$$g_n(\eta, \omega) = g_n^\omega(\eta) := \Psi(X_n(\omega))(\eta).$$

We define random products ($n \geq 1$)

$$P_n^\omega(\eta) = \prod_{k=1}^{n} g_k(\Theta^{k-1}\eta, \omega),$$

and associated random partition functions

$$Z_n^\omega = \int_\Gamma P_n^\omega(\eta)\mu_0(d\eta).$$

We also set

$$\mu_n^\omega(d\eta) = \frac{P_n^\omega(\eta)\mu_0(d\eta)}{Z_n^\omega}.$$

Theorem 1 *Under the above conditions, for almost sure ω, all the limits of weakly convergent subsequences of $\{\mu_n^\omega\}$ are mutually absolutely continuous. Moreover, if μ^ω denote such a limit, then we have the approximation*

$$A\frac{P_n^\omega(\eta)}{2^n Z_n^\omega} \leq \mu^\omega(I_n(\eta)) \leq B\frac{P_n^\omega(\eta)}{2^n Z_n^\omega}$$

where $A = A(\omega) > 0$ and $B = B(\omega) > 0$ are two constants independent of η and n.

In view of the assertions of Theorem 1, for almost sure ω, all the weak limits in the above theorem belong to the same equivalence class(equivalent in the sense of mutually absolute continuity of measures). Fix one representative μ^ω of this class; we shall call it the *random Gibbs measure* defined by $g_n(\eta, \omega)$. The terminology is viewed from its analogy with statistical mechanics. To study μ^ω, we introduce the following random functions

$$Z_n^\omega(\beta) = \int_\Gamma [P_n^\omega(\eta)]^\beta \mu_0(d\eta) \quad \beta \in \mathbf{R}.$$

Theorem 2 *Under the conditions of Theorem 1, suppose further that the random sequence $X_n(\omega)$ is stationary and ergodic(with respect to P). Then, for almost sure ω, the following limit exists for every $\beta \in \mathbf{R}$*

$$\varphi^\omega(\beta) = \lim_{n\to\infty} \frac{1}{n} \log_2 Z_n^\omega(\beta).$$

The limit is actually independent of ω, denote it by $\varphi(\beta)$; we have

$$\varphi(\beta) = \lim_{n\to\infty} \frac{1}{n} \mathbf{E}_\omega \log_2 Z_n^\omega(\beta).$$

The function $\varphi^\omega(\beta)$, $\beta \in \mathbf{R}$, is called the *pressure function* of $\mu^\omega(d\eta)$. The theorem asserts that it is well defined almost surely(due to the stationarity) and is non-random(due to the ergodicity). Yet it may happen that, on a non-empty set of probability zero, the pressure function φ^ω may take different form. These will be illustrated later in §4..

We define a related function of φ,

$$\tau(\beta) = \beta + \beta\varphi(1) - 1 - \varphi(\beta).$$

Now we can state the main result of this section. It concerns with the multifractal property of μ^ω. For a given $\alpha \in \mathbf{R}$, we define random set

$$E_\alpha^\omega = \{\eta \in \Gamma : \lim_{n\to\infty} \frac{\log_2 \mu^\omega(I_n(\eta))}{n} = \alpha\},$$

and the assertion in the following theorem on the Hausdorff dimension(dim) of E_α^ω is usually referred as that *multifractal formalism* holds for the measure μ^ω.

Theorem 3 *Under the conditions of Theorems 1 and 2, suppose further that $\tau'(\theta)$ exists. Let $\alpha = \tau'(\theta)$. Then, for almost sure ω,*

$$\dim E_\alpha^\omega = \tau'(\theta)\theta - \tau(\theta).$$

Theorems 1–3 were stated in Fan(1997), as the random counterparts of the multifractal analysis of (deterministic) Gibbs measures. At there, some statements on the randomness are vague, and here we have clarified them. The proofs of Theorems 1–3 can be proceeded by the sample-wise consideration of the results in Fan(1997). To assert the almost sure existence of the pressure function, we need some Fubini argument and some techniques from convex analysis, see Fan(1997). The differentiability of the function $\varphi(\beta)$ in Theorem 3 is the main difficulty in the analysis. To tackle this, we shall consider a more special subclass, however rich and interesting enough, by using a theorem of Ruelle(1979) concerning the analyticity of the characteristic exponents of random matrix products.

