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計畫名稱: The Leray-Schauder Degree of Mean Field Equations with Exponential Nonlinearity

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主持人: 陳俊全
執行機構及單位名稱:國立台灣大學 數學系

Main Results

Let T be the flat torus with the rectangle fundamental domain $[0, a] \times [0, b]$, $\rho \in \mathbf{R}$, $\alpha_j \geq 0$, δ_{g_j} be the Dirac delta function with mass at q_j and $W(x)$ is a Lipschitz function on T. We consider the following equation

$$\Delta u + \rho \frac{e^u}{\int_T e^u} - \sum_{j=1}^{l} 4\pi\alpha_j\delta_{q_j} + W(x) = 0 \quad \text{on} \quad T. \tag{0.1}$$

Clearly, the condition

$$\sum_{j=1}^{l} 4\pi\alpha_j = \rho + \int_T W\, dv \tag{0.2}$$

is necessary for the existence of solutions to (0.1). Let

$$N = \sum_{j=1}^{l} \alpha_j. \tag{0.3}$$

When $\alpha_j' s$ are nonnegative integers, N is called the vortex number.

One interesting phenomenon of (0.1) is its blow-up behavior. For $N = 0$, by the works of Brezis-Merle, Li-Shafrir, Li and Wolansky, the nonlinear term, after passing to a subsequence, converges to a delta measure with mass 8π near a blow-up point. This implies solutions of (0.1) can blow up only when ρ tends to $8\pi m$ with m a positive integer and the number of blowing up bubbles is exactly equal to m. When $N > 0$, Bartolucci and Tarantello showed that the local mass of the nonlinear term tends to $8\pi(1 + \alpha_j)$ when solutions blow up at q_j.

One method to study the existence problem is to use the Leray-Schauder degree. Let $K(\rho)$ be defined by

$$K(\rho) = \Delta^{-1}\left(\rho \frac{e^u}{\int_T e^u} - \sum_{j=1}^{l} 4\pi\alpha_j\delta_{q_j} + W(x)\right). \tag{0.4}$$

If there is no blow-up of the solutions, the Leray-Schauder degree of (0.1)

$$d_\rho \equiv deg(I + K(\rho), B_R, 0)$$

is well defined on a big ball B_R in a suitable space.

For $N = 0$, we can also consider (0.1) on a compact Riemann surface M without boundary. More generally, let h be a C^2 function on M with $h > 0$. We can consider the following equation

$$\Delta u + \rho \left(\frac{he^u}{\int_M he^u} + \frac{W}{1}\right) = 0 \quad \text{on} \quad M, \tag{0.5}$$

where Δ is the Betrami-Laplace operator on M.

For the case M is the standard sphere, more results about d_ρ were known. For $\rho = 8\pi$, Chang-Yang in [12] obtain a formula for d_ρ when h is a Morse function and $\Delta h \neq 0$ at critical points of h. That formula can be written as follows

$$d_\rho = 1 - \sum_{\nabla h(q)=0, \Delta h(q)<0} (-1)^{ind\, q}, \tag{0.6}$$

where $ind\, q$ is the Morse index of h at q.

Recently, the second author considered the case $h = 1$ in [29] and after an careful study of the orbits of the solutions, was able to obtain

$$\begin{cases} d_\rho = -1 & \text{for } 8\pi < \rho < 16\pi \\ d_\rho = \ \ 0 & \text{for } 16\pi < \rho < 24\pi. \end{cases}$$

For a general Riemann surface M, the authors obtained in [16]and [17] a complete formula for the degree. Let g denote the genus of M and $\chi(M)$ be the Euler characteristic of M, that is, $\chi(M) = 2 - 2g$. For two integers m_1 and m_2 with $m_2 \geq m_1 \geq 0$, let

$$\binom{m_2}{m_1} = \begin{cases} \dfrac{m_2(m_2-1)\cdots(m_2-m_1+1)}{m_1!} & \text{for } m_1 > 0 \\ 1 & \text{for } m_1 = 0 \end{cases}$$

