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Long Time Behaviors of Solutions for the Cauchy Problem to A Relaxation Hyperbolic System with Non-Equilibrium States

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Abstract

In this paper we consider the Cauchy problem for the simplest 2×2 relaxation hyperbolic system of conservation laws with non-equilibrium states. We show that the solutions tend to diffusion waves in L^p-space ($2 \leq p \leq \infty$) time-asymptotically, when the initial perturbations are small, where the diffusion waves are constructed based on the corresponding heat equation and Burgers equation. In particular, we give the L^p-convergence rates in three cases corresponding to the second and third derivatives of nonlinearity at the original point are zero or not. The proof method we adopt is the Fourier transform method and the energy method based on the basic decay estimates of solution to the linearized equation.

1 Introduction and Main Result

We consider the simplest 2×2 hyperbolic system of conservation laws of relaxation model as follows, which was firstly introduced by Jin and Xin [12] for numerical analysis,

$$\begin{cases} v_t + u_x = 0, \\ u_t + av_x = \frac{f(v)-u}{\tau}, \end{cases} \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+ \tag{1.1}$$

with the initial data

$$(v,u)|_{t=0} = (v_0, u_0)(x) \to (0, u_\pm), \quad \text{as } \ x \to \pm\infty, \tag{1.2}$$

where (v,u) belonging to $\mathbb{R}$ are the unknown functions, which represent, for example, the specific volume and the velocity in the model of viscous gas in non-thermodynamic equilibrium, respectively. $\tau > 0$ is the constant of relaxation time, and we may assume $\tau = 1$ without loss of generality. a is a given positive constant. $f(v)$ is the flux function and needed to be smooth, generally say, $f \in C^2$. The state constants $(0, u_\pm)$ satisfy the non-equilibrium relation

$$f(0) \neq u_\pm. \tag{1.3}$$

About nonlinearity $f(v)$, without loss of generality, we may assume

$$f'(0) = 0. \tag{1.4}$$

The most important examples are $f(v) = v^{p+1}/(p+1)$ for all integer $p \geq 1$, where $p = 1$ is the Burgers' case $f(v) = v^2/2$. Others are like $f(v) = \sqrt{1+v^2}$, $f(v) = 1/\sqrt{1+v^2}$ and so on.

Relaxation phenomenon often arise in many physical situations, for example, gases not in thermodynamic equilibrium, kinetic theory, chromatography, river flows, traffic flows, and more general waves, cf. [35]. The general 2×2 relaxation hyperbolic system of conservation laws in the form

$$\begin{cases} v_t + f(v,u)_x = 0 \\ u_t + g(v,u)_x = h(v,u) \end{cases} \tag{1.5}$$

was first analyzed by T.-P. Liu [17] to justify some nonlinear stability criteria for diffusion waves, expansion waves and traveling waves in the Cauchy problem case. After then, the stability of traveling waves, diffusion waves and rarefaction waves for the Cauchy problem or the initial-boundary value problems were studied by many people, for example, see [4, 6, 15, 16, 19, 20, 21, 25, 26, 28, 29, 38]. There are also various works on the limits of relaxation time, or numerical analyses, see [1, 2, 11, 12, 14, 27, 34] and the references therein. All of these works are under consideration of equilibrium states, namely,

$$f(v_\pm) = u_\pm,$$

where $v_\pm = \lim_{x\to\pm\infty} v_0(x)$. When the non-equilibrium states

$$f(v_\pm) \neq u_\pm$$

(our case is $v_\pm = 0$) holds, there are a few works on stability of elementary nonlinear waves, cf. [36] and [37]. Regarding on the nonlinear stability of diffusion waves for the other model equations, we refer to those works in [3, 5, 8, 9, 10, 13, 18, 22, 24, 30, 31, 32].

Now, let us see formally what the asymptotic states of $(v,u)(x,t)$ at $x = \pm\infty$ is. When $x \to \pm\infty$, the second equation of (1.1) behaves

$$\begin{cases} \frac{d}{dt}u(\pm\infty, t) = f(0) - u(\pm\infty, t) \\ \\ u(\pm\infty, 0) = u_\pm \end{cases} \tag{1.6}$$

which solves

$$u(\pm\infty, t) = e^{-t}[u_\pm - f(0)] + f(0). \tag{1.7}$$

While the first equation of (1.1) gives

$$\frac{d}{dt}v(\pm\infty, t) = 0,$$

which implies

$$v(\pm\infty, t) = v(\pm\infty, 0) = v_0(\pm\infty) = 0, \tag{1.8}$$

and we may also have

$$\frac{d}{dt}\int_{-\infty}^{\infty} v(x,t)dx = -\int_{-\infty}^{+\infty} u_x(x,t)dx = -u(+\infty,t) + u(-\infty,t) = -(u_+ - u_-)e^{-t},$$

that is, by integrating above equality over $[0, t]$,

$$\int_{-\infty}^{\infty} v(x,t)dx = \int_{-\infty}^{\infty} v_0(x)dx - (u_+ - u_-) + (u_+ - u_-)e^{-t}. \tag{1.9}$$

Since the initial data $v_0(x)$ is any given, there is no information from (1.9) to let us expect $\int_{-\infty}^{\infty} v(x,t)dx \to 0$ as $t \to \infty$, except for the special case

$$\int_{-\infty}^{\infty} v_0(x)dx - (u_+ - u_-) = 0, \tag{1.10}$$

which is sufficiently used in [36, 37] for the stability of diffusion waves.

We now reduce a new equation from system (1.1). Differentiating the first equation of (1.1) with respect to t and substituting it to the second equation of (1.1), we obtain a scalar wave equation on v with damping

$$v_{tt} + v_t + f(v)_x - av_{xx} = 0. \tag{1.11}$$

In general, the term v_{tt} decays much faster than the other terms do in Eq. (1.11), so the main control part of Eq. (1.11) should be

$$v_t + f(v)_x - av_{xx} = 0. \tag{1.12}$$

This is really reasonable to consider that the time-asymptotic behavior of the solutions to Eq. (1.11) is just as that of its diffusion waves, which are constructed based on the corresponding heat equation or Burgers equation to the parabolic equation (1.12).

Now, let us recall the so-called diffusion waves. We consider the following parabolic equation

$$\begin{cases} \theta_t + [f(0) + f'(0)\theta + \frac{f''(0)}{2}\theta^2]_x - a\theta_{xx} = 0 \\ \theta|_{t=0} = \theta_0(x) \to 0, \quad \text{as } x \to \pm\infty. \end{cases} \tag{1.13}$$

We call the solution of (1.13) as the diffusion wave solution to Eq. (1.12), as well as to Eq. (1.11) or say Eqs. (1.1).

If $f''(0) = 0$, then Eq. (1.13) is equivalent to

$$\begin{cases} \theta_t + f'(0)\theta_x - a\theta_{xx} = 0, \\ \theta|_{t=0} = \theta_0(x), \end{cases} \tag{1.14}$$

which is a linear heat equation, and has a unique solution in the form

$$\theta(x,t) = \frac{1}{\sqrt{4a\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-f'(0)t-y)^2}{4at}} \theta_0(y)dy. \tag{1.15}$$

This solution is called the linear diffusion wave.

If $f''(0) \neq 0$, we may reduce (1.13) into

$$\begin{cases} \theta_t + f'(0)\theta_x + \frac{f''(0)}{2}\theta\theta_x - a\theta_{xx} = 0, \\ \\ \theta|_{t=0} = \theta_0(x). \end{cases} \tag{1.16}$$

By scaling the variables $x' = f''(0)x/(4a)$ and $t' = f''(0)^2 t/(8a)$, and denoting the new variables (x', t') still by (x, t) without confusion, we obtain from Eq. (1.16)

$$\theta_t + \alpha\theta_x + \theta\theta_x - \frac{1}{2}\theta_{xx} = 0, \tag{1.17}$$

where $\alpha = 2f'(0)/f''(0)$. Applying the Hofe-Cole transformation

$$\theta(x,t) = -(\ln\varphi)_x, \quad i.e., \quad \varphi(x,t) = e^{-\int_{-\infty}^{x}\theta(\xi,t)d\xi},$$

to Eq. (1.17), it can be reduced to

$$\begin{cases} \varphi_t + \alpha\varphi_x - \frac{1}{2}\varphi_{xx} = 0, \\ \\ \varphi|_{t=0} = e^{-\int_{-\infty}^{x}\theta_0(y)dy} =: \varphi_0(x). \end{cases} \tag{1.18}$$

It is well-known that the above linear heat equation has a unique solution

$$\varphi(x,t) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\alpha t-y)^2}{2t}} \varphi_0(y)dy. \tag{1.19}$$

Thus, from (1.17) and (1.19), we obtain

$$\begin{aligned} \theta(x,t) &= -(\ln\varphi)_x = -\frac{\varphi_x}{\varphi} \\ &= \frac{\int_{-\infty}^{\infty} \exp\Big(-\frac{(x-\alpha t-y)^2}{2t}\Big)\exp\Big(-\int_{-\infty}^{y}\theta_0(\eta)d\eta\Big)\theta_0(y)dy}{\int_{-\infty}^{\infty} \exp\Big(-\frac{(x-\alpha t-y)^2}{2t}\Big)\exp\Big(-\int_{-\infty}^{y}\theta_0(\eta)d\eta\Big)dy}. \end{aligned} \tag{1.20}$$

This solution is called the nonlinear diffusion wave.