3 Dependence on finite sites

We consider the following two randomizations, in which we take

$$\begin{aligned} \Omega &= \Gamma, \\ X_n(\omega) &= \omega_n, \quad \omega = (\omega_n) \in \Gamma. \end{aligned}$$

The stationarity assumption of X_n is equivalent to assume that P is a certain Θ-invariant probability measure on $\Omega = \Gamma$.

1°. Random sampling of two functions $g^{(0)}(\eta), g^{(1)}(\eta)$:

$$g_n(\eta, \omega) = g^{(\omega_n)}(\eta).$$

2°. Random perturbation of one $g(\eta)$:

$$g_n(\eta, \omega) = g(\eta + \omega_n).$$

We assume also that the $g^{(0)}(\eta), g^{(1)}(\eta), g(\eta)$ depend only on the first ℓ coordinates of η, i.e. $\eta_1, \cdots, \eta_\ell$, where ℓ is a positive integer. This includes the potentials of finite-range interaction.

Since the case 2° can be regarded as a special case of the case 1°($\omega_n = 0, 1$), we concentrate on the case 1°. Suppose $\ell \geq 2$ (the case $\ell = 1$ is simple). Let

$$G^\omega(\eta_1, \cdots, \eta_{\ell-1}; \eta_\ell, \cdots, \eta_{2(\ell-1)}) = \prod_{i=0}^{\ell-2} g_{1+i}(\eta_{1+i}, \cdots, \eta_{\ell+i}; \omega).$$

Consider the following $k \times k$ matrix, $k = 2^{l-1}$,

$$A^\omega = (G^\omega(\eta_1, \cdots, \eta_{\ell-1}; \eta_\ell, \cdots, \eta_{2(\ell-1)})).$$

Also define, for $\beta \in \mathbf{R}$,

$$A^\omega(\beta) = (G^\omega(\eta_1, \cdots, \eta_{\ell-1}; \eta_\ell, \cdots, \eta_{2(\ell-1)})^\beta).$$

Note that the randomness of A^ω and $A^\omega(\beta)$ depend indeed only on $\omega_1, \cdots, \omega_{\ell-1}$.

Now, we are ready to state the main result as follows.

Theorem 4 *Consider the two randomizations mentioned as above. Assume that $g^{(0)}(\eta), g^{(1)}(\eta), g(\eta)$ depend only on the first ℓ coordinates of η and that the random sequence X_n is stationary and ergodic. Then almost surely the pressure function $\varphi(\beta)$ of the random Gibbs measure μ^ω generated by these two randomizations is an analytic function of β; hence μ^ω has a non-random non-trivial multifractal spectrum described as that in Theorem 3.*

Proof It suffices to consider the case 1°. The random function $Z_n^\omega(\beta)$ can be calculated as follows. Note that if $n = m(\ell-1) + r$ with $0 \leq r < \ell$, we have

$$AZ^\omega_{m(\ell-1)}(\beta) \leq Z^\omega_n(\beta) \leq BZ^\omega_{m(\ell-1)}(\beta)$$

where A and B are two constants independent of n. Suppose now $n = m(\ell - 1)$. Then

$$\begin{aligned} Z_n^\omega(\beta) &= \int_\Gamma \prod_{k=1}^{m(\ell-1)} g^{(\omega_k)}(\eta_k, \cdots, \eta_{k+l-1})^\beta \mu_0(d\eta) \\ &= \frac{1}{2^{(m+1)(\ell-1)}} \sum_{\eta_1, \eta_2, \cdots, \eta_{(m+1)(\ell-1)}} \prod_{k=1}^{(m+1)(\ell-1)} g^{(\omega_k)}(\eta_k, \cdots, \eta_{k+\ell-1})^\beta \\ &= \frac{1}{2^{(m+1)(\ell-1)}} \sum_{\eta_1, \eta_2, \cdots, \eta_{(m+1)(\ell-1)}} \prod_{j=0}^{m} G^{\Theta^{(\ell-1)j}\omega}(\eta_{j(\ell-1)+1}, \cdots, \eta_{j(\ell-1)+2(\ell-1)})^\beta \\ &= \frac{1}{2^{(m+1)(\ell-1)}} \|A^\omega(\beta) \cdots A^{\Theta^{m(\ell-1)}\omega}(\beta)\| \end{aligned}$$

where $||A||$ for a matrix $A = (a_{i,j})$ denotes the sum $\sum_{i,j} |a_{i,j}|$. Consequently by Theorem 2 we have

$$\varphi(\beta) = \frac{1}{\ell-1} \lim_{m\to\infty} \frac{1}{m} \mathbf{E}_\omega \log_2 ||A^\omega(\beta) \cdots A^{\Theta^{m(\ell-1)}\omega}(\beta)|| - 1. \qquad (1)$$