Theorem A. *Assume $8\pi m < \rho < 8\pi(m+1)$ with m a nonnegative integer. Then $d_\rho = \binom{m-\chi(M)}{m}$, that is,*

$$d_\rho = \begin{cases} \dfrac{1}{m!}(-\chi(M)+1)(-\chi(M)+2)\cdots(-\chi(M)+m) & \text{for } \ m > 0 \\ 1 & \text{for } \ m = 0. \end{cases}$$

When $\rho = 8\pi m$, the problem becomes more difficult. If h is a Morse type function, then a degree formula similar to (0.6) was obtained also. Another method to find the degree for $\rho = 8\pi m$ is to show that solutions can not blow up when ρ tends to $8\pi m$ from the right (or the left). Then the degree at $8\pi m$ is the same as the one in the right (or the left). For (0.1), one can show that solutions can not blow up when ρ tends to $8\pi m$ from the right. Therefore as an application of Theorem A, we have

Theorem B. *Let d_ρ denote the Leray-Schauder degree for (0.1). Suppose $N = \sum_{j=1}^{l} \alpha_j = 0$. Then $d_\rho = 1$ and (0.1) has a solution for $\rho \in \mathbf{R}$.*

In this paper, we are going to study the more delicate cases with $N > 0$. We focus on the study of the behavior of equation (0.1) when ρ cross the first critical value, 8π. We assume $l = 1$, that is, there is only one delta function in (0.1) $N = 1, 2$ and $0 < \rho < 16\pi$. We will show that a delta function changes the topological property of the solution set. The main results are as follows.

Theorem 1.1. *Let d_ρ denote the Leray-Schauder degree for (0.1). Suppose $l = 1$ and $N = 1$. Then*

$$d_\rho = \begin{cases} 1 & \text{for } \rho < 8\pi \\ 2 & \text{for } 8\pi \le \rho < 16\pi. \end{cases}$$

Theorem 1.2. *Let d_ρ denote the Leray-Schauder degree for (0.1). Suppose $l = 1$ and $N = 2$. Then*

$$d_\rho = \begin{cases} 1 & \text{for } \rho < 8\pi \\ 0 & \text{for } \rho = 8\pi \\ 2 & \text{for } 8\pi < \rho < 16\pi. \end{cases}$$

Comparing to Theorem B, one can see two different features in Theorems 1.1 and 1.2. The first is that the degree is changed for $8\pi \le \rho < 16\pi$ when there is a delta function in the equation. This can be explained as follows. The degree is related to the blowing up of the solutions. The locations of blow-up points can be determined by the critical points of a special function f (see Section 2), which is a sum of $\log h$ (where h is the function defined in 0.5) and Green's functions. If there are l delta functions at $\{q_1, ..., q_l\}$ in (0.1), we need to add more Green's functions with their singularities at $\{q_1, ..., q_l\}$ to the function f. Then the number of the critical points of f changes. This leads to the change of the degree.

The second different feature is in the determination of the side (left or right) from which a blow-up can occur as ρ tends to 8π. Assume u_i is a sequence of blow-up solutions of (0.1) with $\rho = \rho_i$ tending to 8π. It is crucial to know the sign of $\rho_i - 8\pi$ for computing the degree at $\rho = 8\pi$. The dominant term of $\rho_i - 8\pi$ was obtained in [29] and [29]. Unfortunately, it is 0 if $N = 2$. This is exactly the case in Theorem

1.2. Hence we can not decide the sign of $\rho_i - 8\pi$. To overcome a similar difficulty for (0.5) on a bounded domain of $\mathbf{R}^2$ with h = constant, we computed the next order term of $\rho_i - 8\pi$ in []. However, it is still very difficult to know the sign of the next order term for the problem on a torus. The key idea here is to use the Weierstrass P-function. Then the next order term can be considered as the area of the difference between the image of the primitive of a Weierstrass P-function and $\mathbf{R}^2$.

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