Returning back to Eq. (1.13), let us see what the diffusion waves behave. By integrating (1.13) over $(-\infty, +\infty)$, we have

$$\frac{d}{dt}\int_{-\infty}^{\infty}\theta(x,t)dx = 0,$$

namely,

$$\int_{-\infty}^{\infty}\theta(x,t)dx = \int_{-\infty}^{\infty}\theta_0(x)dx. \tag{1.21}$$

This mass doesn't go to zero too, as $t \to \infty$, except for $\int_{-\infty}^{\infty} \theta_0(x)dx = 0$. In fact, we have no such a restriction on $\theta_0(x)$ in this paper.

Making the difference of (1.9) and (1.21), we obtain

$$\int_{-\infty}^{\infty} [v(x,t) - \theta(x,t)]dx = \int_{-\infty}^{\infty} [v_0(x) - \theta_0(x)]dx - (u_+ - u_-) + (u_+ - u_-)e^{-t}. \qquad (1.22)$$

For any given initial value $v_0(x)$, it is completely possible to choose the initial value $\theta_0(x)$ for the corresponding parabolic equation (1.13), so that

$$\int_{-\infty}^{\infty} [v_0(x) - \theta_0(x)]dx - (u_+ - u_-) = 0. \qquad (1.23)$$

This is our essential assumption in this paper.

Now we use Hsiao and Liu's fashion in [8], that is, let

$$m_0(x) \in C_0^\infty(\mathbb{R}) \quad \text{and} \quad \int_{-\infty}^{\infty} m_0(x)dx = 1, \qquad (1.24)$$

and denote

$$\begin{cases} \hat{v}(x,t) = (u_+ - u_-)e^{-t}m_0(x) \\ \hat{u}(x,t) = e^{-t}\Big(u_- + (u_+ - u_-)\int_{-\infty}^{x} m_0(y)dy\Big), \end{cases} \qquad (1.25)$$

it is clear that $(\hat{v}, \hat{u})(x,t)$ satisfy

$$\begin{cases} \hat{v}_t + \hat{u}_x = 0 \\ \hat{u}_t + \hat{u} = 0 \\ \hat{v}(\pm\infty, t) = 0, \quad \hat{u}(\pm\infty, t) = u_\pm e^{-t}. \end{cases} \qquad (1.26)$$

Thus, we obtain the following from (1.22)-(1.25)

$$\begin{aligned} &\int_{-\infty}^{\infty} [v(x,t) - \theta(x,t) - \hat{v}(x,t)]dx \\ &= \int_{-\infty}^{\infty} [v_0(x) - \theta_0(x)]dx - (u_+ - u_-) \qquad (1.27) \\ &= 0. \end{aligned}$$

Notice that, for a parabolic conservation law, Chern and Liu [7], Jeffrey and Zhao [11] studied that, for a given initial data $\theta_0(x) \in L^1 \cap H^2$, the diffusion wave solutions of (1.13) decay in the form $\|\theta(t)\|_{L^2} = O(t^{-1/4})$, for $\int_{-\infty}^{\infty} \theta_0(x)dx \neq 0$, and $\|\theta(t)\|_{L^2} = O(t^{-3/4})$ for $\int_{-\infty}^{\infty} \theta_0(x)\, dx = 0$. On the other hand, applying the decay rate estimates in [23] and [36, 37] for the wave equation with damping (1.11), we can have also the same decay rates of the solution $\|v(t)\|_{L^2} = O(t^{-1/4})$ or $O(t^{-3/4})$ based on the restriction on the initial data $\int_{-\infty}^{\infty} v_0(x)\, dx - (u_+ - u_-) \neq 0$ or $= 0$. So, it is clear that, in the case of $\int_{-\infty}^{\infty} \theta_0(x)dx \neq 0$ and $\int_{-\infty}^{\infty} v_0(x)dx - (u_+ - u_-) \neq 0$, the solution $v(x,t)$ of Eq. (1.11) tends naturally to the diffusion wave $\theta(x,t)$ in the form $\|(v-\theta)(t)\|_{L^2} = O(t^{-1/4})$ due to $\|\theta(t)\|_{L^2} = 0(t^{-1/4})$ and $\|v(t)\|_{L^2} = O(t^{-1/4})$. However, this decay rate is not satisfactory if the initial perturbation $v_0(x) - \theta_0(x)$ satisfies the essential condition (1.23). In fact, we expect that the solution $v(x,t)$ converges to the corresponding diffusion wave $\theta(x,t)$ faster than $O(t^{-1/4})$ in L^2-sense, namely, we will prove that

$$\|(v-\theta)(t)\|_{L^2} = \begin{cases} O(t^{-(3/4)+\sigma}), & \text{for } f''(0) \neq 0 \\ O(t^{-3/4}\ln(2+t)), & \text{for } f''(0) = 0 \text{ but } f'''(0) \neq 0 \\ O(t^{-3/4}), & \text{for } f''(0) = 0 \text{ and } f'''(0) = 0 \end{cases}$$

where σ is any given positive constant, of course, we may let $0 < \sigma \ll 1$. We see also that, the convergence rate in the case $f''(0) \neq 0$ is quite same to that by Chern and Liu [5], wherein they only focused on the nonlinear diffusion waves case, while the results for all three cases are also same to those by Mei and Omata [24], in which they studied the convergence to diffusion waves for the solutions of the Benjamin-Bona-Mahony-Burgers equation.

The decay rates represented in [36, 37] are better than ours, because the stiff condition (1.10) may ensure a better decay rate as we know. However the diffusion wave constructed there is a bit strange. After getting the decay estimates for the original solution of (1.1), they then constructed a solution for a heat equation with an initial value which is dependent on the original solution $v(x,t)$, as the asymptotic profile of $v(x,t)$, and called it the "diffusion wave". Thus, both of two solutions $v(x,t)$ and $\theta(x,t)$ are not independent, and the late is closed dependent on the previous, such a "diffusion wave" is not in the original meaning.

Our proof method is different from those used in the previous works [4, 36, 37] in this direction. By making use of the Fourier transform, we reduce the fundamental solution to the linearized equation. Based on the energy decay estimates on the linearized problem, we further show the decay rates for the nonlinear problem by means of the Duhamal's principle and the elementary energy method.

Now let

$$\varepsilon := \int_{-\infty}^{\infty} (|\theta_0(x)| + |x\theta_0(x)|)\, dx < +\infty, \tag{1.28}$$

we are going to state the main results as follows.

Theorem 1.1 *Suppose that* (1.23) *and*

$$w_0(y) := \int_{-\infty}^{x} [v_0(y) - \theta_0(y)]dy + u_+ - u_- \in (L^1 \cap H^2)(\mathbb{R}) \tag{1.29}$$

$$w_1(y) := \int_{-\infty}^{x} [u_0'(y) - \theta_t(y,0) - \hat{v}_t(y,0)]dy \in (L^1 \cap H^1)(\mathbb{R}) \tag{1.30}$$

hold. Then there exists a positive constant δ_0 such that when $\|(w_0, w_1)\|_{L^1} + \|w_0\|_{H^2} + \|w_1\|_{H^1} + \varepsilon + |u_+ - u_-| \leq \delta_0$, then the Cauchy problem (1.1) *and* (1.2) *has a unique global solution $v(x,t)$ satisfying*

$$v(x,t) - \theta(x,t) \in C^0(0, +\infty; H^1(\mathbb{R})).$$

Furthermore, the different decay rates hold for the following three cases.

1. *If $f''(0) \neq 0$, then, for any $\sigma > 0$, the following estimates hold*

$$\|(v-\theta)(t)\|_{L^2} = O(1)(1+t)^{-\frac{3}{4}+\sigma}, \tag{1.31}$$
$$\|(v-\theta)_x(t)\|_{L^2} = O(1)(1+t)^{-1+\sigma}, \tag{1.32}$$
$$\|(v-\theta)(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{7}{8}+\sigma}. \tag{1.33}$$

2. *If $f''(0) = 0$ but $f'''(0) \neq 0$, the convergence rates are as follows*

$$\|(v-\theta)(t)\|_{L^2} = O(1)(1+t)^{-\frac{3}{4}} \ln(2+t), \tag{1.34}$$
$$\|(v-\theta)_x(t)\|_{L^2} = O(1)(1+t)^{-1}, \tag{1.35}$$
$$\|(v-\theta)(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{7}{8}} \sqrt{\ln(2+t)}. \tag{1.36}$$

3. *If $f''(0) = 0$ and $f'''(0) = 0$, the convergence rates to the diffusion wave are much faster as follows*

$$\|(v-\theta)(t)\|_{L^2} = O(1)(1+t)^{-\frac{3}{4}}, \tag{1.37}$$
$$\|(v-\theta)_x(t)\|_{L^2} = O(1)(1+t)^{-\frac{5}{4}}, \tag{1.38}$$
$$\|(v-\theta)(t)\|_{L^\infty} = O(1)(1+t)^{-1}. \tag{1.39}$$

Using L^2, L^∞-results in Theorem 1.1 and the interposing inequality

$$\|f\|_{L^p} \leq \|f\|_{L^\infty}^{(p-2)/p} \|f\|_{L^2}^{2/p}, \quad \text{for} \quad 2 \leq p \leq \infty,$$

we can obtain immediately L^p-decay rates as follows.