Now, applying Ruelle(1979, Theorem 3.1), we have analyticity of $\varphi(\beta)$. The application of that theorem is legitimate, since the underlying probability space (in our case Γ) is compact. It can be routinely seen that the conditions required in that theorem hold in the present case. □

The following observation is interesting. Let $g^{(0)}(\eta) = g^{(1)}(\eta) = g(\eta)$ in Theorem 4, where g is a potential depending on the first ℓ coordinates. Then Theorem 4 asserts that we can view, via almost every realization of a certain stationary and ergodic sequence(e.g. coin flipping), the multifractal formalism for the Gibbs measure of one-dimensional lattice particles with finite-range. This latter case has been discussed in Fan(1994) and Brown-Michon-Peyriere(1992) by measure-theoretic approach. Finally we mention that the close form of the pressure function in Theorem 4 is in general unknown.

4 Dependence on two sites

In this section, we aim to proceed some calculations on the close form of the pressure function in Theorem 4. We assume the simplest case, i.e. $g^{(0)}(\eta), g^{(1)}(\eta), g(\eta)$ depend only on the two sites η_1, η_2 and P is the infinite product measure of $p\delta_0 + q\delta_1$ with $p + q = 1$(coin flipping). Then ω_n becomes independent and identically distributed and $A^{(0)}$ and $A^{(1)}$ are respectively of the form

$$A^{(0)} = \begin{pmatrix} g^{(0)}(0,0) & g^{(0)}(0,1) \\ g^{(0)}(1,0) & g^{(0)}(1,1) \end{pmatrix} \quad A^{(1)} = \begin{pmatrix} g^{(1)}(0,0) & g^{(1)}(0,1) \\ g^{(1)}(1,0) & g^{(1)}(1,1) \end{pmatrix}.$$

We prove that, if

$$A^{(0)}(\beta)A^{(1)}(\beta) = A^{(1)}(\beta)A^{(0)}(\beta).$$

Then for almost sure ω

$$\varphi^\omega(\beta) = p\log_2 \rho_0(\beta) + q\log_2 \rho_1(\beta) - 1 \qquad (2)$$

where $\rho_0(\beta)$ (resp. $\rho_1(\beta)$) denotes the spectral radius of $A^{(0)}(\beta)$ (resp. of $A^{(1)}(\beta)$).

In fact, by the assumption of commutativity, we have

$$A^{(\omega_1)}(\beta) \cdots A^{(\omega_n)}(\beta) = [A^{(0)}(\beta)]^{n-\omega_1-\cdots-\omega_n}[A^{(1)}(\beta)]^{\omega_1+\cdots+\omega_n}$$

Let u (resp. v) be the left (resp. right) strictly positive eigenvector of $A^{(0)}(\beta)$ (resp. of $A^{(1)}(\beta)$) associated to $\rho_0(\beta)$ (resp. $\rho_1(\beta)$). The existence of u and v is assured by the Perron-Frobenius theorem[Q]. Therefore,

$$\begin{aligned} ||A^{(\omega_1)}(\beta)\cdots A^{(\omega_n)}(\beta)|| &\approx u^t[A^{(0)}(\beta)]^{(n-\omega_1-\cdots-\omega_n)}[A^{(1)}(\beta)]^{\omega_1+\cdots+\omega_n} v \\ &= u^t v \; \rho_0(\beta)^{n-\omega_1-\cdots-\omega_n}\rho_1(\beta)^{\omega_1+\cdots+\omega_n}, \end{aligned}$$

where u^t denotes the transpose of u. So, by the formula (1) in the proof of Theorem 4 with $\ell = 2$, we have

$$\varphi(\beta) = -1 + \log_2 \rho_0(\beta) \lim_{n\to\infty} \mathbf{E}\frac{n - (\omega_1 + \cdots + \omega_n)}{n} + \log_2 \rho_1(\beta) \lim_{n\to\infty} \mathbf{E}\frac{\omega_1 + \cdots + \omega_n}{n}.$$

The formula (2) then follows from the above display.