Corollary 1.1 *Under the assumptions in Theorem 1.1, it follows*

$$\|(v-\theta)(t)\|_{L^p} = \begin{cases} O(1)(1+t)^{-\frac{7}{8}+\frac{1}{4p}+\sigma}, & \text{for } f''(0) \neq 0 \\ O(1)(1+t)^{-\frac{7}{8}+\frac{1}{4p}}(\ln(2+t))^{\frac{1}{2}+\frac{1}{p}}, & \text{for } f''(0) = 0 \text{ but } f'''(0) \neq 0 \\ O(1)(1+t)^{-1+\frac{1}{2p}}, & \text{for } f''(0) = 0 \text{ and } f'''(0) = 0 \end{cases} \tag{1.40}$$

for $2 \leq p \leq \infty$.

Remark 1.1 *As we showed in Theorem 1.1 for the nonlinear stability of diffusion waves, we don't need here the stiff condition* (1.10) *and the subcharacteristic condition*

$$-\sqrt{a} < f'(v) < \sqrt{a} \quad \text{for all } v \text{ under consideration.}$$

But both of them are sufficiently used in [36, 37].

Notations. We now make some notation for simplicity. C always denotes some positive constants without confusion. $\partial_x^k f := \partial^k f/\partial x^k$. L^p presents the Lebesque integral space with the norm $\|\cdot\|_{L^p}$. Especially, L^2 is the square integral space with the norm $\|\cdot\|_{L^2}$, and L^∞ is the essential bounded space with the norm $\|\cdot\|_{L^\infty}$. H^k denotes the usual Sobolev space with the norms $\|\cdot\|_{H^k}$. Suppose that $f(x) \in L^1 \cap L^2(\mathbb{R})$, we define the Fourier transforms of $f(x)$ as follows:

$$F[f](\xi) \equiv \hat{f} = \int_{\mathbb{R}} f(x) e^{-\mathrm{i}x\xi} dx.$$

Let T and B be a positive constant and a Banach space, respectively. $C^k(0,T;B)$ $(k \geq 0)$ denotes the space of B-valued k-times continuously differentiable functions on $[0,T]$, and $L^2(0,T;B)$ denotes the space of B-valued L^2-functions on $[0,T]$. The corresponding spaces of B-valued function on $[0,\infty)$ are defined similarly.

2 Reformulation to the Original Problem

Note (1.27), that is,

$$\int_{-\infty}^{\infty} [v(x,t) - \theta(x,t) - \hat{v}(x,t)] dx = 0,$$

it is reasonable for us to set a new known function as

$$w(x,t) := \int_{-\infty}^{x} [v(y,t) - \theta(y,t) - \hat{v}(y,t)] dy, \tag{2.1}$$

namely,

$$w_x(x,t) = v(x,t) - \theta(x,t) - \hat{v}(x,t). \tag{2.2}$$

Due to Eqs. (1.11), (1.13) and (1.25), we reduce a new equation

$$w_{tt} + w_t - a w_{xx} = F \tag{2.3}$$

with the initial data

$$w|_{t=0} = w_0(x), \quad w_t|_{t=0} = w_1(x), \tag{2.4}$$

where

$$\begin{aligned} F := & -\theta_{xt} + f'(0)\theta_t + f''(0)\theta\theta_t + \hat{u}_x(x,t) \\ & - [f(\theta + \hat{v} + w_x) - f(0) - f'(0)\theta - \frac{1}{2} f''(0)\theta^2], \end{aligned} \tag{2.5}$$

$$w_0(x) := \int_{-\infty}^{x} [v_0(y) - \theta_0(y)] dy - (u_+ - u_-), \tag{2.6}$$

$$w_1(x) := \int_{-\infty}^{x} [u_0'(y) - \theta_t(y,0) - \hat{v}_t(y,0)] dy. \tag{2.7}$$

Here the diffusion wave's equation (1.13) and the definition of $\hat{v}(x,t)$ in (1.25) give

$$\theta_t(x,0) = f'(0)\theta_{0x} + f''(0)\theta_0\theta_{0x} + \theta_{0xx} \tag{2.8}$$
$$\hat{v}_t(x,0) = \hat{u}_x(x,0) = (u_+ - u_-)m_0(x). \tag{2.9}$$

By the Taylor's formula, we have

$$f(\theta + \hat{v} + w_x) = f(\theta) + f'(\theta)(\hat{v} + w_x) + O(1)(\hat{v} + w_x)^2 \tag{2.10}$$

and (using $f'(0) = 0$, see (1.4))

$$f(\theta) = f(0) + \frac{1}{2}f''(0)\theta^2 + \frac{1}{3!}f'''(0)\theta^3 + O(1)\theta^4 \tag{2.11}$$
$$f'(\theta) = f''(0)\theta + \frac{1}{2}f'''(0)\theta^2 + O(1)\theta^3, \tag{2.12}$$

so that

$$\begin{aligned} & f(\theta + \hat{v} + w_x) - f(0) + f'(0)\theta - \frac{1}{2}f''(0)\theta^2 \\ &= \frac{1}{3!}f'''(0)\theta^3 + O(1)\theta^4 + O(1)(\hat{v} + w_x)^2 \\ &\quad + [f''(0)\theta + \frac{1}{2}f'''(0)\theta^2 + O(1)\theta^3](\hat{v} + w_x). \end{aligned} \tag{2.13}$$

Substituting (2.13) into (2.3), we then have

$$\begin{cases} w_{tt} + w_t - aw_{xx} = F_1 + F_2 \\ \\ (w, w_t)|_{t=0} = (w_0(x), w_1(x)), \end{cases} \tag{2.14}$$

where

$$F_1 := -\theta_{xt} + f''(0)\theta\theta_t + \hat{u}_x(x,t), \tag{2.15}$$
$$\begin{aligned} F_2 := & -\frac{1}{3!}f'''(0)\theta^3 - O(1)\theta^4 - O(1)(\hat{v} + w_x)^2 \\ & - [f''(0)\theta + \frac{1}{2}f'''(0)\theta^2 + O(1)\theta^3](\hat{v} + w_x). \end{aligned} \tag{2.16}$$

We state the main theory in this section as follows.

Theorem 2.1 *Under assumptions in Theorem 1.1, then there exists a positive constant δ_1 such that when $\|(w_0, w_1)\|_{L^1} + \|w_0\|_{H^2} + \|w_1\|_{H^1} + \varepsilon \le \delta_1$, then the Cauchy problem (2.14) has a unique global solution $w(x,t)$*

$$w(x,t) \in C^0(0, +\infty; H^2(\mathbb{R}))$$

satisfying that:

1. *If $f''(0) \neq 0$, then, for any $\sigma > 0$, the following estimates hold*

$$\|w(t)\|_{L^2} = O(1)(1+t)^{-\frac{1}{4}+\sigma}, \tag{2.17}$$

$$\|w_x(t)\|_{L^2} = O(1)(1+t)^{-\frac{3}{4}+\sigma}, \tag{2.18}$$

$$\|w_{xx}(t)\|_{L^2} = O(1)(1+t)^{-1+\sigma}. \tag{2.19}$$

2. *If $f''(0) = 0$ but $f'''(0) \neq 0$, the convergence rates are as follows*

$$\|w(t)\|_{L^2} = O(1)(1+t)^{-\frac{1}{4}} \ln(2+t), \tag{2.20}$$

$$\|w_x(t)\|_{L^2} = O(1)(1+t)^{-\frac{3}{4}} \ln(2+t), \tag{2.21}$$

$$\|w_{xx}(t)\|_{L^2} = O(1)(1+t)^{-1}. \tag{2.22}$$

3. *If $f''(0) = 0$ and $f'''(0) = 0$, the convergence rates are much faster as follows*

$$\|w(t)\|_{L^2} = O(1)(1+t)^{-\frac{1}{4}}, \tag{2.23}$$

$$\|w_x(t)\|_{L^2} = O(1)(1+t)^{-\frac{3}{4}}, \tag{2.24}$$

$$\|w_{xx}(t)\|_{L^2} = O(1)(1+t)^{-\frac{5}{4}}. \tag{2.25}$$

By Theorem 2.1 and the well-known inequalities

$$\|w(t)\|_{L^\infty} \leq \sqrt{2}\|w(t)\|_{L^2}^{1/2}\|w_x(t)\|_{L^2}^{1/2},$$

$$\|w_x(t)\|_{L^\infty} \leq \sqrt{2}\|w_x(t)\|_{L^2}^{1/2}\|w_{xx}(t)\|_{L^2}^{1/2},$$

we can obtain the decay rates for $\|w(t)\|_{L^\infty}$ and $\|w_x(t)\|_{L^\infty}$ as follows.

Corollary 2.1 *There follow*

$$\|w(t)\|_{L^\infty} = \begin{cases} O(1)(1+t)^{-(\frac{1}{2}-\sigma)}, & \text{for } f''(0) \neq 0 \\ O(1)(1+t)^{-\frac{1}{2}} \ln(2+t), & \text{for } f''(0) = 0 \text{ but } f'''(0) \neq 0 \\ O(1)(1+t)^{-\frac{1}{2}}, & \text{for } f''(0) = 0 \text{ and } f'''(0) = 0 \end{cases} \tag{2.26}$$

and

$$\|w_x(t)\|_{L^\infty} = \begin{cases} O(1)(1+t)^{-(\frac{7}{8}-\sigma)}, & \text{for } f''(0) \neq 0 \\ O(1)(1+t)^{-\frac{7}{8}} \ln(2+t), & \text{for } f''(0) = 0 \text{ but } f'''(0) \neq 0 \\ O(1)(1+t)^{-1}, & \text{for } f''(0) = 0 \text{ and } f'''(0) = 0 \end{cases} \tag{2.27}$$

Once Theorem 2.1 is proved, we may show Theorem 1.1 as follows. So, to prove Theorem 2.1 is our main effort in the next section.

Proof of Theorem 1.1. Thanks to Theorem 2.1 and Corollary 2.1, and note (2.2),

$$v(x,t) - \theta(x,t) = w_x(x,t) + \hat{v}(x,t),$$

while

$$\|\hat{v}(t)\|_{L^\infty} = O(1)|u_+ - u_-|e^{-t}, \qquad \|\hat{v}(t)\|_{L^2} = O(1)|u_+ - u_-|e^{-t},$$

we can prove Theorem 1.1 immediately. □

3 A Priori Estimates

Now we first reduce a fundamental solution for the linearized equation of (2.14). Let us consider the corresponding linear problem

$$\begin{cases} \psi_{tt} + \psi_t - a\psi_{xx} = 0, & x \in \mathbb{R}, t \in \mathbb{R}_+ \\ \psi|_{t=0} = \psi_0(x), & \psi_t|_{t=0} = \psi_1(x). \end{cases} \tag{3.1}$$

We can represent the solution of (3.1) as

$$\psi(x,t) = K_0(t) * \psi_0 + K_1(t) * \psi_1, \tag{3.2}$$

where $K_i(x,t)$ $(i = 0,1)$ are some functions which will be represented below, the mark $*$ means the convolution $\int_{-\infty}^{\infty} K_i(x-y,t)\psi_i(y)dy$.

Let $R_i(\xi,t)$ be the Fourier transform of $K_1(x,t)$, $i = 0,1$, then R_i satisfies the following ODE

$$\frac{d^2}{dt^2}R_i + \frac{d}{dt}R_i + a\xi^2 R_i = 0, \quad i = 0,1 \tag{3.3}$$

with the initial data

$$R_0(\xi,0) = 1, \qquad \frac{d}{dt}R_0(\xi,0) = 0, \tag{3.4}$$

and

$$R_1(\xi,0) = 0, \qquad \frac{d}{dt}R_1(\xi,0) = 1, \tag{3.5}$$

respectively. Solving these ODEs directly, we obtain the exact solutions as

$$R_1(\xi,t) = \begin{cases} \frac{2e^{-t/2}}{\sqrt{1-4a\xi^2}} \sinh\left(\frac{\sqrt{1-4a\xi^2}}{2}t\right), & |\xi| < \frac{1}{2\sqrt{a}} \\ te^{-t/2}, & |\xi| = \frac{1}{2\sqrt{a}} \\ \frac{2e^{-t/2}}{\sqrt{4a\xi^2-1}} \sin\left(\frac{\sqrt{4a\xi^2-1}}{2}t\right), & |\xi| > \frac{1}{2\sqrt{a}} \end{cases} \tag{3.6}$$

and

$$R_0(\xi,t) = R_1(\xi,t) + R_2(\xi,t) \tag{3.7}$$

where

$$R_2(\xi,t) = \begin{cases} e^{-t/2} \cosh\left(\frac{\sqrt{1-4a\xi^2}}{2}t\right), & |\xi| < \frac{1}{2\sqrt{a}} \\ e^{-t/2}, & |\xi| = \frac{1}{2\sqrt{a}} \\ e^{-t/2} \cos\left(\frac{\sqrt{4a\xi^2-1}}{2}t\right), & |\xi| > \frac{1}{2\sqrt{a}}. \end{cases} \tag{3.8}$$

Thus, we see that the functions $K_i(x,t)$ $(i=0,1)$ can be given by making use of the inverse Fourier transform to $R_i(\xi,t)$ $(i=0,1)$.

Furthermore, for the linear equation with source term

$$\begin{cases} \psi_{tt}+\psi_t-a\psi_{xx}=g(x,t) \\ \psi|_{t=0}=\psi_0, \quad \psi_t|_{t=0}=\psi_1, \end{cases} \tag{3.9}$$

due to the Duhamal's principle, the solution is expressed as

$$\psi(x,t)=K_0(t)*\psi_0+K_1(t)*\psi_1+\int_0^t K_1(t-\tau)*g(\tau)d\tau. \tag{3.10}$$

On this linear problem, we now state some energy decay estimates as follows. Since these can be proved in a quite same way as in [23], we would like to say the following lemma is contributed also by A. Matsumura in [23].

Lemma 3.1 **([23])** *If* $g\in L^1\cap H^j$, *then*

$$\left\|\left(\frac{\partial}{\partial x}\right)^j(K_1(t)*g)\right\|_{L^2}\le C(1+t)^{-\frac{2j+1}{4}}(\|g\|_{L^1}+\|g\|_{H^{j-1}}). \tag{3.11}$$

If $g\in L^1\cap H^{j+1}$, *then*

$$\left\|\left(\frac{\partial}{\partial x}\right)^j(K_0(t)*g)\right\|_{L^2}\le C(1+t)^{-\frac{2j+1}{4}}(\|g\|_{L^1}+\|g\|_{H^j}). \tag{3.12}$$

Now we are going to study the nonlinear problem (2.14). Due to (3.10) we can rewrite (2.14) in the integral form

$$w(x,t)=K_0(t)*\psi_0+K_1(t)*\psi_1+\int_0^t K_1(t-\tau)*(F_1+F_2)(\tau)d\tau. \tag{3.13}$$

Before proving Theorem 2.1, we need several useful lemmas as follows.

Lemma 3.2 **([5, 10])** *Let* $\theta(x,t)$ *be the diffusion waves of* (1.13). *If*

$$\varepsilon=\int_{-\infty}^{\infty}(|\theta_0(x)|+|x\theta_0(x)|)dx<+\infty. \tag{3.14}$$

then

$$\|\partial_x^j\theta(t)\|_{L^2}=O(1)\varepsilon(1+t)^{-\frac{2j+1}{4}}, \tag{3.15}$$

$$\|\theta(t)\|_{L^q}=O(1)\varepsilon(1+t)^{-\frac{q-1}{2q}}, \quad 1\le q\le\infty \tag{3.16}$$

$$\|\partial_x^j\theta_t(t)\|_{L^1}=O(1)\varepsilon(1+t)^{-1-\frac{j}{2}} \tag{3.17}$$

hold for all $t\ge 0$.

Lemma 3.3 ([33]) *Let $a > 0$ and $b > 0$ be constants. If* $\max(a,b) > 1$, *then*

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b}ds \le C(1+t)^{-\min(a,b)}. \tag{3.18}$$

If $\max(a,b) = 1$, *then*

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b}ds \le C(1+t)^{-\min(a,b)}\ln(2+t). \tag{3.19}$$

If $\max(a,b) < 1$, *then*

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b}ds \le C(1+t)^{1-a-b}. \tag{3.20}$$

According to the three cases in Theorem 2.1, we define the solution spaces in the forms

$$X_i(\delta) = \{w \in C^0(0,\infty; H^2(\mathbb{R}))|M_i(w) \le \delta\} \tag{3.21}$$

for $i = 1,2,3$, where

$$M_1(w) = \sup_{0\le t\le\infty}\left\{\sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}-\sigma}\|\partial_x^j w(t)\|_{L^2} + (1+t)^{1-\sigma}\|w_{xx}(t)\|_{L^2}\right\} \tag{3.22}$$

$$M_2(w) = \sup_{0\le t\le\infty}\left\{\sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}}\ln^{-1}(2+t)\|\partial_x^j w(t)\|_{L^2} + (1+t)\|w_{xx}(t)\|_{L^2}\right\} \tag{3.23}$$

$$M_3(w) = \sup_{0\le t\le\infty}\sum_{j=0}^{2}(1+t)^{\frac{2j+1}{4}}\|\partial_x^j w(t)\|_{L^2}. \tag{3.24}$$

Using Lemma 3.1 we obtain immediately the following estimates on the initial values.