In the above argument, it is interesting to observe that: if a sequence $\omega = (\omega_n)$ admits the following limit

$$\alpha(\omega) = \lim_{n\to\infty} \frac{\omega_1 + \cdots \omega_n}{n},$$

we have

$$\varphi^\omega(\beta) = -1 + (1 - \alpha(\omega)) \log_2 \rho_0(\beta) + \alpha(\omega) \log_2 \rho_1(\beta).$$

This may happen in the case that

$$A^{(0)} = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \quad A^{(1)} = \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix},$$

with $a \neq b$. $A^{(0)}$ and $A^{(1)}$ are commutative but have different spectral radii. So, φ^ω varies when $\alpha(\omega)$ varies. If the limit α does not exist, then the function φ is not well defined for this sequence ω (although such an ω occurs with probability zero). This illustrates the remark after Theorem 2 and gives a certain aspect of our random approach.

Next, we consider a strictly positive function $g(\eta) = g(\eta_1, \eta_2)$ depending only on the first two coordinates of η, and define

$$A^{(0)} = \begin{pmatrix} g(0,0) & g(0,1) \\ g(1,0) & g(1,1) \end{pmatrix} \quad A^{(1)} = \begin{pmatrix} g(1,0) & g(1,1) \\ g(0,0) & g(0,1) \end{pmatrix}.$$

We see that $A^{(1)}$ is obtained by interchanging the two rows of $A^{(0)}$. This corresponds to the random perturbation of the first coordinate η_1 of g. We assume that $A^{(0)}$ is defined by one of the following matrices:

$$G_1 = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad G_2 = \begin{pmatrix} a & b \\ a & b \end{pmatrix} \quad G_3 = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$$

Then we have

$$\varphi(\beta) = \log_2(a^\beta + b^\beta) - 1. \tag{3}$$

Let us prove this formula. If $A^{(0)} = G_1$, then $A^{(0)}(\beta)$ commutes with $A^{(1)}(\beta)$. Note also that $\rho_0(\beta) = \rho_1(\beta) = a^\beta + b^\beta$. Thus the desired result follows from the above arguments for proving the formula (2). If $A^{(0)} = G_2$, it is easier because $A^{(0)}(\beta) = A^{(1)}(\beta)$. When $A^{(0)} = G_3$, we have no longer the commutativity if $a \neq b$. However, we may remark the following relations

$$A^{(1)}(\beta)A^{(0)}(\beta) - A^{(0)}(\beta)A^{(1)}(\beta) = (b^{2\beta} - a^{2\beta})C, \quad \text{where } C = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix},$$

and

$$A^{(0)}(\beta)C = A^{(1)}(\beta)C = 0.$$

Thus we have

$$A^{(0)}(\beta)A^{(1)}(\beta)A^{(0)}(\beta) = A^{(0)}(\beta)A^{(0)}(\beta)A^{(1)}(\beta),$$
$$A^{(1)}(\beta)A^{(1)}(\beta)A^{(0)}(\beta) = A^{(1)}(\beta)A^{(0)}(\beta)A^{(1)}(\beta).$$

These identities imply that we can change $A^{(1)}(\beta)A^{(0)}(\beta)$ to $A^{(0)}(\beta)A^{(1)}(\beta)$ whenever there is either $A^{(0)}(\beta)$ or $A^{(1)}(\beta)$ at the left. According to this, we have

$$A^{(\omega_1)}(\beta)\cdots A^{(\omega_n)}(\beta) = A^{(\omega_1)}(\beta)[A^{(0)}(\beta)]^{n-1-\omega_2-\cdots-\omega_n}[A^{(1)}(\beta)]^{\omega_2+\cdots\omega_n}.$$

Note that we have again $\rho_0(\beta) = \rho_1(\beta) = a^\beta + b^\beta$. So, in this case we also have (3).

5 Concluding remarks

Firstly, our main result Theorem 4 concerns only the case of finte-many sites dependence; how about the general case? Furthermore, our theorems are stated in the one-dimensional case; how about the multi-dimensional case, especially for two-dimensional Ising model?

Secondly, in the definition of the random set E_α, we use $\lim_{n\to\infty}$ to define the local dimensions. Whenever the multifracal formalism holds, the existence of the limit is a part of the assertion. However, for the general (random or non-random) measures we may need to consider $\limsup, \liminf$ instead of just lim(the $\limsup, \liminf$ are always mathematically defined). An important case is Brownian local time, which is the cannonical measure supported on Brownian zero set. The lim sup and lim inf then exhibit different multifractal properties; see recent works of Hu-Taylor(1997) and Shieh-Taylor(1998) for this study.

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