Lemma 3.4 *If* $w_0 \in L^1\cap H^2$ *and* $w_1 \in L^1 \cap H^1$, *then*

$$\|\partial_x^j\{K_0(t) * w_0\}\|_{L^2} \le C(1+t)^{-\frac{j+1}{2}}(\|w_0\|_{L^1} + \|w_0\|_{H^2}) \tag{3.25}$$

$$\|\partial_x^j\{K_1(t) * w_1\}\|_{L^2} \le C(1+t)^{-\frac{j+1}{2}}(\|w_1\|_{L^1} + \|w_1\|_{H^1}) \tag{3.26}$$

for $j = 0,1,2$.

We are now going to prove the following estimates.

Lemma 3.5 *In the case:* $f''(0) \neq 0$, *suppose* $w(x,t) \in X_1(\delta)$, *then*

$$\int_0^t \|K_1(t-\tau) * (F_1+F_2)(\tau)\|_{L^2}d\tau \le C(\varepsilon + |u_+ - u_-| + \delta)^2(1+t)^{-\frac{1}{4}+\sigma}, \tag{3.27}$$

$$\int_0^t \|\partial_x\{K_1(t-\tau) * (F_1+F_2)(\tau)\}\|_{L^2}d\tau \le C(\varepsilon + |u_+ - u_-| + \delta)^2(1+t)^{-\frac{3}{4}+\sigma}, \tag{3.28}$$

$$\int_0^t \|\partial_x^2\{K_1(t-\tau) * (F_1+F_2)(\tau)\}\|_{L^2}d\tau \le C(\varepsilon + |u_+ - u_-| + \delta)^2(1+t)^{-1+\sigma}. \tag{3.29}$$

Furthermore, if $w_1(x,t), w_2(x,t) \in X_1(\delta)$, *then*

$$\sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}-\sigma}\int_0^t \|\partial_x^j\{K_1(t-\tau)*(F_2(w_1)-F_2(w_2))(\tau)\}\|_{L^2}d\tau$$
$$+(1+t)^{1-\sigma}\int_0^t \|\partial_x^2\{K_1(t-\tau)*(F_2(w_1)-F_2(w_2))(\tau)\}\|_{L^2}d\tau$$
$$\le C(\varepsilon+|u_+-u_-|+\delta)M_1(w_1-w_2) \tag{3.30}$$

Proof. Let $w(x,t) \in X_1(\delta)$, thanks to Lemma 3.2 and the definition of $X_1(\delta)$, and note the inequality

$$\|w_x(t)\|_{L^\infty} \le \sqrt{2}\|w_x(t)\|_{L^2}^{\frac{1}{2}}\|w_{xx}(t)\|_{L^2}^{\frac{1}{2}} \le \sqrt{2}\delta(1+t)^{-\frac{7}{8}+\sigma},$$

we first have several energy estimates on $\partial_x^j F_1$ and $\partial_x^j F_2$ $(j=0,1)$ in L^1 and L^2 spaces as follows

$$\begin{aligned}\|F_1\|_{L^1} &\le \int_{-\infty}^{\infty}[|\theta_{xt}|+|f''(0)\theta\theta_t|+|\hat{u}_x|]dx\\ &\le C(\|\theta_{xt}(t)\|_{L^1}+\|\theta(t)\|_{L^2}\|\theta_t(t)\|_{L^2}+\|\hat{v}_x(t)\|_{L^1})\\ &\le C[\varepsilon(1+t)^{-\frac{3}{2}}+\varepsilon^2(1+t)^{-\frac{1}{4}}(1+t)^{-\frac{5}{4}}+|u_+-u_-|e^{-t}]\\ &\le C(\varepsilon+|u_+-u_-|)^2(1+t)^{-\frac{3}{2}},\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}\|F_1\|_{L^2} &\le C(\|\theta_{xt}(t)\|_{L^2}+\|\theta(t)\|_{L^\infty}\|\theta_t(t)\|_{L^2}+\|\hat{u}_x(t)\|_{L^2})\\ &\le C[\varepsilon(1+t)^{-\frac{7}{4}}+\varepsilon^2(1+t)^{-\frac{1}{2}}(1+t)^{-\frac{5}{4}}+|u_+-u_-|e^{-t}]\\ &\le C(\varepsilon+|u_+-u_-|)^2(1+t)^{-\frac{7}{4}},\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}\|\partial_x F_1\|_{L^2} &\le C(\|\theta_{xxt}(t)\|_{L^2}+\|\theta_x(t)\|_{L^\infty}\|\theta_t(t)\|_{L^2}+\|\theta(t)\|_{L^\infty}\|\theta_{xt}(t)\|_{L^2}+\|\hat{u}_{xx}(t)\|_{L^2})\\ &\le C[\varepsilon(1+t)^{-\frac{9}{4}}+\varepsilon^2(1+t)^{-\frac{9}{4}}+|u_+-u_-|e^{-t}]\\ &\le C(\varepsilon+|u_+-u_-|)^2(1+t)^{-\frac{9}{4}},\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}\|F_2\|_{L^1} \le& C\int_{-\infty}^{\infty}[|\theta|^3+|\theta|^4+|\hat{v}|^2+|w_x|^2+(|\theta|+|\theta|^2+|\theta|^3)(|\hat{v}|+|w_x|)]dx\\ \le& C[\|\theta\|_{L^3}^3+\|\theta\|_{L^4}^4+\|\hat{v}\|_{L^2}^2+\|w_x\|_{L^2}^2\\ &+(\|\theta\|_{L^2}+\|\theta\|_{L^\infty}\|\theta\|_{L^2}+\|\theta\|_{L^\infty}^2\|\theta\|_{L^2})(\|\hat{v}\|_{L^2}+\|w_x\|_{L^2})]\\ \le& C\{\varepsilon^3(1+t)^{-1}+\varepsilon^4(1+t)^{-\frac{3}{2}}+|u_+-u_-|^2e^{-2t}+\delta^2(1+t)^{-\frac{3}{2}+2\sigma}\\ &+[\varepsilon(1+t)^{-\frac{1}{4}}+\varepsilon^2(1+t)^{-\frac{3}{4}}+\varepsilon^3(1+t)^{-\frac{5}{4}}]|u_+-u_-|e^{-t}\\ &+\varepsilon\delta(1+t)^{-1+\sigma}+\varepsilon^2\delta(1+t)^{-\frac{3}{2}+\sigma}+\varepsilon^3\delta(1+t)^{-2+\sigma}\}\\ \le& C(\varepsilon+|u_+-u_-|+\delta)^2(1+t)^{-1+\sigma}\end{aligned} \tag{3.34}$$

provided $\varepsilon + |u_+ - u_-| < 1$ and $\sigma < 1/2$, and

$$\begin{aligned}
\|F_2\|_{L^2} \le& C[\,\|\theta^3\|_{L^2} + \|\theta^4\|_{L^2} + \|\hat v^2\|_{L^2} + \|w_x^2\|_{L^2} + \|(|\theta| + |\theta|^2 + |\theta|^3)(|\hat v_x| + |w_x|)\|_{L^2}\,] \\
\le& C[\,\|\theta\|_{L^6}^3 + \|\theta\|_{L^8}^4 + \|\hat v\|_{L^\infty}\|\hat v\|_{L^2} + \|w_x\|_{L^\infty}\|w_x\|_{L^2}^2 \\
&+ (\|\theta\|_{L^\infty} + \|\theta\|_{L^\infty}^2 + \|\theta\|_{L^\infty}^3)(\|\hat v\|_{L^2} + \|w_x\|_{L^2})\,] \\
\le& C\{\varepsilon^3(1+t)^{-\frac54} + \varepsilon^4(1+t)^{-\frac74} + |u_+ - u_-|^2 e^{-2t} + \delta^2(1+t)^{-\frac{15}{8}+2\sigma} \\
&+ [\varepsilon(1+t)^{-\frac12} + \varepsilon^2(1+t)^{-1} + \varepsilon^3(1+t)^{-\frac32}]|u_+ - u_-|e^{-t} \\
&+ \varepsilon\delta(1+t)^{-\frac54+\sigma} + \varepsilon^2\delta(1+t)^{-\frac74+\sigma} + \varepsilon^3\delta(1+t)^{-\frac94+\sigma}\} \\
\le& C(\varepsilon + |u_+ - u_-| + \delta)^2(1+t)^{-\frac54+\sigma},
\end{aligned} \tag{3.35}$$

and finally,

$$\begin{aligned}
\|\partial_x F_2\|_{L^2} \le& C\Big(\int_{-\infty}^{\infty} [|\theta^2\theta_x| + |\theta^3\theta_x| + |\hat v\hat v_x| + |w_x w_{xx}| + (|\theta_x| + |\theta\theta_x| + |\theta^2\theta_x|)(|\hat v| + |w_x|) \\
&\qquad + (|\theta| + |\theta^2| + |\theta^3|)(|\hat v_x| + |w_{xx}|)]^2 dx\Big)^{\frac12} \\
\le& C[\,\|\theta\|_{L^\infty}^2\|\theta_x\|_{L^2} + \|\theta\|_{L^\infty}^3\|\theta_x\|_{L^2} + \|\hat v\|_{L^\infty}\|\hat v_x\|_{L^2} + \|w_x\|_{L^\infty}\|w_{xx}\|_{L^2} \\
&+ (\|\theta_x\|_{L^\infty} + \|\theta\|_{L^\infty}\|\theta_x\|_{L^\infty} + \|\theta\|_{L^\infty}^2\|\theta_x\|_{L^\infty})(\|\hat v\|_{L^2} + \|w_x\|_{L^2}) \\
&+ (\|\theta\|_{L^\infty} + \|\theta\|_{L^\infty}^2 + \|\theta\|_{L^\infty}^3)(\|\hat v_x\|_{L^2} + \|w_{xx}\|_{L^2})] \\
\le& C\{\varepsilon^3(1+t)^{-\frac74} + \varepsilon^4(1+t)^{-\frac94} + |u_+ - u_-|^2 e^{-2t} + \delta^2(1+t)^{-\frac{15}{8}+2\sigma} \\
&+ [\varepsilon(1+t)^{-\frac32} + \varepsilon^2(1+t)^{-2} + \varepsilon^3(1+t)^{-\frac52}][|u_+ - u_-|e^{-t} + \delta(1+t)^{-\frac34+\sigma}] \\
&+ [\varepsilon(1+t)^{-\frac12} + \varepsilon^2(1+t)^{-1} + \varepsilon^3(1+t)^{-\frac32}][|u_+ - u_-|e^{-t} + \delta(1+t)^{-1+\sigma}]\} \\
\le& C(\varepsilon + |u_+ - u_-| + \delta)^2(1+t)^{-\frac32+\sigma}.
\end{aligned} \tag{3.36}$$

Applying Lemma 3.1, Lemma 3.3, and above energy estimates on $\partial_x^j F_1$ and $\partial_x^j F_2$ $(j = 0, 1)$, we prove (3.27), (3.28) and (3.29) as follows

$$\begin{aligned}
&\int_0^t \|K_1(t-\tau) * (F_1 + F_2)(\tau)\|_{L^2} d\tau \\
&\le C\int_0^t (1+t-\tau)^{-\frac14}[\|F_1(\tau)\|_{L^1} + \|F_2(\tau)\|_{L^1} + \|F_1(\tau)\|_{L^2} + \|F_2(\tau)\|_{L^2}]d\tau \\
&\le C(\varepsilon + |u_+ - u_-| + \delta)^2 \int_0^t (1+t-\tau)^{-\frac14} \\
&\qquad \cdot [(1+\tau)^{-\frac32} + (1+\tau)^{-1+\sigma} + (1+\tau)^{-\frac74} + (1+\tau)^{-\frac54+\sigma}]d\tau \\
&\le C(\varepsilon + |u_+ - u_-| + \delta)^2 \int_0^t (1+t-\tau)^{-\frac14}(1+\tau)^{-1+\sigma} d\tau \\
&\le C(\varepsilon + |u_+ - u_-| + \delta)^2 (1+t)^{1-\frac14-(1-\sigma)} \\
&= C(\varepsilon + |u_+ - u_-| + \delta)^2 (1+t)^{-\frac14+\sigma},
\end{aligned} \tag{3.37}$$

and

$$\begin{aligned}
&\int_0^t \|\partial_x\{K_1(t-\tau)*(F_1+F_2)(\tau)\}\|_{L^2}d\tau \\
&\leq C\int_0^t (1+t-\tau)^{-\frac{3}{4}}[\|F_1(\tau)\|_{L^1}+\|F_2(\tau)\|_{L^1}+\|F_1(\tau)\|_{L^2}+\|F_2(\tau)\|_{L^2}]d\tau \\
&\leq C(\varepsilon+|u_+-u_-|+\delta)^2\int_0^t (1+t-\tau)^{-\frac{3}{4}} \\
&\qquad\cdot[(1+\tau)^{-\frac{3}{2}}+(1+\tau)^{-1+\sigma}+(1+\tau)^{-\frac{7}{4}}+(1+\tau)^{-\frac{5}{4}+\sigma}]d\tau \\
&\leq C(\varepsilon+|u_+-u_-|+\delta)^2\int_0^t (1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-1+\sigma}d\tau \\
&\leq C(\varepsilon+|u_+-u_-|+\delta)^2(1+t)^{1-\frac{3}{4}-(1-\sigma)} \\
&= C(\varepsilon+|u_+-u_-|+\delta)^2(1+t)^{-\frac{3}{4}+\sigma},
\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}
&\int_0^t \|\partial_x^2\{K_1(t-\tau)*(F_1+F_2)(\tau)\}\|_{L^2}d\tau \\
&\leq C\int_0^t (1+t-\tau)^{-\frac{5}{4}}[\|F_1(\tau)\|_{L^1}+\|F_2(\tau)\|_{L^1}+\|F_1(\tau)\|_{H^1}+\|F_2(\tau)\|_{H^1}]d\tau \\
&\leq C(\varepsilon+|u_+-u_-|+\delta)^2\int_0^t (1+t-\tau)^{-\frac{5}{4}} \\
&\qquad\cdot[(1+\tau)^{-\frac{3}{2}}+(1+\tau)^{-1+\sigma}+(1+\tau)^{-\frac{7}{4}}+(1+\tau)^{-\frac{5}{4}+\sigma}]d\tau \\
&\leq C(\varepsilon+|u_+-u_-|+\delta)^2\int_0^t (1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-1+\sigma}d\tau \\
&\leq C(\varepsilon+|u_+-u_-|+\delta)^2(1+t)^{-\min\{\frac{5}{4},1-\sigma\}} \\
&= C(\varepsilon+|u_+-u_-|+\delta)^2(1+t)^{-1+\sigma}.
\end{aligned} \tag{3.39}$$

On the other hand, if $w_1(x,t),\ w_2(x,t)\in X_1(\delta)$, we are going to prove (3.30). Firstly, we have

$$\begin{aligned}
&F_2(w_1)-F_2(w_2) \\
&= O(1)(2\hat{v}+w_{1x}+w_{2x})(w_{1x}-w_{2x}) \\
&\quad +[f''(0)\theta+\frac{1}{2}f'''(0)\theta^2+O(1)\theta^3](w_{1x}-w_{2x}),
\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}
&\partial_x(F_2(w_1)-F_2(w_2)) \\
&= O(1)(2\hat{v}+w_{1x}+w_{2x})(w_{1xx}-w_{2xx}) \\
&\quad +O(1)(2\hat{v}_x+w_{1xx}+w_{2xx})(w_{1x}-w_{2x}) \\
&\quad +[f''(0)\theta+\frac{1}{2}f'''(0)\theta^2+O(1)\theta^3](w_{1xx}-w_{2xx}) \\
&\quad +[f''(0)\theta_x+f'''(0)\theta\theta_x+O(1)\theta^2\theta_x](w_{1x}-w_{2x}).
\end{aligned} \tag{3.41}$$

Thus, we get the following energy estimates

$$
\begin{aligned}
&\|F_2(w_1)-F_2(w_2)\|_{L^1}\\
&\le C(\|\hat{v}\|_{L^2}+\|w_{1x}\|_{L^2}+\|w_{2x}\|_{L^2})\|w_{1x}-w_{2x}\|_{L^2}\\
&\quad +C(\|\theta\|_{L^2}+\|\theta\|_{L^\infty}\|\theta\|_{L^2}+\|\theta\|_{L^\infty}^2\|\theta\|_{L^2})\|w_{1x}-w_{2x}\|_{L^2}\\
&\le C[|u_+-u_-|e^{-t}+\delta(1+t)^{-\frac{3}{4}+\sigma}]\|w_{1x}-w_{2x}\|_{L^2}\\
&\quad +C[\varepsilon(1+t)^{-\frac{1}{4}}+\varepsilon^2(1+t)^{-\frac{3}{4}}+\varepsilon^3(1+t)^{-\frac{5}{4}}]\|w_{1x}-w_{2x}\|_{L^2}\\
&\le C(\varepsilon+|u_+-u_-|+\delta)(1+t)^{-\frac{1}{4}}\|w_{1x}-w_{2x}\|_{L^2}\\
&\le C(\varepsilon+|u_+-u_-|+\delta)(1+t)^{-1+\sigma}M_1(w_1-w_2),
\end{aligned} \tag{3.42}
$$

and

$$
\begin{aligned}
&\|F_2(w_1)-F_2(w_2)\|_{L^2}\\
&\le C(\|\hat{v}\|_{L^\infty}+\|w_{1x}\|_{L^\infty}+\|w_{2x}\|_{L^\infty})\|w_{1x}-w_{2x}\|_{L^2}\\
&\quad +C(\|\theta\|_{L^\infty}+\|\theta\|_{L^\infty}^2+\|\theta\|_{L^\infty}^3)\|w_{1x}-w_{2x}\|_{L^2}\\
&\le C[|u_+-u_-|e^{-t}+\delta(1+t)^{-\frac{7}{8}+\sigma}]\|w_{1x}-w_{2x}\|_{L^2}\\
&\quad +C[\varepsilon(1+t)^{-\frac{1}{2}}+\varepsilon^2(1+t)^{-1}+\varepsilon^3(1+t)^{-\frac{3}{2}}]\|w_{1x}-w_{2x}\|_{L^2}\\
&\le C(\varepsilon+|u_+-u_-|+\delta)(1+t)^{-\frac{1}{2}}\|w_{1x}-w_{2x}\|_{L^2}\\
&\le C(\varepsilon+|u_+-u_-|+\delta)(1+t)^{-\frac{5}{4}+\sigma}M_1(w_1-w_2),
\end{aligned} \tag{3.43}
$$

and by using the inequality

$$
\begin{aligned}
\|w_{1x}-w_{2x}\|_{L^\infty} &\le \sqrt{2}\|w_{1x}-w_{2x}\|_{L^2}^{\frac{1}{2}}\|w_{1xx}-w_{2xx}\|_{L^2}^{\frac{1}{2}}\\
&\le \frac{\sqrt{2}}{2}(\|w_{1x}-w_{2x}\|_{L^2}+\|w_{1xx}-w_{2xx}\|_{L^2})
\end{aligned}
$$

to get

$$
\begin{aligned}
&\|\partial_x(F_2(w_1)-F_2(w_2))\|_{L^2}\\
&\le C(\|\hat{v}\|_{L^\infty}+\|w_{1x}\|_{L^\infty}+\|w_{2x}\|_{L^\infty})\|w_{1xx}-w_{2xx}\|_{L^2}\\
&\quad +C(\|\hat{v}_x\|_{L^2}+\|w_{1xx}\|_{L^2}+\|w_{2xx}\|_{L^2})\|w_{1x}-w_{2x}\|_{L^\infty}\\
&\quad +C(\|\theta\|_{L^\infty}+\|\theta\|_{L^\infty}^2+\|\theta\|_{L^\infty}^3)\|w_{1xx}-w_{2xx}\|_{L^2}\\
&\quad +C(\|\theta_x\|_{L^2}+\|\theta\|_{L^\infty}\|\theta_x\|_{L^2}+\|\theta\|_{L^\infty}^2\|\theta_x\|_{L^2})\|w_{1x}-w_{2x}\|_{L^\infty}\\
&\le C[|u_+-u_-|e^{-t}+\delta(1+t)^{-\frac{7}{8}+\sigma}]\|w_{1xx}-w_{2xx}\|_{L^2}\\
&\quad +C[|u_+-u_-|e^{-t}+\delta(1+t)^{-1+\sigma}]\|w_{1x}-w_{2x}\|_{L^\infty}\\
&\quad +C[\varepsilon(1+t)^{-\frac{1}{2}}+\varepsilon^2(1+t)^{-1}+\varepsilon^3(1+t)^{-\frac{3}{2}}]\|w_{1xx}-w_{2xx}\|_{L^2}\\
&\quad +C[\varepsilon(1+t)^{-\frac{3}{4}}+\varepsilon^2(1+t)^{-\frac{5}{4}}+\varepsilon^3(1+t)^{-\frac{7}{4}}]\|w_{1x}-w_{2x}\|_{L^\infty}\\
&\le C(\varepsilon+|u_+-u_-|+\delta)(1+t)^{-\frac{1}{2}}(\|w_{1xx}-w_{2xx}\|_{L^2}+\|w_{1x}-w_{2x}\|_{L^\infty})\\
&\le C(\varepsilon+|u_+-u_-|+\delta)(1+t)^{-\frac{1}{2}}(\|w_{1xx}-w_{2xx}\|_{L^2}+\|w_{1x}-w_{2x}\|_{L^2})\\
&\le C(\varepsilon+|u_+-u_-|+\delta)(1+t)^{-\frac{5}{4}}M_1(w_1-w_2).
\end{aligned} \tag{3.44}
$$

Therefore, making use of Lemma 3.1 and Lemma 3.3, we obtain

$$\begin{aligned}
&\sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}-\sigma}\int_0^t \|\partial_x^j\{K_1(t-\tau)*(F_2(w_1)-F_2(w_2))(\tau)\}\|_{L^2}d\tau \\
&+(1+t)^{1-\sigma}\int_0^t \|\partial_x^2\{K_1(t-\tau)*(F_2(w_1)-F_2(w_2))(\tau)\}\|_{L^2}d\tau \\
&\leq C\sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}-\sigma}\int_0^t (1+t-\tau)^{-\frac{2j+1}{4}}[\|F_2(w_1)-F_2(w_2))(\tau)\|_{L^1} \\
&\qquad +\|F_2(w_1)-F_2(w_2))(\tau)\|_{L^2}]d\tau \\
&\quad +C(1+t)^{1-\sigma}\int_0^t (1+t-\tau)^{-\frac{5}{4}}[\|F_2(w_1)-F_2(w_2))(\tau)\|_{L^1} \\
&\qquad +\|F_2(w_1)-F_2(w_2))(\tau)\|_{H^1}]d\tau \\
&\leq CM_1(w_1-w_2)\sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}-\sigma}\int_0^t (1+t-\tau)^{-\frac{2j+1}{4}}(1+\tau)^{-1+\sigma}d\tau \\
&\quad +CM_1(w_1-w_2)(1+t)^{1-\sigma}\int_0^t (1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-1+\sigma}d\tau \\
&\leq CM_1(w_1-w_2)\sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}-\sigma}(1+t)^{-\frac{2j+1}{4}+\sigma} \\
&\quad +CM_1(w_1-w_2)(1+t)^{1-\sigma}(1+t)^{-1+\sigma}d\tau \\
&\leq C(\varepsilon+|u_+-u_-|+\delta)M_1(w_1-w_2).
\end{aligned} \tag{3.45}$$

Thus, our proof is complete. □

Similarly, in the case: $f''(0)=0$ but $f'''(0)\neq 0$, and case: $f''(0)=0$ and $f'''(0)=0$, we may obtain the corresponding estimates in $X_2(\delta)$ and $X_3(\delta)$ in the same way. We state them as follows but omit the proof details.

Lemma 3.6 *In the case: $f''(0)=0$ but $f'''(0)\neq 0$, suppose $w(x,t)\in X_2(\delta)$, then*

$$\int_0^t \|K_1(t-\tau)*(F_1+F_2)(\tau)\|_{L^2}d\tau \leq C(\varepsilon+|u_+-u_-|+\delta)^2(1+t)^{-\frac{1}{4}}\ln(2+t), \tag{3.46}$$

$$\int_0^t \|\partial_x\{K_1(t-\tau)*(F_1+F_2)(\tau)\}\|_{L^2}d\tau \leq C(\varepsilon+|u_+-u_-|+\delta)^2(1+t)^{-\frac{3}{4}}\ln(2+t), \tag{3.47}$$

$$\int_0^t \|\partial_x^2\{K_1(t-\tau)*(F_1+F_2)(\tau)\}\|_{L^2}d\tau \leq C(\varepsilon+|u_+-u_-|+\delta)^2(1+t)^{-1}. \tag{3.48}$$

Furthermore, if $w_1(x,t), w_2(x,t) \in X_2(\delta)$, then

$$\sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}}\ln(2+t)\int_0^t \|\partial_x^j\{K_1(t-\tau)*(F_2(w_1)-F_2(w_2))(\tau)\}\|_{L^2}d\tau$$
$$+(1+t)\int_0^t \|\partial_x^2\{K_1(t-\tau)*(F_2(w_1)-F_2(w_2))(\tau)\}\|_{L^2}d\tau$$
$$\le C(\varepsilon+|u_+-u_-|+\delta)M_2(w_1-w_2). \tag{3.49}$$

Lemma 3.7 *In the case: $f''(0)=0$ and $f'''(0)=0$, suppose $w(x,t)\in X_3(\delta)$, then*

$$\int_0^t \|K_1(t-\tau)*(F_1+F_2)(\tau)\|_{L^2}d\tau \le C(\varepsilon+|u_+-u_-|+\delta)^2(1+t)^{-\frac{1}{4}}, \tag{3.50}$$

$$\int_0^t \|\partial_x\{K_1(t-\tau)*(F_1+F_2)(\tau)\}\|_{L^2}d\tau \le C(\varepsilon+|u_+-u_-|+\delta)^2(1+t)^{-\frac{3}{4}}, \tag{3.51}$$

$$\int_0^t \|\partial_x^2\{K_1(t-\tau)*(F_1+F_2)(\tau)\}\|_{L^2}d\tau \le C(\varepsilon+|u_+-u_-|+\delta)^2(1+t)^{-\frac{5}{4}}. \tag{3.52}$$

Furthermore, if $w_1(x,t), w_2(x,t)\in X_3(\delta)$, then

$$\sum_{j=0}^{2}(1+t)^{\frac{2j+1}{4}}\int_0^t \|\partial_x^j\{K_1(t-\tau)*(F_2(w_1)-F_2(w_2))(\tau)\}\|_{L^2}d\tau$$
$$\le C(\varepsilon+|u_+-u_-|+\delta)M_3(w_1-w_2). \tag{3.53}$$

4 Proof of Main Theorem

We are going to prove our main result-Theorem 2.1. We first prove the case 1: $f''(0)\neq 0$. Set

$$Sw := K_0(t)*w_0+K_1(t)*w_1+\int_0^t K_1(t-\tau)*(F_1+F_2)(\tau)d\tau,$$

we are going to prove that there exists a positive constant δ_1 such that the integral operator S maps contractedly from $X_1(\delta_1)$ into itself, and has a unique fixed point $w(x,t)$ satisfying $w=Sw$ in space $X_1(\delta_1)$, which is just a unique solution of (2.14) as we are looking for.

Step 1. $S: X_1(\delta)\to X_1(\delta)$. Let $\bar{w}(x,t)\in X_1(\delta)$ for some positive constant δ chosen below and $w:=S\bar{w}$, we are going to prove $w=S\bar{w}\in X_1(\delta)$. Thanks to Lemmas 3.4 and 3.5, we have

$$\|\partial_x^j w(t)\|_{L^2}=\|\partial_x^j S\bar{w}(t)\|_{L^2}$$
$$\le \|\partial_x^j\{K_0(t)*w_0\}\|_{L^2}+\|\partial_x^j\{K_1(t)*w_1\}\|_{L^2}+\int_0^t\|\partial_x^j\{K_1(t-\tau)*(F_1+F_2)(\tau)\}\|_{L^2}$$
$$\le C[\|(w_0,w_1)\|_{L^1}+\|w_0\|_{H^2}+\|w_1\|_{H^1}](1+t)^{-\frac{2j+1}{4}}$$
$$+C(\varepsilon+|u_+-u_-|+\delta)^2(1+t)^{-\frac{2j+1}{4}+\sigma}$$
$$\le C[\|(w_0,w_1)\|_{L^1}+\|w_0\|_{H^2}+\|w_1\|_{H^1}+(\varepsilon+|u_+-u_-|+\delta)^2](1+t)^{-\frac{2j+1}{4}+\sigma} \tag{4.1}$$

for $j = 0, 1$, and

$$\begin{aligned}
&\|\partial_x^2 w(t)\|_{L^2} = \|\partial_x^2 S\bar{w}(t)\|_{L^2} \\
&\le \|\partial_x^2\{K_0(t) * w_0\}\|_{L^2} + \|\partial_x^2\{K_1(t) * w_1\}\|_{L^2} + \int_0^t \|\partial_x^2\{K_1(t-\tau) * (F_1 + F_2)(\tau)\}\|_{L^2} \\
&\le C[\|(w_0, w_1)\|_{L^1} + \|w_0\|_{H^2} + \|w_1\|_{H^1}](1+t)^{-\frac{5}{4}} \\
&\quad + C(\varepsilon + |u_+ - u_-| + \delta)^2(1+t)^{-1+\sigma} \\
&\le C[\|(w_0, w_1)\|_{L^1} + \|w_0\|_{H^2} + \|w_1\|_{H^1} + (\varepsilon + |u_+ - u_-| + \delta)^2](1+t)^{-1+\sigma}. \qquad (4.2)
\end{aligned}$$

Thus, multiplying (4.1) by $(1+t)^{-\frac{2j+1}{4}+\sigma}$ $(j = 0, 1)$ and (4.2) by $(1+t)^{-1+\sigma}$, respectively, then we add them to obtain

$$\begin{aligned}
&\sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}-\sigma}\|\partial_x^j w(t)\|_{L^2} + (1+t)^{1-\sigma}\|\partial_x^2 w(t)\|_{L^2} \\
&\le C[\|(w_0, w_1)\|_{L^1} + \|w_0\|_{H^2} + \|w_1\|_{H^1} + (\varepsilon + |u_+ - u_-| + \delta)^2]. \qquad (4.3)
\end{aligned}$$

Now we choose

$$\delta \le \delta_2 := \min\{1, 2/(9C)\},$$

when

$$\|(w_0, w_1)\|_{L^1} + \|w_0\|_{H^2} + \|w_1\|_{H^1} < \delta/(2C) \quad \text{and} \quad \varepsilon + |u_+ - u_-| < \delta/2,$$

from (4.3), we prove

$$\sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}-\sigma}\|\partial_x^j w(t)\|_{L^2} + (1+t)^{1-\sigma}\|\partial_x^2 w(t)\|_{L^2} < \delta,$$

namely, $w = S\bar{w} \in X_1(\delta)$. This means that the operator S maps $X_1(\delta)$ into $X_1(\delta)$ for $\delta \le \delta_2$.

Step 2. S is contraction in $X_1(\delta)$. Now let $\bar{w}_1(x,t)$ and $\bar{w}_2(x,t)$ be in $X_1(\delta)$, and denote $w_i(x,t) := S\bar{w}_i$ $(i = 1, 2)$. Thanks to Lemma 3.5, and note that the term F_1 is independent of $w(x,t)$, we have

$$\begin{aligned}
&\sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}-\sigma}\|\partial_x^j (w_1 - w_2)(t)\|_{L^2} + (1+t)^{1-\sigma}\|\partial_x^2 (w_1 - w_2)(t)\|_{L^2} \\
&= \sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}-\sigma}\|\partial_x^j (S\bar{w}_1 - S\bar{w}_2)(t)\|_{L^2} + (1+t)^{1-\sigma}\|\partial_x^2 (S\bar{w}_1 - S\bar{w}_2)(t)\|_{L^2} \\
&\le \sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}-\sigma}\int_0^t \|\partial_x^j\{K_1(t-\tau) * [F_2(\bar{w}_1) - F_2(\bar{w}_2)](\tau)\}\|_{L^2} \\
&+ (1+t)^{1-\sigma}\int_0^t \|\partial_x^2\{K_1(t-\tau) * [F_2(\bar{w}_1) - F_2(\bar{w}_2)](\tau)\}\|_{L^2} \\
&\le C(\varepsilon + |u_+ - u_-| + \delta)M_1(\bar{w}_1 - \bar{w}_2), \qquad (4.4)
\end{aligned}$$

namely, we prove

$$M_1(w_1 - w_2) \leq C(\varepsilon + |u_+ - u_-| + \delta) M_1(\bar{w}_1 - \bar{w}_2).$$

Let

$$\delta \leq \delta_3 := \min\{1, 2/(3C)\}$$

and

$$\varepsilon + |u_+ - u_-| < \delta/2,$$

then we prove

$$M_1(w_1 - w_2) < M_1(\bar{w}_1 - \bar{w}_2).$$

This means that the operator S is a contraction map in $X_1(\delta)$ for $\delta \leq \delta_3$.

Now let δ_1 in Theorem 2.1 be

$$\delta_1 := \min\{\delta_2, \delta_3\},$$

due to above two steps, we proved that S maps $X_1(\delta_1)$ into itself, and is contraction in $X_1(\delta_1)$. Thus, applying the Banach's fixed point theorem, there exists a unique fixed point $w(x,t)$ for the operator S in the space $X_1(\delta_2)$ so that $w = Sw$. Such a fixed point $w(x,t)$ is just the unique solution of Eq. (2.14). Thus, we have completed the proof of the first case in Theorem 2.1.

In the same way, making use of Lemmas 3.4, 3.6 and 3.7, we can prove the other two cases in Theorem 2.1. The details are omitted here.

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訪問報告

1. 2002/8/17-8/19

此次訪問，先到廈門大學訪問許傳矩教授，他是廈大數學系計算科學實驗室主任，在 spectral-element 方法上有很多建樹。此次訪問主要是了解他及廈大計算科學實驗室之具體工作及未來發展，並商談他們到台灣訪問及合作事宜。

2. 2002/8/19-8/26

到北京訪問中科院應數所常謙順教授，同時參加世界數學家大會（ICM）。訪問常教授是要完成我們合作的一篇文章“Acceleration Methods for Total variation based Image Restoration”，同時也討論另一合作工作。其他時間則參加 ICM。大會邀請的一小時報告中，D. Arnold 談到數值的 De Rham 定理及廣義相對論的計算以及 Bresson 的 Conservation Laws 近年的進展最令我感興趣。

3.大陸在舉辦 ICM 期間，數學新聞頻上媒體、電視台每日並有數學家的故事，以及數學應用之一小時專訪報導。對提昇社會大眾對數學的認知有很大幫助。